Mem. Fac. Sci., Shimane Univ.,
12, pp. 1–4 Dec. 20, 1978

The Embedding of Generalized Inverse Semigroup Amalgams

Teruo IMAOKA
Department of Mathematics, Shimane University, Matsue, Japan
(Received September 6, 1978)

Let $[\{S_1, S_2\}; U]$ be a [left, right] generalized inverse semigroup amalgam. In [4], the author has shown that the amalgam is embedded in a [left, right] generalized inverse semigroup if S_1 is isomorphic to S_2. In this paper, we shall give a sufficient condition for the amalgam to be embedded in a [left, right] generalized inverse semigroup. Further, we shall consider the embedding of [left, right] normal band amalgams. The notation and terminologies are those of [1] and [5], unless otherwise stated.

Let $\mathscr{A}$ be a class of algebras. We define that an $\mathscr{A}$-amalgam $[\{S_i: i \in I\}; U]$ consists of an algebra U in $\mathscr{A}$ and a family of algebras $\{S_i \in \mathscr{A}: S_i \supset U, i \in I\}$. This definition is slightly different to Howie's [3]. An $\mathscr{A}$-amalgam $[\{S_i: i \in I\}; U]$ is said to be embedded in an algebra in $\mathscr{A}$ if there exist an algebra T in $\mathscr{A}$ and monomorphisms $\phi_i: S_i \to T$, $i \in I$, such that

(i) $\phi_i | U = \phi_j | U$ for all $i, j \in I$,

(ii) $S_i\phi_i \cap S_j\phi_j = U\phi_i$ for all $i, j \in I$ with $i \neq j$.

We now state an extension of [left, right] normal bands which we need later.

LEMMA 1. *Let Δ be a subsemilattice of a semilattice Γ. Let $E \equiv \Sigma\{E_\alpha: \alpha \in \Delta\}$ be a [left, right] normal band. Then there exist a [left, right] normal band $B \equiv \Sigma\{B_\alpha: \alpha \in \Gamma\}$ and a monomorphism $\phi: E \to B$ such that $E_\alpha\phi \subset B_\alpha$ for all $\alpha \in \Delta$.*

PROOF. Let E be a normal band. Since each E_α is a rectangular band, it is the direct product of a left zero semigroup L_α and a right zero semigroup R_α, $\alpha \in \Delta$. We denote an element of E_α by (l_α, r_α), $l_\alpha \in L$, $r_\alpha \in R$. Let $F = \{(e_\alpha, f_\alpha): \alpha \in \Gamma\}$ be a semilattice whose multiplication is defined by

$$(e_\alpha, f_\alpha)(e_\beta, f_\beta) = (e_{\alpha\beta}, f_{\alpha\beta}) \qquad \text{for all} \quad \alpha, \beta \in \Gamma.$$

Then $F' = \{(e_\alpha, f_\alpha)\ \alpha \in \Delta\}$ is a subsemilattice of F. Let $E' = E \cup F'$ and its multiplication is defined as follows:

$$(x_\alpha, y_\alpha)(z_\beta, w_\beta) = \begin{cases} (x_\alpha l_{\alpha\beta}, r_{\alpha\beta} w_\beta) & \text{if } (x_\alpha, y_\alpha), (z_\beta, w_\beta) \in E, \\ (x_\alpha l_{\alpha\beta}, f_{\alpha\beta}) & \text{if } (x_\alpha, y_\alpha) \in E, (z_\beta, w_\beta) = (e_\beta, f_\beta), \end{cases}$$

$$\begin{cases} (e_{\alpha\beta}, r_{\alpha\beta}w_\beta) & \text{if } (x_\alpha, y_\alpha)=(e_\alpha, f_\alpha),\ (z_\beta, w_\beta)\in E, \\ (e_{\alpha\beta}, f_{\alpha\beta}) & \text{if } (x_\alpha, y_\alpha)=(e_\alpha, f_\alpha),\ (z_\beta, w_\beta)=(e_\beta, f_\beta). \end{cases}$$

It is clear that E' is a normal band whose structure decomposition is $E'\equiv\Sigma\{E_\alpha\cup\{(e_\alpha, f_\alpha)\}: \alpha\in\Delta\}$. Since $E'\cap F=F'$, it follows from [4] that there exists the free procuct B of E' and F amalgamating F' in the class of normal bands, and that the structure semilattice of B is Γ.

