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On the λ_1- and λ_2-Branching of Predator-Prey Equations Simulating an Immune Response

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This paper is concerned with the predator-prey equations simulating an immune response which G. I. Bell and G. H. Pimbley, Jr. studied. In [1], G. H. Pimbley, Jr. has shown that the α_1-branching of the periodic solutions of this equations appears. In this paper, it will be shown that the λ_1- and λ_2-branching appears under more restricted conditions.

§1. Introduction

In this paper we are concerned with the λ_1- and λ_2-branching of the following system

$$\frac{du}{ds}=u\left[-\lambda_2-k\lambda_2 u+k(\alpha_2-\lambda_2)v-\frac{k\alpha_2}{\theta}uv\right]$$

$$\frac{dv}{ds}=v[\lambda_1-k(\alpha_1-\lambda_1)u+k\lambda_1)u+k\lambda_1 v], \tag{1}$$

where $\alpha_1>0$, $\alpha_2>0$, $\lambda_1>0$, $\lambda_2>0$, $\theta>0$ and $k>0$.

For the role of the variables u, v and s, and the parameters α_1, α_2, λ_1, λ_2, θ and k, see [1].

Every point of intersection of the two curves

$$v=\frac{\lambda_2(1+ku)}{k\left(\alpha_2-\lambda_2-\frac{\alpha_2}{\theta}u\right)} \tag{2a}$$

$$v=\frac{1}{k\lambda_1}[-\lambda_1+k(\alpha_1-\lambda_1)u] \tag{2b}$$

is the singular point of the system. From equations (2), we can obtain that the vertical coordinates of the points of intersection of curves (2) satisfy the following quadratic equation

$$\frac{k\lambda_1\alpha_2}{\theta}v^2-\left[k(\alpha_1-\lambda_1)(\alpha_2-\lambda_2)-k\lambda_1\lambda_2-\frac{\lambda_1\alpha_2}{\theta}\right]v+\lambda_2\alpha_1=0 \tag{3}$$

We will need the following condition;

(A) The two intersections of (2) exist in the set $A=\{(u, v);\ u\geqq 0,\ v\geqq 0\}$
We will need the following

LEMMA. *Fix the parameters $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ and k such that $\alpha_1>\lambda_1+\lambda_2$, $\alpha_2>\lambda_1+\lambda_2$. There exists a value $\theta_0=\theta_0(\alpha_1, \alpha_2, \lambda_1, \lambda_2, k)$ such that if $\theta>\theta_0$, then assumption (A) is satisfied for the parameters $\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta$ and k.*

PROOF. Let us consider the function $B=B(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ and $D=D(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ defined by

$$B(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)=k(\alpha_1-\lambda_1)(\alpha_2-\lambda_2)-k\lambda_1\lambda_2-\frac{\lambda_1\alpha_2}{\theta} \tag{4}$$

$$D(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)=B^2-\frac{4k\alpha_1\alpha_2\lambda_1\lambda_2}{\theta}. \tag{5}$$

From that $B(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ and $D(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ are monotone increasing functions of θ and that $\alpha_1>\lambda_1$, we obtain this lemma.

Denote by (u_f, v_f) the singular point that is the intersection of the two curves, of which vertical coordinates is smaller than the other. By solving (2), we have

$$v_f=\frac{k\theta(\alpha_2-\lambda_2)(\alpha_1-\lambda_1)-\alpha_2\lambda_1-k\theta\lambda_1\lambda_2-\sqrt{\{k\theta(\alpha_2-\lambda_2)-\alpha_2\lambda_1-k\theta\lambda_1\lambda_2\}^2-4k\alpha_1\alpha_2\lambda_1\lambda_2\theta}}{2k\alpha_2\lambda_1}. \tag{6}$$

For the system we define the function $F=F(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$, by

$$F(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)=u_f\lambda_2+\frac{\alpha_2}{\theta}u_fv_f-v_f\lambda_1.$$

G. H. Pimbley, Jr. [1] proved

THEOREM. *Let condition (A) be satisfied for all the parameters.*

(i) If F changes its signature from positive (negative) to negative (positive), then the branching of the periodic solutions which are asymptotically orbitally stable from the constant solution (u_f, v_f) appear (vanish) at the zeros of F.

