A convergence property of Picard iteration, a fixed set theorem, and Crouzeix characterization in set-valued convex analysis

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February 20, 2018

Introduction

In convex analysis and set-valued analysis, the notions of convexities, quasiconvexities and fixed point for set-valued maps is important. For example, the subdifferential of a convex function is set-valued maps and a minimal of convex optimization problems is characterized the resolvents of the subdifferential of a convex function. The Nash equilibrium is expressed by set-valued maps, a proof of the existance of the Nash equilibrium is used the Kakutani fixed point theorem and the assumption of convexity.

The notions of convexities and quasiconvexities for set-valued maps behave good roles to consider set-valued analysis and these were introduced by many authors in the literature, respectively. However, these notions do not classify properly, and we do not know number of quasiconveixities. Also Kuroiwa, Popovici and Rocca(2015, [44]) gave a characterization of C-quasiconvexity for set-valued maps which extend a characterization of quasiconvexty for real-valued function by Crouzeix(1997, [15]). There are no results of other type quasiconvexities. In the fixed point researches, there are two type fixed point theorems which are fixed point theorems in a complete metric space and fixed point theorems of continuous maps on a compact convex set. Fixed point theorems in a complete metric space were categorized into the following four types with respect to Picard iteration by Suzuki(2008), see [37]:

(T1) Leader-type : T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$;

(T2) Unnamed-type : T has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point;

(T3) Subrahmanyam-type : T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point of T for all $x \in X$; and

(T4) Caristi-type : T may have more one than fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point of T,

where X is a complete metric space and $T : X \to X$. The Banach contraction principle(Banach, 1922 [2]) and almost of its generalizations belong to (T1). There are few fixed point theorems which belong to (T3). As far as I know, an useful sufficient condition is only Subramanyam fixed point theorem(Subramanyam, 1974 [11]). On the other hand the Brouwer fixed point theorem(Brouwer, 1910, [1]) is the oldest theorem in fixed point theorem for continuous functions on a compact convex set. The Schauder fixed point theorem(Cauty, 2001, [30]) extended the Brouwer fixed point theorem for compact convex subset of a Hausdorff topological vector space. Also Kakutani fixed point theorem(Kakutani, 1941, [3]), which extended the Brouwer fixed point theorem, is famous one for set-valued maps and its application the existance of the Nash equilibrium. However we do not find fixed point theorems for set-to-set maps which extend the Schauder fixed point theorem as far as I know.

This paper is written by reconstructing the following two papers:

- A convergence theorem of the Picard iteration whose mapping has multiple fixed points, Kazuki Seto and Daishi Kuroiwa, Advances in Fixed Point Theory, 2015;5(4):387–395

- A systematization of convexity and quasiconvexity concepts for set-valued maps, defined by l-type and u-type preorder relations, Kazuki Seto, Daishi Kuroiwa and Nicolae Popovici, Accepted to Optimization.

- A fixed set theorem for set-to-set maps, Kazuki Seto, Daishi Kuroiwa, Accepted to Applied Analysis and Opitmization.

In section 1, we explain some properties in convex analysis which are needed to give our results. In section 2, we propose a systematization of quasiconvexity for set-valued maps and we obtain Crouzeix characterizations for set-valued maps which are generalizations of previous one in [42]. In section 3, we obtain a sufficient condition which guarantees the convergence of every Picard iteration $\{T^n x\}$ to a fixed point. Furthermore, we observe a fixed point theorem for set-valued maps based on our results. Also we study fixed sets for set-to-set maps and we give fixed set theorems in term of $T(A) = A$.

Contents

Chapter 1

Preliminaries

In this chapter, we will explain some properties in convex analysis which are needed to give main results based on [29]. At first, we mention some properties of convexity of sets and functions. Secondly, we explain the Crouzeix characterization for real-valued function. Thirdly, we mention a fixed point theorem for nonexpansive maps. Finally, we explain a relationship between convex optimization problems and fixed points.

1.1 Convex function

In this part, we mention basic properties of convex analysis. Let X be a vector space over $\mathbb{R}$. For any $A, B \subset X$ and $\lambda \in \mathbb{R}$, we define $A + B$ and λA as follows:

$$A + B := \{a + b \mid a \in A, b \in B\};$$

$$\lambda A := \{\lambda a \mid a \in A\}.$$

Definition 1.1.1. A set $C \subset X$ is said to be convex if

$$\bigcup_{t \in [0,1]} (1 - t)C + tC \subset C, \text{ that is,}$$

for any $x_0, x_1 \in C$ and $t \in (0, 1)$,

$$(1 - t)x_0 + tx_1 \in C.$$

Remark 1. The empty set is convex.

Definition 1.1.2. A set $C \subset X$ is said to be a cone if

$$tC \subset C \text{ for any } t \in [0, +\infty), \text{ that is,}$$

for any $x \in C$ and $t \in [0, +\infty)$,

$$tx \in C.$$

Definition 1.1.3. A function $f : C \to \mathbb{R}$ is said to be convex if for any $x_0, x_1 \in C$ and $t \in (0, 1)$,

$$f((1-t)x_0 + tx_1) \leq (1-t)f(x_0) + tf(x_1),$$

alternatively, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for any $x_0, x_1 \in \mathrm{dom} f$ and $t \in (0, 1)$,

$$f((1-t)x_0 + tx_1) \leq (1-t)f(x_0) + tf(x_1),$$

where $\mathrm{dom} f := \{x \in X \mid f(x) < +\infty\}$. Furthermore, these are equivalent to the following definition: A function f is said to be convex if the set

$$\mathrm{epi} f := \{(x, \mu) \mid f(x) \leq \mu\},$$

which is called the epigraph of f, is convex.

Proposition 1.1.4. *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex, then for any $\alpha \in \mathbb{R}$, $\{x \in X \mid f(x) \leq \alpha\}$, which is called the level set at α, is convex.*

Remark 2. The reverse of Proposition 1.1.4 does not hold.

Definition 1.1.5. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be closed if for any $\alpha \in \mathbb{R}$, $\{x \in X \mid f(x) \leq \alpha\}$ is closed.

Definition 1.1.6. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be proper if the set $\mathrm{dom} f := \{x \in X \mid f(x) < +\infty\}$ is nonempty.

Let H be a Hilbert space, let $f : H \to \mathbb{R} \cup \{+\infty\}$ be proper convex and let $x \in H$. Then the set

$$\partial f := \{z \in H \mid f(y) - f(x) \geq \langle z, y - x \rangle \text{ for any } y \in H\}$$

is said to be the subdifferential of f.

Theorem 1.1.7. *Let C be a nonempty closed convex subset of H. and let $f : C \to \mathbb{R} \cup \{+\infty\}$ be a proper closed convex function satisfying $\|x_n\| \to \infty$ implies $f(x_n) \to \infty$. Then there exists $x_0 \in \mathrm{dom} f$ such that*

$$f(x_0) = \inf_{x \in X} f(x).$$

1.2 A fixed point theorem for nonepansive maps

In this part, we explain a fixed point theorem for nonepansive maps. Let H be a Hilbert space and let C be a nonempty closed convex subset of H.

Definition 1.2.1. Let $T : C \to H$. An element $\bar{x} \in C$ is said to be a fixed point of T if

$$T(\bar{x}) = \bar{x},$$

and $F(T)$ denotes the set of all fixed points of T.

Definition 1.2.2. A map $T : C \to H$ is said to be nonexpansive if for any $x_0, x_1 \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Theorem 1.2.3. *Let H be a Hilbert space, let C be a nonempty closed subset and let $T : C \to C$ be a nonexpansive map. Then $F(T)$ is closed convex.*

Theorem 1.2.4. *Let H be a Hilbert space, let C be a nonempty closed bounded convex subset and let $T : C \to C$ be a nonexpansive map. Then $F(T) \neq \emptyset$.*

1.3 A relationship between convex optimization problems and fixed points

Definition 1.3.1. Let X be a subset of Hilbert space H. A map $T : X \to 2^X$ is said to be accretive if for any $x_0, x_1 \in \mathrm{dom}T$ and $y_0 \in Tx_0$, $y_1 \in Tx_1$,

$$\langle x_0 - x_1, y_0 - y_1 \rangle \geq 0.$$

Let $T : X \to 2^X$ be accretive. Consider

$$J_r(x) := \{z \in H \mid x \in z + rTz\} = (I + rT)^{-1}(x) \text{ for all } r > 0,$$

which are called resolvents of T. Let

$$R(I + rT) = \{(I + rT)(z) \mid z \in H\}.$$

For any $x \in R(I + rT)$, $J_r(x)$ consists of one point and we may consider J_r is a map from $R(I + rT)$ to H.

Proposition 1.3.2. *Let $T : X \to 2^X$ be accretive, let $r > 0$ and let J_r be a resolvent of T. Then the following holds:*

(i) $T_r x \in TJ_r x$ *for any* $x \in R(I + rT)$;

(ii) $\langle x - y, T_r x - T_r y\rangle \geq r\|T_r x - T_r y\|^2$ *for any* $x, y \in R(I + rT)$;

(iii) $\|J_r x - J_r y\|^2 \leq \|x - y\|^2 - \|(I - J_r)x - (I - J_r)y\|^2$ *for any* $x, y \in R(I + rT)$;

(iv) $\|T_r x\| \leq \inf\{\|z\| \mid z \in Tx\}$ *for any* $x \in \mathrm{dom}T \cap R(I + rT)$,

where $\mathrm{dom}T := \{x \in X \mid Tx \neq \emptyset\}$ *and* $T_r := \frac{1}{r}(I - J_r)$

From (iii) of Proposition 1.3.2, we see that J_r is nonexpansive.

Definition 1.3.3. An accretive map $T : X \to 2^X$ is said to be m-accretive if there exists $r > 0$ such that $R(I + rT) = H$.

Proposition 1.3.4. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function. Then ∂f is m-accretive.*

The following shows the relationship between zero point and fixed point:

Proposition 1.3.5. *If $T : H \to 2^H$ be m-accretive and $r > 0$, then the following holds:*

$$0 \in Tz \iff J_r(z) = z.$$

Now, we consider the following problem:

$$(\mathrm{P}) \quad \begin{array}{ll} \text{Min} & f(x) \\ \text{subject to} & x \in C \end{array}$$

where $f : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function and $C \subset H$ is a nonempty convex subset. We define

$$\delta_C := \begin{cases} 0 & x \in C, \\ +\infty & x \notin C. \end{cases}$$

Then $0 \in \partial(f + \delta_C)(\bar{x})$ if and only if $\bar{x}$ is a minimizer of (P). Also we put $T := \partial(f + \delta_C)$ then we can see that T is accretive. Hence, we can consider resolvent $J_r = (I + rT)^{-1}$ for any $r > 0$. Since J_r is nonexpansive, if C is closed bounded then there exists $\bar{x}$ such that $J_r(\bar{x}) = \bar{x}$ from Theorem 1.2.4. Furthermore, if f is closed then T is m-accretive from Proposition 1.3.4. We summarize these results as follows:

Theorem 1.3.6. *Let $C \subset H$ be a nonempty closed bounded convex and let $f : C \to \mathbb{R} \cup \{+\infty\}$ be a proper closed convex function. Then there exist $\bar{x} \in C$ such that*

$$f(\bar{x}) = \min_{x \in C} f(x).$$

Therefore, finding a fixed point of J_r leads solving (P). Actually, we need fixed point approximation methods to find the fixed point. The Picard iteration is one of famous fixed point approximation methods. Many researchers gave fixed point theorems whose mapping has the unique fixed point and the convergence of every Picard iterations $\{T^n x\}$ to the fixed point. However, since a function does not have always the unique minimizer, we need to give fixed point theorems whose mapping has multiple fixed point. Hence we give a sufficient condition which guarantees the existence of multiple fixed points of T and the convergence of every Picard iteration $\{T^n x\}$ to the fixed point.

Chapter 2

C-quasiconvexity for set-valued maps

In this chapter, we will obtain Crouzeix characterization for set-valued maps based on a systematization of quasiconvexity for set-valued maps which will be proposed by us. At first, we introduce Crouzeix characterization for real-valued maps. Next, we focus on l-type and u-type set-relations which was given by Kuroiwa, see [43]. We introduce the concepts of quasiconvexity for set-valued maps which was obtained in [26, 46, 24, 27, 32], and we will propose a systematization for set-valued maps. Finally, we will obtain Crouzeix characterization for set-valued maps which is a generalization of the previous one in [42].

2.1 The Crouzeix characterization

In this part, we explain the Crouzeix characterization. At first, we give the notion of quasiconvexity.