Now we give a sufficient condition for a [left, right] generalized inverse semigroup amalgam to be embedded in a [left, right] generalized inverse semigroup. Simplifying the proof, we consider the case of left generalized inverse semigroups. In the other cases, we can similarly obtain the analogous results.

Let Γ be an inverse semigroup whose basic semilattice is Δ, that is, $E(\Gamma)=\Delta$. Hereafter, we denote it by $\Gamma(\Delta)$. Let S_1 and S_2 be left generalized inverse semigroups with a common orthodox semigrup U. We can assume without loss of generality that $S_1\cap S_2=U$. By [6], there exist inverse semigroups $\Gamma_1(\Delta_1)$, $\Gamma_2(\Delta_2)$ and $\Xi(\Lambda)$ and left normal bands $L_i\equiv\Sigma\{L_i^\alpha: \alpha\in\Delta_i\}$, $i=1, 2$, and $U\equiv\Sigma\{U_\alpha: \alpha\in\Lambda\}$ such that $S_i=Q(L_i\otimes\Gamma_i; \Delta_i)$, $i=1, 2$, $U=Q(U\otimes\Xi; \Lambda)$, $L_1\cap L_2=V$, $\Gamma_1\cap\Gamma_2=\Xi$ and $\Delta_1\cap\Delta_2=\Lambda$.

It is well-known [2] that the inverse semigroup amalgam $[\{\Gamma_1, \Gamma_2\}; \Xi]$ is embedded in an inverse semigroup Γ, that is, there exist monomorphisms $\phi_i: \Gamma_i\to\Gamma$, $i=1, 2$, such that $\phi_1|\Xi=\phi_2|\Xi$, $\Gamma_1\phi_1\cap\Gamma_2\phi_2=\Xi\phi_1$. Let Δ be the basic semilattice of Γ and let Σ be the subsemilattice of generated by $\Delta_1\phi_1\cup\Delta_2\phi_2$.

Theorem 2. *We use the notations above. Let Ω be a subsemilattice of Δ containing Σ. Assume that the amalgam $[\{L_1, L_2\}; V]$ is embedded in a left normal band $B\equiv\Sigma\{B_\alpha; \alpha\in\Omega\}$, that is, there exist monomorphisms $\psi_i: L_i\to B$, $i=1, 2$, satisfying $\psi_1|V=\psi_2|V$ and $L_1\psi_1\cap L_2\psi_2=V\psi_1$. Under each ψ_i, $L_i^\alpha\psi_i$ is contained in some B_β. Let λ_i; $\Delta_i\to\Omega$ be a mapping such that $\alpha\lambda_i=\beta$. If $\lambda_i=\phi_i|\Delta_i$, $i=1, 2$, the amalgam $[\{S_1, S_2\}; U]$ is embedded in a left generalized inverse semigroup.*

Proof. It follows from Lemma 1 that there exist a left normal band $N\equiv\Sigma\{N_\alpha: \alpha\in\Delta\}$ and a monomorphism $\rho: B\to N$ such that $B_\alpha\rho\subset N_\alpha$ for all $\alpha\in\Omega$. Let T be the left quasi-direct product $Q(N\otimes\Gamma; \Delta)$, and define mappings $\mu_i: S_i\to T$, $i=1, 2$, as follows:

$$(a_i, \alpha_i)\mu_i=(a_i\psi_i\rho, \alpha_i\phi_i) \qquad \text{for all} \quad (a_i, \alpha_i)\in S_i.$$

Let $\gamma=\alpha_i\alpha_i^{-1}$. Then $a_i\psi_i\rho\in L_i^\gamma\psi_i\rho\subset L_i^{\gamma\lambda_i}\rho\subset N_{\gamma\lambda_i}=N_{\gamma\phi_i}=N_{\alpha_i\phi_i(\alpha_i\phi_i)}-1$, and we have $(a_i, \alpha_i)\psi_i\in T$. Since ρ, ψ_i and ϕ_i are all monomorphisms, we can easily show that each μ_i is a monomorphism, $i=1, 2$. Since $\psi_1\rho|V=\psi_2\rho|V$ and $\phi_1|\Xi=\phi_2|\Xi$, it is clear that $\mu_1|U=\mu_2|U$. Let (a, α) and (b, β) be elements of S_1 and S_2, respectively, such that $(a, \alpha)\mu_1=(b, \beta)\mu_2$. It follows from the definition of the μ_i that $a\psi_1\rho=b\psi_2\rho$ and $\alpha\phi_1=\beta\phi_2$. Since ρ is a monomorphism, we have $a\psi_1=b\psi_2$. By the definitions of the ψ_i and the ϕ_i, we have $a=b\in V$ and $\alpha=\beta\in\Xi$. Then $S_1\mu_1\cap S_2\mu_2=U\mu_1$. Therefore, the amalgam $[\{S_1, S_2\}; U]$ is embedded in T, and hence we have the theorem.