(ii) If F changes its signature from positive to negative as one parameter is increased (decreased) with other parameters fixed, then the direction of the branching is towards higher (lower) values of the parameter; i.e. a right (left) branching with respect to the parameter.

REMARK. For the more realistic model simulating an immune response, see [2].

§2. λ_1- and λ_2-branching

From (2a), we obtain

$$\lambda_2 u_f + \frac{\alpha_2}{\theta} u_f v_f = (\alpha_2 - \lambda_2) v_f - \frac{\lambda_2}{k}$$

Thus we have the following

LEMMA 1. $F = (\alpha_2 - \lambda_2 - \lambda_1) v_f - \frac{\lambda_2}{k}$. (7)

We will need the following

LEMMA 2. *Fix the parameters* $\alpha_1, \alpha_2, \lambda_2$ *and* k *such that* $\alpha_1 > \alpha_2$, $\alpha_2 > \lambda_2$. *There exists a value* $\theta_1 = \theta_1(\alpha_1, \alpha_2, \lambda_2, k)$ *such that if* $\theta > \theta_1$, *then condition* (*A*) *is satisfied for every* λ_1 *such that* $0 < \lambda_1 < \alpha_2 - \lambda_2$.

PROOF. Since $B(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ is a monotone decreasing function of λ_1 and $B(\alpha_1, \alpha_2, \alpha_2 - \lambda_2, \lambda_2, \theta, k) = k(\alpha_2 - \lambda_2)(\alpha_1 - \alpha_2) - \frac{(\alpha_2 - \lambda_2)\alpha_2}{\theta}$, we obtain the inequality $B(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k) > k(\alpha_2 - \lambda_2)(\alpha_1 - \alpha_2) - \frac{(\alpha_2 - \lambda_2)\alpha_2}{\theta}$. There exists a value θ_3 such that if $\theta > \theta_3$, then $B(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k) > 0$. Let us consider the function $D(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$. Also about D, there exists a value θ_4 such that if $\theta > \theta_4$, then $D > 0$. Define $\theta_1 = \theta_1(\alpha_1, \alpha_2, \lambda_2, k)$ by $\theta_1 = \max\{\theta_3, \theta_4\}$. The proof is completed.

Thus we the following

THEOREM 1. *Fix the parameters* $\alpha_1, \alpha_2, \lambda_2$ *and* k *such that* $\alpha_1 > \alpha_2$, $\alpha_2 > \lambda_2$. *Fix the parameter* θ *such that* $\theta > \theta_1 = \theta_1(\alpha_1, \alpha_2, \lambda_2, k)$. *If there exists a value* λ_1^0 *such that* $F(\alpha_1, \alpha_2, \lambda_1^0, \lambda_2, \theta, k) > 0$, *then there exists a value* λ_1^1 *such that* $\lambda_1^0 < \lambda_1^1 < \alpha_2 - \lambda_2$, $F(\alpha_1, \alpha_2, \lambda_1^1, \lambda_2, \theta, k) = 0$ *and as* λ_1 *is increased through* λ_1^1, $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ *changes its signature from positive to negative; i.e. a* λ_1-*(right)-branching appears at* λ_1^1.

PROOF. Lemma 2 shows that λ_1 can be increased to $\alpha_2 - \lambda_2$, satisfying assumption (A). From the boundedness of v_f, we obtain

$$\lim_{\lambda_1 \to \alpha_2 - \lambda_2} F = -\frac{\lambda_2}{k} < 0$$

This shows the existence of a value λ_1^1.

Similarly to Lemma 2 and Theorem 1, we can prove the following

LEMMA 3. *Fix the parameters* $\alpha_1, \alpha_2, \lambda_1$ *and* k *such that* $\alpha_1 > \alpha_2$, $\alpha_2 > \lambda_1$. *There exists a value* $\theta_2 = \theta_2(\alpha_1, \alpha_2, \lambda_1, k)$ *such that if* $\theta > \theta_2$, *then condition* (*A*) *is satisfied for every* λ_2 *such that* $0 < \lambda_2 < \alpha_2 - \lambda_1$.