Definition 2.1.1. A function f is said to be affine if f and $-f$ are convex.

Definition 2.1.2. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be quasiconvex if for any $x_0, x_1 \in \mathrm{dom} f$ and $t \in (0, 1)$,

$$f((1-t)x_0 + tx_1) \leq \max\{f(x_0), f(x_1)\}.$$

Remark 3. We can see that if f is convex then f is quasiconvex.

The notion of quasiconvexity can be expressed by the level sets.

Proposition 2.1.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$. Then the following are equivalent:*

- *f is quasiconvex;*
- *the level set at α is convex for any $\alpha \in \mathbb{R}$.*

The following is an interesting characterization of convexity which is given by Crouzeix[15]:

Theorem 2.1.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$. Then the following are equivalent:*

- *f is convex;*
- *$f + g$ is quasiconvex for any affine $g : X \to \mathbb{R}$.*

The latter condition is equivalent to

$$\{x \in X \mid f(x) \leq g(x)\} \text{ is convex for any affine } g : X \to \mathbb{R}.$$

2.2 Preliminaries for Chapter 2

In this part, we explain some notions of set-valued analysis. Also we introduce some concepts of quasiconvexity for vector-valued maps which is a generalization of quasiconvexity for real-valued maps.

Let X be a nonempty convex subset of a real vector space and Y be a real vector space. Consider a convex cone $C \subset Y$, i.e. $0 \in C = tC = C + C$ for all $t \in [0, +\infty)$. Then C induces on Y a partial ordering (i.e., a reflexive and transitive binary relation, which is compatible with the linear structure of Y, cf. Jahn [40]), defined for any $y_0, y_1 \in Y$ by

$$y_0 \leq_C y_1 \; :\Longleftrightarrow \; y_1 \in y_0 + C.$$

Moreover, C induces on 2^Y two binary relations, defined for any $A, B \in 2^Y$ by

$$A \leq_C^l B :\Longleftrightarrow A + C \supset B,$$
$$A \leq_C^u B :\Longleftrightarrow A \subset B - C.$$

These relations were introduced by Kuroiwa [26]. Obviously, they are reflexive and transitive, therefore we will call them the l-type and u-type preorder relations induced by C.

Definition 2.2.1. A vector-valued function $f : X \to Y$ is said to be:

- C-convex (convex in the sense of Luenberger [6]) if for any $x_0, x_1 \in X$ and $t \in (0, 1)$ we have

$$f((1-t)x_0 + tx_1) \leq_C (1-t)f(x_0) + tf(x_1); \tag{2.1}$$

- C-quasiconvex (strongly quasiconvex w.r.t. C in the sense of Borwein [10]) if for any $y \in Y$, the level set

$$f^{-1}(y - C) := \{x \in X \mid f(x) \leq_C y\} = \{x \in X \mid f(x) + (-y) \leq_C 0\} \tag{2.2}$$

  is convex; in other words (see, e.g., Luc [21]), f is C-quasiconvex if and only if for any $x_0, x_1 \in X$ and $y \in Y$

$$f(x_0) \leq_C y,\ f(x_1) \leq_C y \Rightarrow f((1-t)x_0 + tx_1) \leq_C y \text{ for any } t \in (0, 1); \tag{2.3}$$

- properly C-quasiconvex (in the sense of Ferro [16]) if for any $x_0, x_1 \in X$, and $t \in (0, 1)$,

$$f((1-t)x_0 + tx_1) \leq_C f(x_0) \text{ or } f((1-t)x_0 + tx_1) \leq_C f(x_1); \tag{2.4}$$

- natural C-quasiconvex (in the sense of Tanaka [25]) if for any $x_0, x_1 \in X$, and $t \in (0, 1)$, there exists $\lambda \in [0, 1]$ such that

$$f((1-t)x_0 + tx_1) \leq_C (1-\lambda)f(x_0) + \lambda f(x_1); \tag{2.5}$$

- quasiconvex (in the sense of Jahn [17, 40]) or quasiconvex w.r.t. C (cf. Borwein [10]), if for any $x_0, x_1 \in X$,

$$f(x_0) \leq_C f(x_1) \Rightarrow f((1-t)x_0 + tx_1) \leq_C f(x_1) \text{ for any } t \in (0, 1). \tag{2.6}$$

The following result, obtained in [44] is extended the classical Crouzeix characterization:

Theorem 2.2.2. *A vector-valued function $f : X \to Y$ is C-convex if and only if $f + g$ is C-quasiconvex, for every linear operator (restricted to X) $g : X \to Y$.*

2.3 Convexity and quasiconvexity concepts for set-valued maps, defined by the l-type and u-type preorder relations

As usual in set-valued analysis, given any map $F : X \to 2^Y$, we define its domain by

$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}.$$

The following notions of generalized convexity for set-valued maps was introduced by Kuroiwa [26].

Definition 2.3.1. A set-valued map $F : X \to 2^Y$ is said to be:

- l-type C-convex, if for any $x_0, x_1 \in \operatorname{dom} F$ and $t \in (0, 1)$,

$$F((1-t)x_0 + tx_1) \leq_C^l (1-t)F(x_0) + tF(x_1);$$

- u-type C-convex, if for any $x_0, x_1 \in \operatorname{dom} F$ and $t \in (0, 1)$,

$$F((1-t)x_0 + tx_1) \leq_C^u (1-t)F(x_0) + tF(x_1).$$

In the sequel it will be convenient to introduce the following sets:

$$\mathcal{C}_l(X, Y, C) \text{ and } \mathcal{C}_u(X, Y, C)$$

the classes of l-type C-convex and u-type C-convex set-valued maps, respectively.

Remark 4. a) The notion of l-type C-convexity coincide with the C-convexity in the sense of Borwein [14]. In particular, when $C = \{0\}$, the l-type $\{0\}$-convexity corresponds to the notion of convexity introduced by Robinson [12] (called θ-convexity in [14]). It is easily seen that

$$\mathcal{C}_l(X, Y, \{0\}) \subset \mathcal{C}_l(X, Y, C). \tag{2.7}$$

b) The notion of u-type C-convexity coincide with the K-concavity in the sense of Nikodem [23] for $K := -C$. In particular, when $C = \{0\}$, the u-type $\{0\}$-convexity corresponds to the notion of concavity introduced by Nikodem in [20]. Notice that

$$\mathcal{C}_u(X, Y, \{0\}) \subset \mathcal{C}_u(X, Y, C). \tag{2.8}$$

c) When F is a single-valued map, i.e.,

$$F(x) = \{f(x)\} \text{ for any } x \in X,$$

where $f : X \to Y$ is a vector-valued function, then both l-type C-convexity and l-type C-convexity of F are equivalent to the usual C-convexity (cf. Definition 2.2.1). However, in general, these notions are distinct. For instance, when $X = [0, 1]$, $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the set-valued map $F : X \to 2^Y$ defined by

$$F(x) = [0, 1 - x^2]$$

is l-type $\{0\}$-convex, hence l-type C-convex in view of (2.7), but F is not u-type C-convex, while the set-valued map $G : X \to 2^Y$ defined by

$$G(x) = [1 - x^2, 1]$$

is u-type $\{0\}$-convex and therefore u-type C-convex in view of (2.8), but G is not l-type C-convex.

d) The l-type C-convexity of a set-valued map F assures the convexity of dom$F$, but the u-type C-convexity of F does not.

In the next two definitions we present several concepts of quasiconvexity for set-valued maps, that extend the classical concept of quasiconvex real-valued function. First we use the l-type preorder relation.

Definition 2.3.2. A set-valued map $F : X \to 2^Y$ is said to be:

- (l1)-type C-quasiconvex, if for any convex $A \in 2^Y$ the set

$$\left\{x \in \mathrm{dom}F \;\middle|\; F(x) + A \leq^l_C \{0\}\right\} \text{ is convex;}$$

- (l2)-type C-quasiconvex, if for any $y \in Y$ the set

$$\left\{x \in \mathrm{dom}F \;\middle|\; F(x) \leq^l_C \{y\}\right\} \text{ is convex,}$$

  which equivalently means that for any convex $A \in 2^Y$ the set

$$\{x \in \mathrm{dom}F \mid F(x) \leq^l_C A\} \text{ is convex;}$$

- (l3)-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0, 1)$,

$$F((1-t)x_0 + tx_1) \leq^l_C (F(x_0) + C) \cap (F(x_1) + C);$$

- ($l4$)-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$,

$$F((1-t)x_0 + tx_1) \leq_C^l F(x_0) \text{ or } F((1-t)x_0 + tx_1) \leq_C^l F(x_1);$$

- ($l5$)-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$F((1-t)x_0 + tx_1) \leq_C^l (1-\lambda)F(x_0) + \lambda F(x_1);$$

- ($l6$)-type C-quasiconvex, if for any convex $A \in 2^Y$ and any $x_0, x_1 \in \mathrm{dom}F$,

$$(1-\lambda)F(x_0) + \lambda F(x_1) + A \leq_C^l \{0\} \text{ for any } \lambda \in [0,1]$$

implies

$$F((1-t)x_0 + tx_1) + A \leq_C^l \{0\} \text{ for any } t \in (0,1);$$

- ($l7$)-type C-quasiconvex, if for any $y \in Y$ and $x_0, x_1 \in \mathrm{dom}F$,

$$(1-\lambda)F(x_0) + \lambda F(x_1) \leq_C^l \{y\} \text{ for any } \lambda \in [0,1]$$

implies

$$F((1-t)x_0 + tx_1) \leq_C^l \{y\} \text{ for any } t \in (0,1);$$

- ($l8$)-type C-quasiconvex if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$,

$$F((1-t)x_0 + tx_1) \leq_C^l \bigcap_{\lambda \in [0,1]} ((1-\lambda)F(x_0) + \lambda F(x_1) + C)\,;$$

- ($l9$)-type C-quasiconvex if for any $x_0, x_1 \in \mathrm{dom}F$,

$$F(x_0) \leq_C^l F(x_1) \Rightarrow F((1-t)x_0 + tx_1) \leq_C^l F(x_1) \text{ for any } t \in (0,1).$$

In a similar way, one can introduce quasiconvexity concepts with respect to the u-type preorder relation.

Definition 2.3.3. A set-valued map $F : X \to 2^Y$ is said to be:

- ($u1$)-type C-quasiconvex, if for any convex $A \in 2^Y$ the set

$$\left\{x \in \mathrm{dom}F \;\middle|\; F(x) \leq_C^u A\right\} \text{ is convex;}$$

- $(u2)$-type C-quasiconvex, if for any $y \in Y$ the set
$$\left\{x \in \mathrm{dom}F \;\middle|\; F(x) \leq_C^u \{y\}\right\} \text{ is convex;}$$
- $(u3)$-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$,
$$F((1-t)x_0 + tx_1) \leq_C^u F(x_0) \cup F(x_1);$$
- $(u4)$-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$,
$$F((1-t)x_0 + tx_1) \leq_C^u F(x_0) \text{ or } F((1-t)x_0 + tx_1) \leq_C^u F(x_1);$$
- $(u5)$-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that
$$F((1-t)x_0 + tx_1) \leq_C^u (1-\lambda)F(x_0) + \lambda F(x_1);$$
- $(u6)$-type C-quasiconvex, if for any convex $A \in 2^Y$ and any $x_0, x_1 \in \mathrm{dom}F$,
$$(1-\lambda)F(x_0) + \lambda F(x_1) \leq_C^u A \text{ for any } \lambda \in [0,1]$$
implies
$$F((1-t)x_0 + tx_1) \leq_C^u A \text{ for any } t \in (0,1);$$
- $(u7)$-type C-quasiconvex, if for any $y \in Y$ and $x_0, x_1 \in \mathrm{dom}F$,
$$(1-\lambda)F(x_0) + \lambda F(x_1) \leq_C^u \{y\} \text{ for any } \lambda \in [0,1]$$
implies
$$F((1-t)x_0 + tx_1) \leq_C^u \{y\} \text{ for any } t \in (0,1);$$
- $(u8)$-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$,
$$F((1-t)x_0 + tx_1) \leq_C^u \bigcup_{\lambda \in [0,1]} \left((1-\lambda)F(x_0) + \lambda F(x_1)\right);$$
- $(u9)$-type C-quasiconvex, if for any $x_0, x_1 \in \mathrm{dom}F$,
$$F(x_0) \leq_C^u F(x_1) \Rightarrow F((1-t)x_0 + tx_1) \leq_C^u F(x_1) \text{ for any } t \in (0,1).$$