Let N_1 and N_2 be [left, right] normal bands with a common [left, right] normal band V. Let Δ_1, Δ_2 and Λ be the structure semilattices of N_1, N_2 and V, respectively. Let Ω be a semilattice in which the amalgam $[\{\Delta_1, \Delta_2\}; \Lambda]$ is embedded. However, there does not always exist a [lift, right] normal band B in which the amalgam $[\{N_1, N_2\}; V]$ is embedded and whose structure semilattice is Ω. For example, let $N_1=\{u, v, a\}$, $N_2=\{u, v, b, c\}$ and $U=\{u, v\}$ be left normal bands whose multiplications are defined as follows:

	u	v	a
u	u	u	a
v	v	v	a
a	a	a	a

	u	v	b	c
u	u	u	b	b
v	v	v	c	c
b	b	b	b	b
c	c	c	c	c

Let $\Delta_1=\{\delta, \alpha: \delta>\alpha\}$, $\Delta_2=\{\delta, \beta: \delta>\beta\}$ and $\Lambda=\{\delta\}$. Then $N_1\equiv\Sigma\{N_\mu: \mu\in\Delta_1\}$, $N_2\equiv\Sigma\{N_\mu: \mu\in\Delta_2\}$ and $V=\{N_\delta\}$ where $N_\delta=\{u, v\}$, $N_\alpha=\{a\}$ and $N_\beta=\{b, c\}$. It is obvious that the amalgam $[\{\Delta_1, \Delta_2\}; \Lambda]$ is embedded in $\Omega=\{\delta, \alpha, \beta: \delta>\alpha>\beta\}$. Assume that the amalgam $[\{N_1, N_2\}: V]$ is embedded in a left normal band $B\equiv\Sigma\{B_\mu: \mu\in\Omega\}$. It follows from [6] that there exists an element f in B_β such that $ax=f$ for all x in B_β. Then

$$b=ub=(ub)a=uab=ab=f.$$

Similarly, we have $c=f$, and hence $b=c$, contradiction.

In [4], we have seen that the amalgam $[\{N_1, N_2\}; V]$ is embedded in a [left, right] normal band whose structure semilattice is the free product of Δ_1 and Δ_2 amalgamating Λ in the class of semilattices. Then we have the following corollary of Theorem 2.

COROLLARY 3. *Let S_1 and S_2 be [left, right] generalized inverse semigroups with a common orthodox subsemigroup U. Let $\Gamma_1(\Delta_1)$, $\Gamma_2(\Delta_2)$ and $\Xi(\Lambda)$ be the structure inverse semigroups of S_1, S_2 and U, respectively. Let $\Gamma[\Delta]$ together with monomorphisms $\phi_i: \Gamma_i\to\Gamma[\psi_i: \Delta_i\to\Delta]$, $i=1, 2$, be the free product of $\Gamma_1[\Delta_1]$ and $\Gamma_2[\Delta_2]$ amalgamating $\Xi[\Lambda]$ in the class of inverse semigroups [semilattices]. If there exists a monomorphism $\mu: \Delta\to\Gamma$ such that $\psi_i\mu=\phi_i|\Delta_i$, $i=1, 2$, then the amalgam $[\{S_1, S_2\}, U]$ is embedded in a [left, right] generalized inverse semigroup.*

References

[1] Clifford, A. H. and G. B. Preston, The algebraic theory of semigroups, Math. Surveys, No. 7, Amer. Math. Soc., Providence, R. I., Vol. I (1961), Vol. II (1967).

[2] Hall, T. E., Free products with amalgamation of inverse semigroups, J. Algebra **34** (1975), 375–385.

[3] Howie, J. M., Introduction to semigroup theory, Academic Press Inc., New York, 1976.

[4] Imaoka, T., Free products with amalgamation of bands, Mem. Fac. Lit. & Sci., Shimane Univ., Nat. Sci. **10** (1976), 7–17.

[5] ———, Free products with amalgamation of semigroups, Doctoral Dissertation, Monash University, 1978.

[6] Yamada, M., Introduction to semigroups (in Japanese), Maki Shyoten, Tokyo, Japan, 1976.