THEOREM 2. *Fix the parameters* $\alpha_1, \alpha_2, \lambda_1$ *and* k *such that* $\alpha_1>\alpha_2, \alpha_2>\lambda_1$. *Fix the parameter* θ *such that* $\theta>\theta_2=\theta_2(\alpha_1, \alpha_2, \lambda_1, k)$. *If there exists a value* λ_2^0 *such that* $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2^0, \theta, k)>0$, *then there exists a value* λ_2^1 *such that* $\lambda_2^0<\lambda_2^1<\alpha_2-\lambda_1$, $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2^1, \theta, k)=0$ *and as* λ_2 *is increased through* λ_2^1, $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ *changes its signature from positive to negative*; *i.e. a* λ_2*-(right)-branching appears at* λ_2^1.

Define the function $G=G(\alpha_1, \alpha_2, \lambda_2, \theta, k)$ by $k\theta(\alpha_2-\lambda_2)(\alpha_2-\alpha_1)+\alpha_2^2$. The consideration of F in the neighborhood of $\lambda_1=0$ gives the following

THEOREM 3. *Fix the parameters* $\alpha_1, \alpha_2, \lambda_2, k$ *such that* $\alpha_2>\lambda_2, \alpha_1>\lambda_2$. *If there exists a value* λ_1^0 *and* θ *such that* $\theta>\theta_0=\theta_0(\alpha_1, \alpha_2, \lambda_1^0, \lambda_2, k)$, $F(\alpha_1, \alpha_2, \lambda_1^0, \lambda_2, \theta, k)<0$ *and* $G(\alpha_1, \alpha_2, \lambda_2, \theta, k)>0$, *then there exists a value* λ_1^1 *such that* $0<\lambda_1^1<\lambda_1^0$, $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)=0$ *and as* λ_1 *is decreased through* λ_1^1, $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ *changes its signature from negative to positive*; *i.e. a* λ_1*-(right)-branching vanishes at* λ_1^1.

PROOF. From that $B(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ and $D(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)$ are monotone decreasing functions of λ_1, it follows that λ_1 can be decreased satisfying assumption (A). Notice that the function F is considered as a function of λ_1 only and that $\frac{dF}{d\lambda_1}(\alpha_1, \alpha_2, 0, \lambda_2, \theta, k)=\frac{\lambda_2}{k^2\theta\alpha_1(\alpha_2-\lambda_2)^2}G(\alpha_1, \alpha_2, \lambda_2, \theta, k)$. The condition that $G>0$ completes the proof.

Define the function $H=H(\alpha_1, \alpha_2, \theta, k)$ by $2k\theta(\alpha_2-\alpha_1)+\alpha_2$. Similarly to Theorem 3, we can prove the following

THEOREM 4. *Fix the parameters* $\alpha_1, \alpha_2, \lambda_2$, *and* k *such that* $\alpha_2>\lambda_2, \alpha_1>\lambda_2$. *If there exists a value* λ_2^0 *and* θ *such that* $\theta>\theta_0=\theta_0(\alpha_1, \alpha_2, \lambda_1, \lambda_2^0, k)$, $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2^0, \theta, k)<0$ *and* $H(\alpha_1, \alpha_2, \theta, k)>0$, *then there exists a value* λ_2^1 *such that* $0<\lambda_2^1<\lambda_2^0$, $F(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta, k)=0$ *and as* λ_2 *is decreased through* λ_2^1, $F(\alpha_1, \alpha_2, \lambda_1, \alpha_2, \theta, k)$ *changes its signature from negative to positive*; *i.e. a* λ_2*-(right)-branching vanishes at* λ_2^1.

REMARK. We give an example of parameters which satisfy the assumption of Theorem 3; $\alpha_1=0.7$, $\alpha_2=0.699$, $\lambda_1=0.3$, $\lambda_2=0.3$, $\theta=1000$, $k=1$. For this example, we get $F=-0.005315$ and $G=0.089601$.

References

[1] G. H. Pimbley, Jr., Periodic solutions of predator-prey equations simulating an immune response I., Math. Biosci. **20** (1974), 27–51.

[2] G. H. Pimbley, Jr., Periodic solutions of third order predator-prey equations simulating an immune response., Arch. Rational Mech. Anal. **55** (1974), 93–123.