Remark 5. a) The notions of ($l1$)- and ($l2$)-type C-quasiconvexity are based on (2.2). The notions of ($l3$)-, ($l4$)-, ($l5$)-type C-quasiconvexity are based on (2.3), (2.4) and (2.5), respectively. The concepts of ($l6$)-, ($l7$)- and ($l8$)-type C-quasiconvexity can be seen as counterparts of the notions of ($l1$)-, ($l2$)- and ($l3$)-type C-quasiconvexity via ($l5$). The notion of ($l9$)-type C-quasiconvexity is based on Jahn's quasiconvexity (2.6).

b) Similarly to a), the notions of u-type quasiconvexity given in Definition 2.3.3 are based on the corresponding vector-valued counterparts presented in Definition 2.2.1.

c) Some of the notions introduced in Definitions 2.3.2 and 2.3.3 were already considered by several authors, by using a different terminology. For instance, the notions of ($l1$)- and ($u1$)-type C-quasiconvexity were used recently by Crespi, Kuroiwa and Rocca [46], the notion of ($l2$)-type C-quasiconvexity was studied by Luc and Cristobal [24], the notion of ($l3$)-type C-quasiconvexity has been defined by Kuroiwa [27] and studied by others, for example Benoist and Popovici [32]. The notions of ($l4$)- and ($l5$)-type C-quasiconvexity were also defined in [27]. The notions of ($u2$)-, ($u3$)-, ($u4$)-, ($u5$)-, and ($u8$)-type C-quasiconvexity were introduced by Kuroiwa [26].

d) As far as we know, the notions of ($l9$)- and ($u9$)-type C-quasiconvexity are new and they have no equivalent counterpart in the existing literature.

e) If $C - C = Y$ (which actually means that $(y_1 + C) \cap (y_2 + C) \neq \emptyset$ for all $y_1, y_2 \in Y$) and the set-valued map $F : X \to 2^Y$ is ($l3$)-type C-quasiconvex, then dom$F$ is convex (cf. Benoist and Popovici [32]).

f) For single-valued maps, as defined in Remark 4 c), the ($l1$)-type [resp. ($l2$)-, ($l4$)-, ($l5$)-, ($l6$)- ($l7$)-, and ($l9$)-type] C-quasiconvexity is equivalent to ($u1$)-type [resp. ($u2$)-, ($u4$)-, ($u5$)- ($u6$)-, ($u7$)-, and ($u9$)-type] C-quasiconvexity, due to the fact that $\{y\} \leq_C^l \{y'\} \Leftrightarrow \{y\} \leq_C^u \{y'\}$, and $\{y\} + A \leq_C^l \{0\} \Leftrightarrow \{y\} \leq_C^u -A$, for any $y, y' \in Y$ and $A \in 2^Y$. Moreover, if $Y = C \cup (-C)$, then the ($l3$)-type [resp. ($l8$)-type] C-quasiconvexity is also equivalent to ($u3$)-type [resp. ($u8$)-type] C-quasiconvexity for single-valued maps. The latter equivalences hold due to the fact that any two (outcome) points $y, y' \in Y$ are comparable, i.e., $y - y' \in C \cup (-C)$. More precisely, assuming without loss of generality that $y - y' \in C$, we get

$$(y + C) \cap (y' + C) = y + C \left[= \bigcap_{\lambda\in[0,1]} ((1-\lambda)y + \lambda y' + C) \right],$$

$$(y - C) \cup (y' - C) = y - C \left[= \bigcup_{\lambda\in[0,1]} ((1-\lambda)y + \lambda y') - C \right],$$

leading to the desired equivalences.

In the sequel it will be convenient to introduce the following set:

$$\mathcal{Q}_{l1} := \mathcal{Q}_{l1}(X, Y, C), \dots, \mathcal{Q}_{l9} := \mathcal{Q}_{l9}(X, Y, C)$$

the classes of all set-valued maps, acting from the set X to the space Y, that are (l1)-type C-quasiconvex, ..., (l9)-type C-quasiconvex, respectively. Similarly,

$$\mathcal{Q}_{u1} := \mathcal{Q}_{u1}(X, Y, C), \dots, \mathcal{Q}_{u9} := \mathcal{Q}_{u9}(X, Y, C)$$

will represent the corresponding classes of u-type C-quasiconvex set-valued maps.

Proposition 2.3.4. *Among the nine classes of l-type quasiconvex set-valued maps defined above, at most five are distinct. More precisely, we have*

$$\mathcal{Q}_{l4} \subset \mathcal{Q}_{l5} \subset \mathcal{Q}_{l1} = \mathcal{Q}_{l6} \subset \mathcal{Q}_{l2} = \mathcal{Q}_{l3} = \mathcal{Q}_{l7} = \mathcal{Q}_{l8} \subset \mathcal{Q}_{l9}.$$

In other words, we can summarize these relationships as follows

$$\boxed{(l4)} \Longrightarrow \boxed{(l5)} \Longrightarrow \boxed{(l1),\ (l6)} \Longrightarrow \boxed{(l2),\ (l3),\ (l7),\ (l8)} \Longrightarrow \boxed{(l9)}$$

Proof. We present the detailed proofs for the main implications only.

(l4)⇒(l5) Assume that F is (l4)-type C-quasiconvex. Let $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0, 1)$ be arbitrarily chosen. Since F is (l4)-type C-quasiconvex, we have $F((1-t)x_0 + tx_1) \leq^l_C (1-\lambda)F(x_0) + \lambda F(x_1)$ when $\lambda = 0$ or $\lambda = 1$. Thus F is (l5)-type C-quasiconvex.

(l5)⇒(l1) Assume that F is (l5)-type C-quasiconvex. Consider any convex set $A \in 2^Y$. Let $x_0, x_1 \in \mathrm{dom}F$ with $F(x_0)+A \leq^l_C \{0\}$ and $F(x_1)+A \leq^l_C \{0\}$ and let $t \in (0,1)$ be arbitrarily chosen. Since F is $(l5)$-type C-quasiconvex, there is $\lambda \in [0,1]$ such that

$$F((1-t)x_0 + tx_1) \leq^l_C (1-\lambda)F(x_0) + \lambda F(x_1).$$

Hence we have

$$\begin{aligned} F((1-t)x_0 + tx_1) + A + C &\supset (1-\lambda)F(x_0) + \lambda F(x_1) + A + C \\ &= (1-\lambda)(F(x_0) + A + C) + \lambda(F(x_1) + A + C) \\ &\supset (1-\lambda)\{0\} + \lambda\{0\} \\ &= \{0\}. \end{aligned}$$

Thus F is $(l1)$-type C-quasiconvex.

$(l1)\Rightarrow(l6)$ Assume that F is $(l1)$-type C-quasiconvex. Consider a convex set $A \in 2^Y$ and let $x_0, x_1 \in \mathrm{dom}F$ be arbitrarily chosen. Assume that

$$(1-\lambda)F(x_0) + \lambda F(x_1) + A \leq^l_C \{0\} \text{ for any } \lambda \in [0,1].$$

In particular, letting $\lambda \in \{0,1\}$, we get

$$F(x_0) + A \leq^l_C \{0\} \text{ and } F(x_1) + A \leq^l_C \{0\}.$$

Since F is $(l1)$-type C-quasiconvex, we conclude that

$$F((1-t)x_0 + tx_1) + A \leq^l_C \{0\} \text{ for any } t \in (0,1).$$

Thus F is $(l6)$-type C-quasiconvex.

$(l6)\Rightarrow(l1)$ Assume that F is $(l6)$-type C-quasiconvex. Consider a convex set $A \in 2^Y$. Let $x_0, x_1 \in \mathrm{dom}F$ with $F(x_0) + A \leq^l_C \{0\}$ and $F(x_1) + A \leq^l_C \{0\}$, and let $t \in (0,1)$ be arbitrarily chosen. It is easy to check that

$$(1-\lambda)F(x_0) + \lambda F(x_1) + A \leq^l_C \{0\} \text{ for any } \lambda \in [0,1].$$

Since F is $(l6)$-type C-quasiconvex, we can see that

$$F((1-t)x_0 + tx_1) + A \leq^l_C \{0\} \text{ for any } t \in (0,1).$$

Thus F is $(l1)$-type C-quasiconvex.

($l1$)⇒($l2$) It is easily verified.

($l8$)⇒($l2$) Assume that F is ($l8$)-type C-quasiconvex. Let $y \in Y$ and $x_0, x_1 \in \mathrm{dom}F$ with $F(x_0) \leq_C^l \{y\}$ and $F(x_1) \leq_C^l \{y\}$, and let $t \in (0,1)$ be arbitrarily chosen. We have

$$(1-\lambda)F(x_0) + \lambda F(x_1) + C \leq_C^l \{y\} \text{ for any } \lambda \in [0,1],$$

that is,

$$\bigcap_{\lambda\in[0,1]} ((1-\lambda)F(x_0) + \lambda F(x_1) + C) \supset \{y\}.$$

Since F is ($l8$)-type C-quasiconvex, we have

$$F((1-t)x_0 + tx_1) + C \supset \bigcap_{\lambda\in[0,1]} ((1-\lambda)F(x_0) + \lambda F(x_1) + C).$$

Therefore we can conclude that

$$F((1-t)x_0 + tx_1) + C \supset \{y\}.$$

Thus F is ($l2$)-type C-quasiconvex.

($l2$)⇒($l7$) Can be proven in a similar way as ($l1$)⇒($l6$).

($l7$)⇒($l3$) Assume that F is ($l7$)-type C-quasiconvex. Let $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$ be arbitrarily chosen. For any $y \in (F(x_0) + C) \cap (F(x_1) + C)$, we have

$$y \in (1-\lambda)(F(x_0) + C) + \lambda(F(x_1) + C) \text{ for any } \lambda \in [0,1].$$

Since F is ($l7$)-type C-quasiconvex, we conclude that

$$F((1-t)x_0 + tx_1) + C \supset \{y\}.$$

Thus F is ($l3$)-type C-quasiconvex.

($l3$)⇒($l8$) It is easily verified, since

$$(F(x_0) + C) \cap (F(x_1) + C) \supset \bigcap_{\lambda\in[0,1]} ((1-\lambda)F(x_0) + \lambda F(x_1) + C).$$

($l3$)⇒($l9$) Assume that F is ($l3$)-type C-quasiconvex. Let $x_0, x_1 \in \mathrm{dom}F$ with $F(x_0) \leq_C^l F(x_1)$ and $t \in (0,1)$ be arbitrarily chosen. Since F is ($l3$)-type C-quasiconvex, we have

$$F((1-t)x_0 + tx_1) \leq_C^l (F(x_0) + C) \cap (F(x_1) + C)$$

and therefore

$$\begin{aligned} F((1-t)x_0 + tx_1) + C &\supset (F(x_0) + C) \cap (F(x_1) + C) \\ &\supset (F(x_1) + C) \cap (F(x_1) + C) \\ &= F(x_1) + C \\ &\supset F(x_1). \end{aligned}$$

Thus F is ($l9$)-type C-quasiconvex.

□

Remark 6. The converse implications in Proposition 2.3.4 are not true in general, as the following examples show.

Example 2.3.5. In each of the following four instances, $X := [0,1]$ while $Y := \mathbb{R}^2$ is endowed with the usual ordering cone $C := \mathbb{R}^2_+$.

1. [($l5$) $\not\Rightarrow$ ($l4$)] Consider the set-valued map $F : X \to 2^Y$ defined by

$$F(x) = \{(x, 1-x)\}.$$

For any $x_0, x_1 \in \mathrm{dom}F = [0,1]$ and $t \in (0,1)$, we put $\lambda = t$. Since we have $F((1-t)x_0 + tx_1) = (1-\lambda)F(x_0) + \lambda F(x_1)$, F is ($l5$)-type C-quasiconvex. Also, letting $x_0 = 0, x_1 = 1$, we do not have $F((1-t)x_0 + tx_1) \leq_C^l F(0)$ and $F((1-t)x_0 + tx_1) \leq_C^l F(1)$ for any $t \in (0,1)$. Therefore F is not ($l4$)-type C-quasiconvex.

2. [($l1$) $\not\Rightarrow$ ($l5$)] Consider the set-valued map $F : X \to 2^Y$ defined by

$$F(x) = \begin{cases} \mathrm{co}\{(-1,1),(0,0)\} & \text{if } x = 0, \\ \{(0,0)\} & \text{if } x \in (0,1), \\ \mathrm{co}\{(0,0),(1,-1)\} & \text{if } x = 1. \end{cases}$$

Consider a convex set $A \in 2^Y$. Let $x_0, x_1 \in \{x \in \mathrm{dom}F \mid F(x) + A \leq_C^l \{(0,0)\}$ and $t \in (0,1)$ be arbitrarily chosen. It is clear that F satisfies the condition requested in the definition of ($l1$)-type C-quasiconvexity. Therefore we consider only the case when $x_0 = 0$ and $x_1 = 1$. Since $F(0) + A \leq^l_C \{(0,0)\}$ and $F(1) + A \leq^l_C \{(0,0)\}$, there exist $y_0 \in F(x_0)$ and $y_1 \in F(x_1)$ such that

$$y_0 + A + C \supset \{(0,0)\} \text{ and } y_1 + A + C \supset \{(0,0)\}.$$

Also there exists $\lambda \in (0,1)$ such that $(1-\lambda)y_0 + \lambda y_1 = (0,0) \in F((1-t)x_0 + tx_1)$. Since A and C are convex, we have

$$F((1-t)x_0 + tx_1) + A + C \supset (1-\lambda)y_0 + \lambda y_1 + A + C \supset \{(0,0)\}.$$

Therefore F is ($l1$)-type C-quasiconvex. On the other hand, if $x_0 = 0$ and $x_1 = 1$, then for any $t \in (0,1)$ and $\lambda \in [0,1]$, we have $F((1-t)x_0 + tx_1) \not\supset (1-\lambda)F(0) + \lambda F(1)$. Therefore F is not ($l5$)-type C-quasiconvex.

3. [($l3$) $\not\Rightarrow$ ($l1$)] Consider the set-valued map $F : X \to 2^Y$ defined by

$$F(x) = \begin{cases} \left\{\left(2x - \frac{1}{2}, 1\right)\right\} - \{y\} & \text{if } x \leq \frac{1}{2}, \\ \left\{\left(\frac{1}{2}, -4x + 3\right)\right\} - \{y\} & \text{if } x > \frac{1}{2}. \end{cases}$$

where $y = \left(\frac{7}{16}, \frac{15}{16}\right)$. Clearly, F is ($l3$)-type C-quasiconvex. Consider $A = \operatorname{co}\left\{\left(-\frac{1}{8}, \frac{1}{8}\right), \left(\frac{1}{8}, -\frac{1}{8}\right)\right\}$. Then $F(0) + A \leq^l_C \{0\}$ and $F(1) + A \leq^l_C \{0\}$, but $F(\frac{1}{2}) + A + C \not\supset \{0\}$. Hence F is not ($l1$)-type C-quasiconvex.

4. [($l9$) $\not\Rightarrow$ ($l2$)] Consider the set-valued map $F : X \to 2^Y$ defined by

$$F(x) = \begin{cases} \{(-1,0)\} & \text{if } x = 0, \\ \{(1,1)\} & \text{if } x \in (0,1), \\ \{(0,-1)\} & \text{if } x = 1. \end{cases}$$

We can check easily that F is ($l9$)-type C-quasiconvex. However, F is not ($l2$)-type C-quasiconvex, because $F(0) \leq^l_C \{(0,0)\}$, $F(1) \leq^l_C \{(0,0)\}$, and $F(\frac{1}{2}) \not\leq^l_C \{(0,0)\}$.

Remark 7. In view of Proposition 2.3.4 and Remark 5.e), if Y is directed with respect to C, then all ($l1$)-type,...,($l8$)-type C-quasiconvex set-valued maps have a convex domain. However, the domain of a ($l9$)-type C-quasiconvex map is not necessarily convex, even if Y is directed with respect to C. For instance, when $X = [0,1]$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$, the set-valued map $F : X \to 2^Y$, given by

$$F(x) = \begin{cases} \{(x, 1-x)\} & \text{if } x \in \{0,1\}, \\ \emptyset & \text{if } x \in (0,1), \end{cases}$$

is $(l9)$-type C-quasiconvex while $\mathrm{dom}F = \{0,1\}$ is not convex.

Proposition 2.3.6. *Among the nine classes of u-type quasiconvex set-valued maps defined above, at most seven are distinct. More precisely,*

$$\mathcal{Q}_{u4} \subset \mathcal{Q}_{u3} \cap \mathcal{Q}_{u5} \subset \mathcal{Q}_{u3} \cup \mathcal{Q}_{u5} \subset \mathcal{Q}_{u8} \subset \mathcal{Q}_{u1} = \mathcal{Q}_{u6} \subset \mathcal{Q}_{u2} = \mathcal{Q}_{u7}$$

and, for any $F \in \mathcal{Q}_{u6}$ such that $F(x) - C$ is convex for all $x \in \mathrm{dom}F$, we have $F \in \mathcal{Q}_{u8} \cap \mathcal{Q}_{u9}$. In other words, we can summarize these relationships as follows

$$\begin{tikzcd}
\boxed{(u4)} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \boxed{(u5)} \arrow[r, Rightarrow] & \boxed{(u8)} \arrow[r, Rightarrow, shift right] & \boxed{(u1),\ (u6)} \arrow[l, Rightarrow, shift right, "(*)"'] \arrow[r, Rightarrow] \arrow[d, Rightarrow, "(*)"] & \boxed{(u2),\ (u7)} \\
 & \boxed{(u3)} \arrow[ur, Rightarrow] & & \boxed{(u9)} &
\end{tikzcd}$$

where $(*)$ requires the convexity of $F(x) - C$ for all $x \in \mathrm{dom}F$.

Proof. First we prove the general implications, in absence of the assumption $(*)$.

$(u4)\Rightarrow(u5)$ Assume that F is $(u4)$-type C-quasiconvex. Let $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$ be arbitrarily chosen. Since F is $(u4)$-type C-quasiconvex, we have $F((1-t)x_0 + tx_1) \leq^u_C (1-\lambda)F(x_0) + \lambda F(x_1)$ when $\lambda = 0$ or $\lambda = 1$. Thus F is $(u5)$-type C-quasiconvex.

$(u4)\Rightarrow(u3)$ It is easily verified.

$(u5)\Rightarrow(u8)$ Follows from the fact that

$$(1-\lambda)F(x_0) + \lambda F(x_1) \subset \bigcup_{\lambda \in [0,1]} ((1-\lambda)F(x_0) + \lambda F(x_1)) \text{ for any } \lambda \in [0,1].$$

$(u3)\Rightarrow(u8)$ It is easily verified, since

$$F(x_0) \cup F(x_1) \subset \bigcup_{\lambda \in [0,1]} ((1-\lambda)F(x_0) + \lambda F(x_1)).$$

($u8$)⇒($u1$) Assume that F is ($u8$)-type C-quasiconvex. Consider a convex set $A \in 2^Y$. Let $x_0, x_1 \in \mathrm{dom}F$ with $F(x_0) \leq_C^u A$ and $F(x_1) \leq_C^u A$, and let $t \in (0,1)$ be arbitrarily chosen. Since F is ($u8$)-type C-quasiconvex,

$$F((1-t)x_0 + tx_1) \leq_C^u \bigcup_{\lambda \in [0,1]} ((1-\lambda)F(x_0) + \lambda F(x_1)),$$

that is, for any $y \in F((1-t)x_0 + tx_1)$, there exists $\bar{\lambda} \in [0,1]$ such that $y \in (1-\bar{\lambda})F(x_0) + \bar{\lambda}F(x_1) - C$. Hence

$$\begin{aligned} y &\in (1-\bar{\lambda})F(x_0) + \bar{\lambda}F(x_1) - C \\ &\subset (1-\bar{\lambda})(A - C) + \bar{\lambda}(A - C) - C \\ &= A - C. \end{aligned}$$

Therefore we have $F((1-t)x_0 + tx_1) \leq_C^u A$. Thus F is ($u8$)-type C-quasiconvex.

($u1$)⇒($u6$) Assume that F is ($u1$)-type C-quasiconvex. Consider a convex set $A \in 2^Y$ and let $x_0, x_1 \in \mathrm{dom}F$ be arbitrarily chosen. Assume that

$$(1-\lambda)F(x_0) + \lambda F(x_1) \leq_C^u A \text{ for any } \lambda \in [0,1].$$

In particular, letting $\lambda \in \{0,1\}$, we get

$$F(x_0) \leq_C^u A \text{ and } F(x_1) \leq_C^u A.$$

Since F is ($u1$)-type C-quasiconvex, we conclude that

$$F((1-t)x_0 + tx_1) \leq_C^u A \text{ for any } t \in (0,1).$$

Thus F is ($u6$)-type C-quasiconvex.

($u6$)⇒($u1$) Assume that F is ($u6$)-type C-quasiconvex. Consider a convex set $A \in 2^Y$. Let $x_0, x_1 \in \mathrm{dom}F$ with $F(x_0) \leq_C^u A$ and $F(x_1) \leq_C^u A$, and let $t \in (0,1)$ be arbitrarily chosen. It is easy to check that

$$(1-\lambda)F(x_0) + \lambda F(x_1) \leq_C^u A \text{ for any } \lambda \in [0,1].$$

Since F is ($u6$)-type C-quasiconvex,

$$F((1-t)x_0 + tx_1) \leq_C^u A \text{ for any } t \in (0,1).$$

Thus F is ($u1$)-type C-quasiconvex.

(u1)⇒(u2) It is easily verified.

(u2)⇔(u7) Can be proven in a similar way as (u1)⇔(u6).

Now we prove the two implications requiring the assumption ($*$).

(u1)⇒(u8) Assume that F is (u1)-type C-quasiconvex while $F(x) - C$ is convex for any $x \in \mathrm{dom}F$. Let $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0,1)$ be arbitrarily chosen. We have $F(x_0) \leq^u_C \bigcup_{\lambda\in[0,1]}((1-\lambda)F(x_0) + \lambda F(x_1))$ and $F(x_0) \leq^u_C \bigcup_{\lambda\in[0,1]}((1-\lambda)F(x_0) + \lambda F(x_1))$. Since F is (u1)-type C-quasiconvex and $\bigcup_{\lambda\in[0,1]}((1-\lambda)F(x_0) + \lambda F(x_1)) - C$ is convex, we conclude that

$$F((1-t)x_0 + tx_1) \leq^u_C \bigcup_{\lambda\in[0,1]}((1-\lambda)F(x_0) + \lambda F(x_1)).$$

Thus F is (u8)-type C-quasiconvex.

(u1)⇒(u9) Assume that F is (u1)-type C-quasiconvex while $F(x) - C$ is convex for any $x \in \mathrm{dom}F$. Consider a convex set $A \in 2^Y$ and let $x_0, x_1 \in \mathrm{dom}F$ with $F(x_0) \leq^u_C F(x_1)$ and $t \in (0,1)$ be arbitrarily chosen. Since F is (u1)-type C-quasiconvex and $F(x_0) \leq^u_C F(x_1) - C$ and $F(x_1) \leq^u_C F(x_1) - C$, we conclude that

$$F((1-t)x_0 + tx_1) \leq^u_C F(x_1).$$

Thus F is (u9)-type C-quasiconvex.

□

Remark 8. The converse implications in Proposition 2.3.6 are not true in general, as shown by the following examples.

Example 2.3.7. As in Example 2.3.5, consider the particular framework where $X := [0,1]$, $Y := \mathbb{R}^2$ and $C := \mathbb{R}^2_+$.

1. [(u5) $\not\Rightarrow$ (u4) and (u8) $\not\Rightarrow$ (u3)] The set-valued map F in Example 2.3.5.1 is (u5) and (u8)-type C-quasiconvex, but it is neither (u3) nor (u4)-type C-quasiconvex.

2. [(u3) $\not\Rightarrow$ (u4) and (u8) $\not\Rightarrow$ (u5)] Let $F : X \to 2^Y$ be the set-valued map defined by

$$F(x) = \begin{cases} \{(0,1)\} & \text{if } x = 0, \\ \{(-1,1),(1,-1)\} & \text{if } x \in (0,1), \\ \{(1,0)\} & \text{if } x = 1. \end{cases}$$

We can check easily that F is $(u3)$ and $(u8)$-type C-quasiconvex. However F is neither $(u4)$ nor $(u5)$-type C-quasiconvex. Indeed, $F(x) \not\leq^u_C F(0)$ and $F(x) \not\leq^u_C F(1)$ for any $x \in (0,1)$. Therefore we have $F(x) \not\leq^u_C (1-\lambda)F(0) + \lambda F(1)$ for any $\lambda \in [0,1]$.

3. $[(u2) \not\Rightarrow (u1)$ and $(u9) \not\Rightarrow (u1)]$ Let $F : X \to 2^Y$ be the set-valued map defined by

$$F(x) = \begin{cases} \{(0,1)\} & \text{if } x = 0, \\ \{(1,1)\} & \text{if } x \in (0,1), \\ \{(1,0)\} & \text{if } x = 1. \end{cases}$$

We can check easily that F is both $(u2)$-type and $(u9)$-type C-quasiconvex. On the other hand, by considering $A = \mathrm{co}\{(0,1),(1,0)\}$ we have $F(0) \leq^u_C A$, $F(1) \leq^u_C A$, but $F(x) \not\leq^u_C A$ for any $x \in (0,1)$. Thus F is not $(u1)$-type C-quasiconvex.

4. $[(u2) \not\Rightarrow (u9)]$ Consider the set-valued map $F : X \to 2^Y$ defined by

$$F(x) = \begin{cases} \mathrm{co}\{(-1,0),(0,-1)\} & \text{if } x \in \{0,1\}, \\ \{(0,0)\} & \text{if } x \in (0,1). \end{cases}$$

It is easy to check that F is $(u2)$-type C-quasiconvex. However, F is not $(u9)$-type C-quasiconvex, because $F(1) \leq^u_C F(0)$ and $F(\frac{1}{2}) \not\leq^u_C F(0)$.

5. $[(u9) \not\Rightarrow (u2)]$ It follows from Example 2.3.5.4, since a single-valued map is $(l2)$-type (resp. $(l9)$-type) C-quasiconvex if and only if it is $(u2)$-type (resp. $(u9)$-type) C-quasiconvex, as mentioned in Remark 5 f).

2.4 Characterizations of l-/u-type convexity in terms of l-/u-type quasiconvexity

We start this section by pointing out the relationship between the l-/u-type C-convexity and certain notions of l-/u-type C-quasiconvexity. It is easily seen that

$$C_l(X,Y,C) \subset \mathcal{Q}_{l5}(X,Y,C) \text{ and } C_u(X,Y,C) \subset \mathcal{Q}_{u5}(X,Y,C). \tag{2.9}$$

As the following example shows, in general we have

$$\mathcal{C}_l(X, Y, C) \not\subset \mathcal{Q}_{l4}(X, Y, C) \text{ and } \mathcal{C}_u(X, Y, C) \not\subset \mathcal{Q}_{u3}(X, Y, C),$$

hence, in view of Propositions 2.3.4 and 2.3.6,

$$\mathcal{C}_u(X, Y, C) \not\subset \mathcal{Q}_{u4}(X, Y, C).$$

Example 2.4.1. Let $X = [0, 1] \subset \mathbb{R}$ and let $Y = \mathbb{R}^2$ be endowed with the usual ordering cone $C = \mathbb{R}^2_+$. As in Example 2.3.5, consider the set-valued map $F : X \to 2^Y$, defined by

$$F(x) = \{(x, 1 - x)\}.$$

It is clear that F is l-type C-convex as well as u-type C-convex. We have already seen in Example 2.3.5 that F is not ($l4$)-type C-quasiconvex. On the other hand, for $x_0 = 0, x_1 = 1$ and any $t \in (0, 1)$, we have $F((1-t)x_0+tx_1)+C \not\subset (F(x_0) \cup F(x_1)) - C$, hence F is not ($u3$)-type C-quasiconvex.

Definition 2.4.2. A set-valued map $F : X \to 2^Y$ is said to be affine (cf. Gorokhovik [36]) if for any $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0, 1)$,

$$F((1 - t)x_0 + tx_1) = (1 - t)F(x_0) + tF(x_1).$$

We denote by $\mathcal{A}(X, Y)$ the class of all affine set-valued maps. Notice that

$$\mathcal{A}(X, Y) = \mathcal{C}_l(X, Y, \{0\}) \cap \mathcal{C}_u(X, Y, \{0\}) \subset \mathcal{C}_l(X, Y, C) \cap \mathcal{C}_u(X, Y, C). \qquad (2.10)$$

The following result is a generalization of the classical Crouzeix characterization of convexity for set-valued maps. It was obtained by Kuroiwa, Popovici and Rocca [42], but we present it in a slightly different form, by using the terminology l-type convexity/quasiconvexity.

Theorem 2.4.3. *Let X be a nonempty convex subset of a real vector space, let Y be a real topological vector space, let $C \subset Y$ be a convex cone, and let $F : X \to 2^Y$ be a set-valued map. If $F(x) + C$ is closed and convex while $F(x)$ is bounded for all $x \in \mathrm{dom}F$ (in particular, if C is closed and $F(x)$ is convex and compact for any $x \in X$), then the following are equivalent:*

1° $F \in \mathcal{C}_l(X, Y, C)$.

2° $F + G \in \mathcal{Q}_{l2}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

3° $F + G \in \mathcal{Q}_{l2}(X, Y, C)$ *for any* $G \in \mathcal{C}_l(X, Y, C)$.

Based on Propositions 2.3.4 and 2.3.6 we can identify new classes of generalized quasiconvex set-valued maps for which Crouzeix type characterizations hold. In particular, we will recover Theorem 2.4.3 as a particular instance of our new results. To this aim, we will need the following cancellation law, obtained by Urbański [13]:

Lemma 2.4.4. *Let* A, A' *and* B *be any subsets of* Y *such that*

$$A + B \subset \mathrm{cl}(A' + B).$$

If A' *is closed and convex while* B *is non empty and bounded, then* $A \subset A'$.

Theorem 2.4.5. *Let* X *be a nonempty convex subset of a real vector space, let* Y *be a real topological vector space, let* $C \subset Y$ *be a convex cone, and let* $F : X \to 2^Y$ *be a set-valued map. If* $F(x) + C$ *is closed convex and* $F(x)$ *is bounded for all* $x \in \mathrm{dom}F$, *then the following are equivalent:*

1° $F \in \mathcal{C}_l(X, Y, C)$

2° $F + G \in \mathcal{Q}_{l5}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

3° $F + G \in \mathcal{Q}_{l1}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

4° $F + G \in \mathcal{Q}_{l2}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

5° $F + G \in \mathcal{Q}_{l9}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

Proof. We just have to prove the implication 5° ⇒ 1°, since the implications 1° ⇒ 2° ⇒ 3° ⇒ 4° ⇒ 5° are direct consequences of Proposition 2.3.4, the inclusion $\mathcal{C}_l(X, Y, C) + \mathcal{A}(X, Y) \subset \mathcal{C}_l(X, Y, C)$, and (2.9). Assume that 5° holds and let $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0, 1)$ be arbitrarily chosen. We may assume that $x_0 \neq x_1$. Denoting by $[x_0, x_1]$ the line-segment joining the points x_0 and x_1, we define a bijective vector-valued function $h_{x_0,x_1} : [0, 1] \to [x_0, x_1]$ as

$$h_{x_0,x_1}(t) := (1 - t)x_0 + tx_1 \text{ for all } t \in [0, 1].$$

By means of its inverse, $h^{-1}_{x_0,x_1} : [x_0, x_1] \to [0, 1]$, we introduce a set-valued map $G : X \to 2^Y$, defined for all $x \in X$ as

$$G(x) := \begin{cases} h^{-1}_{x_0,x_1}(x)(F(x_0) + C) + (1 - h^{-1}_{x_0,x_1}(x))(F(x_1) + C) & \text{if } x \in [x_0, x_1] \\ \emptyset & \text{if } x \notin [x_0, x_1]. \end{cases}$$

We can show easily that G is affine. Also we have

$$(F + G)(x_0) = F(x_0) + F(x_1) + C = (F + G)(x_1),$$

that is, $(F + G)(x_0) \leq^l_C (F + G)(x_1)$. Now, let $t \in (0, 1)$ be arbitrarily chosen.

Since $F + G$ is $(l9)$-type C-quasiconvex, we have

$$(F + G)((1 - t)x_0 + tx_1) \leq^l_C (F + G)(x_1).$$

Taking into account that

$$(F + G)((1 - t)x_0 + tx_1) = F((1 - t)x_0 + tx_1) + tF(x_0) + (1 - t)F(x_1) + C$$

and

$$(F + G)(x_1) = (1 - t)F(x_0) + tF(x_1) + tF(x_0) + (1 - t)F(x_1) + C,$$

we have

$$F((1-t)x_0+tx_1)+tF(x_0)+(1-t)F(x_1)+C \supset (1-t)F(x_0)+tF(x_1)+tF(x_0)+(1-t)F(x_1)+C.$$

Finally, we infer by Lemma 2.4.4 that

$$F((1 - t)x_0 + tx_1) \leq^l_C (1 - t)F(x_0) + tF(x_1).$$

Therefore, F is l-type C-convex. □

As a direct consequence of Theorem 2.4.5, in view of the relations (2.9), (2.10) and $C_l(X, Y, C) + C_l(X, Y, C) \subset C_l(X, Y, C)$, we obtain the following result:

Corollary 2.4.6. *Under the hypotheses of Theorem 2.4.5 the following are equivalent:*

1° $F \in C_l(X, Y, C)$.

2° $F + G \in \mathcal{Q}_{l5}(X, Y, C)$ *for any* $G \in \mathcal{C}_l(X, Y, C)$.

3° $F + G \in \mathcal{Q}_{l1}(X, Y, C)$ *for any* $G \in \mathcal{C}_l(X, Y, C)$.

4° $F + G \in \mathcal{Q}_{l2}(X, Y, C)$ *for any* $G \in \mathcal{C}_l(X, Y, C)$.

5° $F + G \in \mathcal{Q}_{l9}(X, Y, C)$ *for any* $G \in \mathcal{C}_l(X, Y, C)$.

A characterization of l-type C-convex maps similar to Theorem 2.4.5 does not hold with respect to ($l4$)-type C-quasiconvexity, as Example 2.4.1 shows.

Now we present a counterpart of Crouzeix theorem for u-type C-convexity.

Theorem 2.4.7. *Let X be a nonempty convex subset of a real vector space, let Y be a real topological vector space, let $C \subset Y$ be a closed convex cone and let $F : X \to 2^Y$ be a set-valued map. If $F(x)$ is compact convex for all $x \in \mathrm{dom}F$, then the following are equivalent:*

1° $F \in \mathcal{C}_u(X, Y, C)$.

2° $F + G \in \mathcal{Q}_{u5}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

3° $F + G \in \mathcal{Q}_{u1}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

4° $F + G \in \mathcal{Q}_{u9}(X, Y, C)$ *for any* $G \in \mathcal{A}(X, Y)$.

Proof. We only need to prove the implication 4° $\Rightarrow$ 1° similarly to the proof of Theorem 2.4.5. Suppose that 4° holds and let $x_0, x_1 \in \mathrm{dom}F$ and $t \in (0, 1)$ be arbitrarily chosen. We may assume that $x_0 \neq x_1$. Define $h_{x_0,x_1} : [0, 1] \to [x_0, x_1]$ in the same way as in the proof of Theorem 3.3. Consider the set-valued map $G : X \to 2^Y$ defined for all $x \in X$ by

$$G(x) := \begin{cases} h^{-1}_{x_0,x_1}(x)(F(x_0) - C) + (1 - h^{-1}_{x_0,x_1}(x))(F(x_1) - C) & \text{if } x \in [x_0, x_1] \\ \emptyset & \text{if } x \notin [x_0, x_1]. \end{cases}$$

We can show easily that G is affine. Also we have

$$(F + G)(x_0) = F(x_0) + F(x_1) - C = (F + G)(x_1),$$

hence $(F + G)(x_0) \leq^u_C (F + G)(x_1)$. Now let $t \in (0, 1)$ be arbitrarily chosen.

Since $F+G$ is $(u9)$-type C-quasiconvex, we have

$$(F+G)((1-t)x_0+tx_1) \leq_C^u (F+G)(x_1).$$

Taking into account that

$$(F+G)((1-t)x_0+tx_1) = F((1-t)x_0+tx_1)+tF(x_0)+(1-t)F(x_1)-C$$

and

$$(F+G)(x_1) = (1-t)F(x_0)+tF(x_1)+tF(x_0)+(1-t)F(x_1)-C,$$

we have

$$F((1-t)x_0+tx_1)+tF(x_0)+(1-t)F(x_1)-C \subset (1-t)F(x_0)+tF(x_1)+tF(x_0)+(1-t)F(x_1)-C.$$

Since under the hypotheses of Theorem 2.4.7 the set $(1-t)F(x_0)+tF(x_1)$ is closed, we infer by Lemma 2.4.4 that

$$F((1-t)x_0+tx_1) \leq_C^u (1-t)F(x_0)+tF(x_1).$$

Therefore F is u-type C-convex, i.e., 1° holds. □

As a direct consequence of Theorem 2.4.7, in view of the relations (2.9), (2.10) and $\mathcal{C}_u(X,Y,C)+\mathcal{C}_u(X,Y,C) \subset \mathcal{C}_u(X,Y,C)$, we obtain the following result:

Corollary 2.4.8. *Under the hypotheses of Theorem 2.4.7 the following are equivalent:*

1° $F \in \mathcal{C}_u(X,Y,C)$.

2° $F+G \in \mathcal{Q}_{u5}(X,Y,C)$ *for any* $G \in \mathcal{C}_u(X,Y,C)$.

3° $F+G \in \mathcal{Q}_{u1}(X,Y,C)$ *for any* $G \in \mathcal{C}_u(X,Y,C)$.

4° $F+G \in \mathcal{Q}_{u9}(X,Y,C)$ *for any* $G \in \mathcal{C}_u(X,Y,C)$.

A characterization of u-type C-convex maps similar to Theorem 2.4.7 does not hold neither in terms of $(u3)$- and $(u4)$-type C-quasiconvexity, as Example 2.4.1 shows, nor in terms of $(u2)$ i.e. $(u7)$-type C-quasiconvexity, as shown by the following example.

Example 2.4.9. Let $X = [0,1] \subset \mathbb{R}$ and let $Y = \mathbb{R}^2$ be endowed with the usual ordering cone $C = \mathbb{R}^2_+$. Consider the set-valued map $F : X \to 2^Y$ given by

$$F(x) = \mathrm{co}\left\{(0, \sqrt{x}), (\sqrt{x}, 0)\right\}.$$

Then $F + G$ is $(u2)$-type C-quasiconvex for any affine map $G : X \to 2^Y$, but F is not u-type C-convex. Indeed, let $G : X \to 2^Y$ be an affine map. Let $y \in \mathbb{R}^2$ and let $x_0, x_1 \in [0,1]$ be such that $(F + G)(x_0) \leq^u_C \{y\}$ and $(F + G)(x_1) \leq^u_C \{y\}$. Then, for any $t \in (0,1)$, we have $(F + G)((1-t)x_0 + tx_1) \leq^u_C \{y\}$. Hence $F + G$ is $(u2)$-type C-quasiconvex. However, by choosing $x_0 = 0$, $x_1 = 1$ and $t = \frac{1}{2}$, we get $\frac{1}{2}F(0) + \frac{1}{2}F(1) = \mathrm{co}\{(0, \frac{1}{2}), (\frac{1}{2}, 0)\}$ while $F(\frac{1}{2}) = \mathrm{co}\left\{(0, \sqrt{1/2}), (\sqrt{1/2}, 0)\right\}$. Therefore F is not u-type C-convex.

Chapter 3

Fixed point theorem in a complete metric space

In this chapter, we give fixed point theorems whose map have multiple fixed points in a complete metric space. At first, we mention some fixed point theorems which are extended the Banach contraction principle. Next, we will mention the main results and its motivations, and we give some examples. Finally, we will study fixed point theorems for set-valued maps.

3.1 Fixed point theorems in a complete metric space

The following theorem is called the Banach contraction principle which behaves good roles in many fields of mathematics and applied mathematics:

Theorem 3.1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a map satisfying there exists $r \in (0, 1)$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq rd(x, y).$$

Then T has a unique fixed point $\bar{x} \in X$ and $T^n x \to \bar{x}$ for any $x \in X$.

The Banach contraction principle was extended by many authors in [7, 5, 31]. The following are one of famous generalization of the Banach contraction principle:

Theorem 3.1.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a map satisfying there exists $\phi : [0, +\infty) \to [0, +\infty)$ satisfying $\phi(0) = 0$, $\phi(s) < s$ for any $s > 0$ and ϕ is upper semi continuous from the right such that*

$$d(Tx, Ty) \leq \phi(d(x, y)).$$

Then T has a unique fixed point $\bar{x} \in X$ and $T^n x \to \bar{x}$ for any $x \in X$.

Theorem 3.1.2 were extended two types fixed theorems in [7, 31], and it is well know that these theorems are equivalent.

Theorem 3.1.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be a map satisfying for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon.$$

Then T has a unique fixed point $\bar{x} \in X$ and $T^n x \to \bar{x}$ for any $x \in X$.

In [31], the above theorem were characterized by L-function.

Definition 3.1.4. A map $\phi : [0, +\infty) \to [0, +\infty)$ is said to be L-function if the following conditions hold:

(i) $\phi(0) = 0$;

(ii) for any $\varepsilon > 0$, $\phi(\varepsilon) > 0$; and

(iii) for any $\varepsilon > 0$, there exists $\eta > 0$ for any $t \in [\varepsilon, \varepsilon + \eta]$, $\phi(t) \leq \varepsilon$.

Proposition 3.1.5. *Let (X, d) be a metric space and let $T : X \to X$. T holds the condition of Theorem 3.1.3 if and only if there exists L-function ϕ such that*

$$d(x, y) > 0 \Rightarrow d(Tx, Ty) < \phi(d(x, y)).$$

In a complete metric space (X, d), fixed point theorems are categorized four types as follows[37]: let T be a self-mapping on X,

(T1) Leader-type : T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$;

(T2) Unnamed-type : T has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point;

(T3) Subrahmanyam-type : T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point of T for all $x \in X$; and

(T4) Caristi-type : T may have more one than fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point of T.

Theorems 3.1.1, 3.1.2 and 3.1.3 belong to (T1). The following theorem shows a necessary and sufficient condition for (T1) in [37]:

Theorem 3.1.6. *Let T be a mapping on a complete metric space (X, d). T belongs to (T1) if and only if T satisfies the following two conditions:*

1. *For $x, y \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathbb{N}$ such that*

$$d(T^i x, T^j y) < \varepsilon + \delta \textit{ implies } d(T^{i+v} x, T^{j+v} y) < \varepsilon$$

for all $i, j \in \mathbb{N} \cup \{0\}$.

2. *For $x, y \in X$, there exist $v \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, +\infty)$ such that*

$$d(T^i x, T^j y) < \alpha_n \textit{ implies } d(T^{i+v} x, T^{j+v} y) < \frac{1}{n}$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

3.2 Fixed point theorems for (T3)

Fixed point theorems, which belong to (T1), were studied as generalizations of the Banach contraction principle. In general, we imagine that fixed point theorems, which belong to (T3), can be studied in a similar way to (T1). However, there are few fixed point theorems which belong to (T3) as far as I know. The following theorem is a famous one which belongs to (T3), see [11]:

Theorem 3.2.1. *Let (X, d) be a complete metric space, and let T be a self-mapping on X. Assume that there exists $r \in [0, 1)$ such that for all $x \in X$,*

$$d(T^2 x, Tx) \leq r d(Tx, x).$$

Then T has at least one fixed point and $\{T^n x\}$ converges to a fixed point of T for all $x \in X$.

Similar to (T1), a necessary and sufficient condition for (T3) were given in [38] as follows:

Theorem 3.2.2. *Let T be a mapping on a complete metric space (X, d). T belongs to (T3) if and only if T satisfies the following two conditions:*

1. *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathbb{N}$ such that*

$$d(T^i x, T^j x) < \varepsilon + \delta \text{ implies } d(T^{i+v} x, T^{j+v} x) < \varepsilon$$

for all $i, j \in \mathbb{N} \cup \{0\}$;

2. *For $x, y \in X$, there exist $v \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, +\infty)$ such that*

$$d(T^i x, T^j y) < \alpha_n \text{ implies } d(T^{i+v} x, T^{j+v} y) < \frac{1}{n}$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

Indeed, the above theorem shows a necessary and sufficient for (T3). However, it is not easy to check that a map T holds these conditions. We believe that fixed point theorems, which belongs to (T3), are needed for convex optimization problem. Hence, we will give a Meir and Keeler type sufficient condition for (T3) in the next section.

3.3 Main result

In this part, we give a main result with respect to a sufficient condition for a self-mapping T on X which has multiple fixed points satisfying the Picard iteration $\{T^n x\}$ converges to a fixed point of T for every starting point x in a given subset of X.

Theorem 3.3.1. *Let (X, d) be a complete metric space, let T be a self-mapping on X, and let B be a subset of X satisfies $T(B) \subseteq B$. Assume that for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\text{for all } x, y \in B,\ \varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon. \tag{3.1}$$

Then there exists $\bar{x} \in X$ such that $\{T^n x\} \to \bar{x}$ for all $x \in B$.

Proof. At first, we prove that

$$\text{for all } x, y \in B \text{ with } x \neq y,\ d(Tx, Ty) < d(x, y).$$

If not, there exist $x_0, y_0 \in B$ with $x_0 \neq y_0$ such that $d(Tx_0, Ty_0) \geq d(x_0, y_0)$. Put $\varepsilon_0 = d(x_0, y_0) > 0$, then there exists $\delta_0 > 0$ such that (3.1) holds by the assumption. From $\varepsilon_0 = d(x_0, y_0) < \varepsilon_0 + \delta_0$, we have $d(Tx_0, Ty_0) < \varepsilon_0 = d(x_0, y_0)$. This is a contradiction. Next, for any given $x \in B$, define a sequence $\{x_n\}$ as

$$x_0 = x,\ x_n = Tx_{n-1}\ (n = 1,\ 2, \ldots).$$

If $x_n = x_{n-1}$ holds, x_{n-1} is the fixed point. Then we may assume that $x_n \neq x_{n-1}$ for all n. Put $c_n = d(x_n, x_{n-1})$ for all $n \in \mathbb{N}$. Since $c_n \geq 0$ and

$$c_{n+1} = d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) < d(x_n, x_{n-1}) = c_n,$$

$\{c_n\}$ is a lower bounded and decreasing sequence. Then there exists $c \in [0, +\infty)$ such that $c_n \to c$. We show $c = 0$. If $c > 0$, by putting $\varepsilon_1 = c$, there exists $\delta_1 > 0$ such that (3.1) holds. From $c \leq c_n$ for all $n \in \mathbb{N}$ and $c_n \to c$, we have $c \leq c_n < c + \delta_1$ for sufficiently large n. Since $\varepsilon_1 \leq d(x_n, x_{n-1}) < \varepsilon_1 + \delta_1$, then $c_{n+1} = d(Tx_n, Tx_{n-1}) < \varepsilon_1 = c$ and this is a contradiction. Therefore $c = 0$. Now we show that $\{x_n\}$ is a Cauchy sequence. If not, there exists $\varepsilon_2 > 0$ such that for all $N \in \mathbb{N}$, there exist $l, m \geq N$ such that $d(x_l, x_m) > 2\varepsilon_2$ and $l < m$. Also there exists $\delta_2 > 0$ such that (3.1) holds. Put $\delta' = \min\{\varepsilon_2, \delta_2\}$. We have $\varepsilon_2 \leq d(x, y) < \varepsilon_2 + \delta'$ implies $d(Tx, Ty) < \varepsilon_2$. Form $c_n \to 0$, there exists $M \in \mathbb{N}$ such that $c_n < \delta'/3$, for all $n \geq M$. Put $N = M$, then there exist $l, m \geq M$ such that $l < m$ and $d(x_l, x_m) > 2\varepsilon_2$. Also we have, for all $j \in \{l, l+1, \ldots, m\}$,

$$|d(x_l, x_j) - d(x_l, x_{j+1})| \leq d(x_j, x_{j+1}) = c_j < \frac{\delta'}{3}.$$

From this and

$$c_l = d(x_l, x_{l+1}) < \frac{\delta'}{3} < \varepsilon_2 + \frac{2}{3}\delta' < \varepsilon_2 + \delta^{'} \leq 2\varepsilon_2 < d(x_l, x_m),$$

there exists $k \in \mathbb{N}$ such that $\varepsilon_2 + 2\delta'/3 < d(x_l, x_k) < \varepsilon_2 + \delta'$. Then we have $d(x_{l+1}, x_{k+1}) < \varepsilon_2$, and then

$$\begin{aligned} d(x_l, x_k) &\leq d(x_l, x_{l+1}) + d(x_{l+1}, x_{k+1}) + d(x_{k+1}, x_k) \\ &< c_l + \varepsilon_2 + c_k \\ &< \varepsilon_2 + \frac{2}{3}\delta'. \end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. Next, we prove that $T^n x \to \bar{x}$ for all $x \in B$. Assume that there exist $x_0, y_0 \in B$ such that $T^n x_0 \to \bar{x}$ and $T^n y_0 \to \bar{y}$ where $\bar{x} \neq \bar{y}$. Put $\varepsilon_3 = d(\bar{x}, \bar{y}) > 0$, then there exists $\delta_3 > 0$ such that (3.1) holds. From $\{d(T^n x_0, T^n y_0)\}$ is a lower bounded and decreasing sequence, we have $\varepsilon_3 \leq d(T^n x_0, T^n y_0)$ for all $n \in \mathbb{N}$. Since $T^n x \to \bar{x}$ and $T^n y \to \bar{y}$, we have $d(T^n x_0, \bar{x}) < \delta_3/2$ and $d(T^n y_0, \bar{y}) < \delta_3/2$ for sufficiently large n. Using (3.1), we have $d(T^{m_0+1} x_0, T^{m_0+1} y_0) < \varepsilon_3$. This is a contradiction. Finally, we prove that $\bar{x} \in F(T)$. Assume that $\bar{x} \notin F(T)$.

$$
\begin{aligned}
0 < d(\bar{x}, T\bar{x}) &\leq d(\bar{x}, T^n x) + d(T^n x, T\bar{x}) \\
&< d(\bar{x}, T^n x) + d(T^{n-1} x, \bar{x}) \\
&\to 0.
\end{aligned}
$$

This is a contradiction. □

In the above theorem, if B is closed then Theorem 3.3.1 is equivalent to Theorem 3.1.3, however B may not be closed. The following example shows Theorem 3.3.1 can be applied to a mapping T but Theorems from 3.1.1 to 3.1.6 can not be applied to a mapping T.

Example 3.3.2. Let $(\mathbb{R}^n, d)$, and let T be defined as follows:

$$
Tx = \begin{cases} \frac{1}{2}x & x \in (0, +\infty)^n, \\ 2x & x \notin (0, +\infty)^n. \end{cases}
$$

Then we can apply Theorem 3.3.1 for open set $B = (0, +\infty)^n$. Indeed, for all $\varepsilon > 0$, by putting $\delta = \varepsilon$, for all $x, y \in B$ satisfying $\varepsilon \leq d(x, y) < \varepsilon + \delta$,

$$
\begin{aligned}
d(Tx, Ty) = \|Tx - Ty\| &= \left\| \frac{1}{2}x - \frac{1}{2}y \right\| \\
&= \frac{1}{2} \left\| x - y \right\| \\
&= \frac{1}{2} d(x, y) < \frac{1}{2}(\varepsilon + \delta) = \varepsilon.
\end{aligned}
$$

Hence T holds the condition of Theorem 3.3.1. Therefore T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point of T for all $x \in B$. However, Theorems from 3.1.1 to 3.1.6 can not be applied because $\{T^n x\}$ does not converge when $x \in X \setminus (B \cup \{0\})$ and also B is not closed.

In the following example, we give a self-mapping T which hold the conditions of Theorem 3.3.1:

Example 3.3.3. Let $(\mathbb{R}^2, d)$, and let T be defined as follows:

$$Tx = \frac{1}{2}(x + P_A(x)),$$

where $A = [-1, 1]^2$, $P_A(x)$ is the point y of $\mathbb{R}^2$ satisfying $d(x, y) \le d(x, z)$ for all $z \in A$. Let $F(T)$ be the set of all fixed points of T, then we can see $F(T) = A$, that is, T has multiple fixed points. Since

$$T^n x = \frac{1}{2^n}x + \left(1 - \frac{1}{2^n}\right)P_A(x) \to P_A(x) \in A = F(T)$$

for all $x \in X$, (T3) holds for T. Let $B_{(1,1)} := \{x \in \mathbb{R}^2 \mid T^n x \to (1, 1)\}$. Then we can check that the condition of Theorem 3.3.1 for $B = B_{(1,1)}$ holds. Indeed, for all $\varepsilon > 0$, by putting $\delta = \varepsilon$, for all $x, y \in B_{(1,1)}$, $P_A(x) = P_A(y) = (1, 1)$. Therefore

$$\begin{aligned}
\varepsilon \le d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) &= d\left(\frac{1}{2}(x + P_A(x)), \frac{1}{2}(y + P_A(y))\right) \\
&= \left\| \frac{1}{2}(x + P_A(x)) - \frac{1}{2}(y + P_A(x)) \right\| \\
&= \frac{1}{2}\|x - y\| \\
&= \frac{1}{2}d(x, y) \\
&< \frac{1}{2}(\varepsilon + \delta) = \varepsilon.
\end{aligned}$$

Also we have $T^n x \to (1, 1)$ for all $x \in B_{(1,1)}$ and $B_{(1,1)} = [1, +\infty)^2$. In a similar way, for each $z \in A$, let $B_z := \{x \in \mathbb{R}^2 \mid T^n x \to z\}$, then we have the condition of Theorem 3.3.1 for $B = B_z$ holds and $T^n x \to P_A(x) \in A = F(T)$ for all $x \in B_z$.

Motivated by Example 3.3.3, we give a result of (T3) from Theorem 3.3.1 by putting a certain subset B of X. For $A \subset X$ and $n \in \mathbb{N}$, denote that $T^{-n}A := (T^n)^{-1}A$ and $T^0 A := A$.

Proposition 3.3.4. *Let (X, d) be a complete metric space, and let T be a self-mapping on X. Assume that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X \setminus \bigcup_{n \in \mathbb{N} \cup \{0\}} T^{-n}(F(T))$,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon.$$

Then there exists $\bar{x} \in X$ such that $\{T^n x\} \to \bar{x}$ for all $x \in X \setminus \bigcup_{n \in \mathbb{N} \cup \{0\}} T^{-n}(F(T))$.

Proof. Let

$$B = X \setminus \bigcup_{n \in \mathbb{N} \cup \{0\}} T^{-n}(F(T)).$$

We show $T(B) \subset B$. If there exists $y \in T(B)$ such that $y \notin B$, then there exists $x \in B$ such that $y = Tx$. Since $y = Tx \notin B$, $Tx \in \bigcup_{n \in \mathbb{N} \cup \{0\}} T^{-n}(F(T))$, and this shows $Tx \in T^{-n_0}(F(T))$ for some $n_0 \in \mathbb{N} \cup \{0\}$, that is,

$$x \in T^{-n_0-1}(F(T)) \subset \bigcup_{n \in \mathbb{N} \cup \{0\}} T^{-n}(F(T)).$$

This contradicts to $x \in B$. Using Theorem 3.3.1, $\{T^n x\}$ converges to a fixed point of T for all $x \in B$. On the other hand, when $x \notin B$, since

$$x \in \bigcup_{n \in \mathbb{N} \cup \{0\}} T^{-n}(F(T)),$$

there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $T^{n_0} x \in F(T)$, that is, $T^n x = T^{n_0} x$ hold for all $n \geq n_0$. This means $\{T^n x\}$ converges to a fixed point of T. This completes the proof. $\square$

Example 3.3.5. Consider $(\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}, d)$ where $d(x, y) = |x - y|$ and T define by

$$Tx = \begin{cases} 0 & x = 0 \\ \frac{1}{n+1} & x = \frac{1}{n}. \end{cases}$$

We put $B = X$. Then T holds the assumption of Theorem 3.3.1 but T does not hold the assumption of Theorem 3.2.1. We check only the assumption of Theorem 3.3.1. At first, we see that for any $\varepsilon > 0$, we can show the existence of $\alpha := \min\{\frac{1}{n} - \frac{1}{m} \mid \varepsilon \leq \frac{1}{n} - \frac{1}{m}\ n, m \in \mathbb{N}\}$. In the similar way, for $\alpha > 0$, we can show the existence of $\beta := \min\{\frac{1}{n} - \frac{1}{m} \mid \alpha < \frac{1}{n} - \frac{1}{m},\ n, m \in \mathbb{N}\}$.

Put $\delta = \frac{\alpha+\beta}{2} > 0$, then there exist $n_0, m_0 \in \mathbb{N}$ such that $\varepsilon \leq \frac{1}{n_0} - \frac{1}{m_0} < \varepsilon + \delta$, but there does not exist $\bar{n}, \bar{m} \in \mathbb{N}$ such that $\varepsilon \leq \frac{1}{\bar{n}} - \frac{1}{\bar{m}} < \varepsilon + \delta$ and $\frac{1}{\bar{n}} - \frac{1}{\bar{m}} = \frac{1}{n_0} - \frac{1}{m_0}$. Also we can prove $\frac{1}{n_0+1} - \frac{1}{m_0+1} < \varepsilon$ by contradiction. We assume that $\varepsilon \leq \frac{1}{n_0+1} - \frac{1}{m_0+1}$. Since $\frac{1}{n_0} - \frac{1}{m_0}$ is minimum of $\{\frac{1}{n} - \frac{1}{m} \mid \varepsilon \leq \frac{1}{n} - \frac{1}{m}\ n, m \in \mathbb{N}\}$, we can see that $\frac{1}{n_0} - \frac{1}{m_0} \leq \frac{1}{n_0+1} - \frac{1}{m_0+1}$. Hence, we can calculate $m_0(m_0+1) \leq n_0(n_0+1) < m_0(m_0+1)$, this is contradiction.

We give an observation between our result and the previous ones. Define a binary relation $\sim$ on X by for every $x, y \in X$,

$$x \sim y \text{ if and only if } T^n x \to z \text{ and } T^n y \to z \text{ for some } z \in X \text{ or}$$
$$\text{both } \{T^n x\} \text{ and } \{T^n y\} \text{ do not converge.}$$

Then $\sim$ is an equivalence relation on X, that is, for all x, y and $z \in X$,

1. $x \sim x$;
2. if $x \sim y$ then $y \sim x$; and
3. if $x \sim y$ and $y \sim z$ then $x \sim z$.

Let the equivalence class of x and the quotient set be

$$[x] = \{y \in X \mid x \sim y\} \text{ and } X/\!\sim = \{[x] \mid x \in X\},$$

respectively, and define a function $\varphi : (X/\!\sim) \setminus \{N(T)\} \to X$ by

$$\varphi(x) = \lim_{n\to+\infty} T^n x,$$

where $N(T) = \{x \mid \{T^n x\} \text{ does not converge}\}$. By using the notations, fixed point theorems can be categorized as follows:

1. $N(T) = \emptyset$, $|F(T)| = 1$, and $\varphi(X/\!\sim) \subset F(T)$;
2. $N(T) = \emptyset$, $F(T) \neq \emptyset$, and $\varphi(X/\!\sim) \subset F(T)$; and
3. $N(T) \cap B = \emptyset$, $F(T) \neq \emptyset$, and $\varphi(B/\!\sim) \subset F(T)$;

where $B/\!\sim = \{[x] \mid x \in B\}$. We can see that (1) is equivalent to (T1), (2) is equivalent to (T3), and (3) is equivalent to the result of Theorem 3.3.1. If $B = X$ then (3) coincide with (T3), and if $B = X = B_z$ then (3) coincide with (T1) where $z \in X$. As we have seen in Examples 3.3.2 and 3.3.3, including the situation $N(T) \neq \emptyset$, Theorem 3.3.1 is useful to observe the limits of the Picard iteration.

3.4 Fixed point theorem for set-valued maps

In this section, we mention fixed point theorem for set-valued maps, and we give a fixed point theorems for set-valued maps by using Theorem 3.3.1.

Let (X, d) be a metric space. $C(X)$ denotes the family of all closed subsets of X. $CB(X)$ denotes the family of all closed bounded subsets of X. For any $A, B \in C(X)$,

$$H(A, B) := \max\{\sup_{x\in B} d(x, A), \sup_{y\in A} d(y, B)\},$$

where $d(x, A) := \inf_{y\in A} d(x, y)$. The following theorems, which are a generalization of the Banach contraction principle, is a fixed point theorem for set valued-maps in [8]:

Theorem 3.4.1. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ Assume that there exists $r \in (0, 1)$ such that for any $x, y \in X$,*

$$H(Tx, Ty) \leq rd(x, y).$$

Then T has a fixed point.

The following are a generalization of Theorem 3.4.1 in [22]:

Theorem 3.4.2. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$. Assume that there exists $\phi : (0, +\infty) \to (0, 1)$ such that*

$$\limsup_{r\to t^+} \phi(r) < 1 \text{ for any } t \in (0, +\infty);$$

and

$$H(Tx, Ty) \leq \phi(d(x, y))d(x, y) \text{ for any } x, y \in X \text{ with } x \neq y.$$

Then T has a fixed point.

We can give the following theorems by using Theorem 3.3.1:

Theorem 3.4.3. *Let (X, d) be a complete metric space and let $F : X \to 2^X$. Assume that there exists $f : X \to X$ satisfying (3.1) and $f(x) \in F(x)$ for any $x \in X$. Then T has a fixed point.*

3.5 Fixed set theorems

In this section, we study fixed set theorems for set-to-set maps. Let X be a compact convex subset of a normed space. For a set-valued map $T : X \to 2^X$, $\bar{x} \in X$ is said to be a fixed point of T if $T(\bar{x}) \ni \bar{x}$. Nadler established a fixed point theorem for set-valued maps in [8] which is an extension of the Banach contraction principle, Mizoguchi and Takahashi have extended Nadler's results in [22]. Also Fakhar, Soltani and Zafarani gave a maximal invariant set (fixed set) theorem for set-valued maps in [45].

On the other hand, for a set-to-set map $T : 2^X \to 2^X$ and a nonempty set $A \in 2^X$, there are four type *fixed set* notions which are generalizations of the fixed point notion:

1. $T(A) = A$;
2. $T(A) \subset A$;
3. $T(A) \supset A$;
4. $T(A) \cap A \neq \emptyset$.

We can find the following previous works for such fixed set theorems: Pradip, Binayak and Murchana showed a fixed set theorem in term of $T(A) \supset A$ in [47], which is a generalization of Nadler's result, and Robert, Klaus and Bradon showed a fixed set theorem in term of $T(B) = B$ for a monotone map T under the existence of A such that $T(A) \subset A$ in [39], and applied to study of a boundary value problem for a system of differential equations. In this paper, we give another fixed set theorem for set-to-set maps, by using an embedding idea in [4], which is a generalization of the following Schauder fixed point theorem, see [29]:

Theorem 3.5.1. *Let X be a nonempty convex subset of a normed space E, and let T be a continuous self-mapping on X. If $T(X)$ is compact, then there exists $\bar{x} \in X$ such that $T(\bar{x}) = \bar{x}$.*

Throughout this section, let E be a normed space, let X be a nonempty compact convex subset of E, and let C_X be the family of all nonempty compact convex subsets of X.

Lemma 3.5.2. *Define* $H : \mathcal{C}_X \times \mathcal{C}_X \to [0, +\infty)$ *by*

$$H(A, B) := \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\},$$

for any $A, B \in \mathcal{C}_X$*. Then* H *is a metric on* $\mathcal{C}_X$*, which is called the Hausdorff metric, and the metric space* $(\mathcal{C}_X, H)$ *is compact.*

Proof. We give a proof based on the non-convex version, see [28]. Since X is compact, that is, X is totally bounded, for any $\varepsilon > 0$, there exists a finite set $Y \subset X$ such that

$$\min_{y\in Y} d(x, y) < \varepsilon \text{ for any } x \in X.$$

For any $C \in \mathcal{C}_X$, put $S = \{y \in Y \mid d(C, y) < \varepsilon\}$, then $H(C, S) < \varepsilon$ holds, that is, $H(C, \mathrm{co}S) < \varepsilon$ holds. Put a finite subfamily $\mathcal{T} = \{\mathrm{co}S \mid S \in 2^Y\}$, then $\mathcal{T} \subset \mathcal{C}_X$ and

$$\min_{T\in\mathcal{T}} H(C, T) < \varepsilon \text{ for any } C \in \mathcal{C}_X.$$

This shows that $(\mathcal{C}_X, H)$ is also total bounded. Next, for any Cauchy sequence $\{A_n\} \subset \mathcal{C}_X$, define

$$A := \{x \in X \mid \exists\{x_n\} \subset X \text{ s.t. } x_n \to x, x_n \in A_n\ \forall n \in \mathbb{N}\},$$

then we can see that A is a nonempty compact convex subset of X and $\{A_n\}$ converges to A with respect to the Hausdorff metric H. Then $(\mathcal{C}_X, H)$ is complete, and consequently $(\mathcal{C}_X, H)$ is compact. □

Now we give the main theorem.

Theorem 3.5.3. *Let* $\mathcal{A}$ *be a subfamily of* $\mathcal{C}_X$ *satisfying*

$$A, B \in \mathcal{A}, \lambda \in (0, 1) \Rightarrow (1 - \lambda)A + \lambda B \in \mathcal{A}, \tag{3.2}$$

and let $T : \mathcal{A} \to \mathcal{A}$ *be continuous with respect to the Hausdorff metric* H*. If either the following (i) or (ii) holds:*

(i) $\mathcal{A}$ *is closed with respect to the Hausdorff metric* H*,*

(ii) $T(\mathcal{A}) := \{T(A) \mid A \in \mathcal{A}\}$ *is closed with respect to the Hausdorff metric* H*,*

then T *has a fixed set, that is, there exists* $\bar{A} \in \mathcal{A}$ *such that* $T(\bar{A}) = \bar{A}$*.*

Proof. We may assume (ii). Indeed, if (i) holds, then $\mathcal{A}$ is compact because $\mathcal{A}$ is closed and C_X is compact with respect to the Hausdorff metric H, therefore, the image $T(\mathcal{A})$ is also compact because T is continuous.

Let C be the family of all nonempty compact convex subsets of E, and define a binary relation $\equiv$ on C^2 by, for all $(A, B), (C, D) \in C^2$,

$$(A, B) \equiv (C, D) \text{ if } A + D = B + C,$$

then $\equiv$ is an equivalence relation on C^2. The cancellation low on C, that is,

$$A + B \subset A + C \Rightarrow B \subset C$$

is essential to show the equivalence. Define the quotient space

$$C^2/{\equiv} := \{[A, B] \mid (A, B) \in C^2\},$$

where

$$[A, B] := \{(C, D) \in C^2 \mid (A, B) \equiv (C, D)\},$$

and define the following addition and scholar multiplication on $C^2/{\equiv}$ by

$$[A, B] + [C, D] = [A + C, B + D],$$

$$\lambda[A, B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0 \\ [-\lambda B, -\lambda A] & \text{if } \lambda < 0, \end{cases}$$

for any $[A, B], [C, D] \in C^2/{\equiv}$ and $\lambda \in \mathbb{R}$, then $C^2/{\equiv}$ is a vector space over $\mathbb{R}$. Also define

$$\|[A, B]\| = H(A, B)$$

for each $[A, B] \in C^2/{\equiv}$, then $(C^2/{\equiv}, \|\cdot\|)$ becomes a normed space. For details about these arguments, see [43, 4].

Define

$$\begin{array}{cccc} \psi : & \mathcal{A} & \rightarrow & C^2/{\equiv} \\ & \Cup & & \Cup \\ & A & \longmapsto & [A, \{0\}]. \end{array}$$

Note that

$$\|\psi(A) - \psi(B)\| = \|[A, \{0\}] - [B, \{0\}]\| = \|[A, B]\| = H(A, B),$$

for any $A, B \in C_X$. Consequently, ψ is continuous because

$$\|\psi(A_n) - \psi(A)\| = H(A_n, A) \rightarrow 0$$

for a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ converges to $A \in \mathcal{A}$ with respect to the Hausdorff metric H. Also $\psi(\mathcal{A})$ is a convex subset of $C^2/\equiv$. Indeed, for any $\psi(A), \psi(B) \in \psi(\mathcal{A})$ and $\lambda \in (0,1)$, from

$$(1-\lambda)\psi(A) + \lambda\psi(B) = [(1-\lambda)A + \lambda B, \{0\}] = \psi((1-\lambda)A + \lambda B)$$

and $(1-\lambda)A + \lambda B \in \mathcal{A}$, then $(1-\lambda)\psi(A) + \lambda\psi(B) \in \psi(\mathcal{A})$.

Consider a self-mapping on convex set $\psi(\mathcal{A})$ defined by

$$\begin{array}{rccc} \mathcal{T}: & \psi(\mathcal{A}) & \to & \psi(\mathcal{A}) \\ & \Downarrow & & \Downarrow \\ & [A,\{0\}] & \longmapsto & [T(A),\{0\}], \end{array}$$

then $\mathcal{T}$ is continuous. Indeed, if a sequence $\{\psi(A_n)\} \subset \psi(\mathcal{A})$ converges to $\psi(A) \in \psi(\mathcal{A})$, that is $\|\psi(A_n) - \psi(A)\| \to 0$, then $H(A_n, A) \to 0$ and

$$\|\mathcal{T}(\psi(A_n)) - \mathcal{T}(\psi(A))\| = \|[T(A_n),\{0\}] - [T(A),\{0\}]\| = H(T(A_n), T(A)).$$

Since T is continuous with respect to H, then $H(T(A_n), T(A)) \to 0$. This shows $\mathcal{T}$ is continuous. Also $\mathcal{T}(\psi(\mathcal{A}))$ is compact because $T(\mathcal{A})$ is compact, ψ is continuous, and

$$\begin{aligned} \mathcal{T}(\psi(\mathcal{A})) &= \{\mathcal{T}(\psi(A)) \mid A \in \mathcal{A}\} \\ &= \{\mathcal{T}([A,\{0\}]) \mid A \in \mathcal{A}\} \\ &= \{[T(A),\{0\}] \mid A \in \mathcal{A}\} \\ &= \{\psi(T(A)) \mid A \in \mathcal{A}\} \\ &= \psi(T(\mathcal{A})). \end{aligned}$$

By using Theorem 3.5.1, there exists $\bar{A} \in \mathcal{A}$ such that $\mathcal{T}(\psi(\bar{A})) = \psi(\bar{A})$, that is, $T(\bar{A}) = \bar{A}$. □

Remark 9. It is clear that Theorem 3.5.3 is different from the previous fixed set theorems in [39, 47].

We can obtain the following corollaries by using Theorem 3.5.3:

Corollary 3.5.4. *Let T be a continuous self-mapping on C_X with respect to the Hausdorff metric H. Then T has a fixed set, that is, there exist $\bar{A} \in C_X$ such that $T(\bar{A}) = \bar{A}$.*

Proof. Put $\mathcal{A} := C_X$, then we can see that $\mathcal{A}$ is closed with respect to H satisfying (3.2). Therefore we can apply Theorem 3.5.3 to show the existence of fixed sets of T. □

Corollary 3.5.5 (Theorem 3.5.1)**.** *Let X be a nonempty convex subset of a normed space E, and let T be a continuous self-mapping on X. If $T(X)$ is compact, then there exists $\bar{x} \in X$ such that $T(\bar{x}) = \bar{x}$.*

Proof. Put $\mathcal{A} := \{\{x\} \mid x \in X\}$, then we can see that $\mathcal{A}$ is closed with respect to H satisfying (3.2) and $\hat{T} : \mathcal{A} \to \mathcal{A}$, defined by $\hat{T}(\{x\}) = \{T(x)\}$, is continuous with respect to the Hausdorff metric H. Therefore we can apply Theorem 3.5.3 to show the existence of fixed sets of T. □

Remark 10. Theorem 3.5.3 does not guarantee an existence fixed set $\bar{A}$ which is a non-singleton set. However by constructing $\mathcal{A}$ which does not include singleton sets, every existence fixed set becomes non-singleton. We give the following examples to explain this remark:

Example 3.5.6. Let $X = [0, 2]^2$ and

$$\mathcal{A} = \{B(x_1, x_2, r) \mid B(x_1, x_2, r) \subset X, (x_1, x_2) \in \mathbb{R}^2, r \geq 0\},$$

where $B(x_1, x_2, r) = \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq r^2\}$. Then $\mathcal{A} \subset C_X$ is closed with respect to the Hausdorff metric H. Hence each continuous self-mapping on $\mathcal{A}$ with respect to H has a fixed set from Theorem 3.5.3. For example, define

$$T(B(x_1, x_2, r)) = B(x_2, x_1, r^2),$$

then we can check that $T : \mathcal{A} \to \mathcal{A}$ is continuous with respect to H and then there exists a fixed set $\bar{A} \in \mathcal{A}$ such that $T(\bar{A}) = \bar{A}$. However, we can not see whether an existence fixed set $\bar{A}$ is a non-singleton set or not. Indeed, $B(1, 1, 1)$ and $B(x, x, 0)$, $0 \leq x \leq 2$, are fixed sets of T. On the other hand, let

$$\mathcal{A}' = \{B(x_1, x_2, r) \mid B(x_1, x_2, r) \subset X, (x_1, x_2) \in \mathbb{R}^2, r > 0\}$$

and assume that a self-mapping T on $\mathcal{A}'$ has a fixed set $\bar{A}$, then $\bar{A}$ should be non-singleton. However $\mathcal{A}'$ is not closed with respect to H and Theorem 3.5.3 can not be applied to the situation.

Example 3.5.7. Let $X = [0,4] \times [0,4] \subset \mathbb{R}^2$ and let

$$\mathcal{A} = \{[a,b] \times [c,d] \mid 0 \le a \le b \le 4, 0 \le c \le d \le 4, (b-a)(d-c) = 1\}.$$

Consider a self-mapping $T : \mathcal{A} \to \mathcal{A}$ defined by

$$T([a,b] \times [c,d]) = [a + t_W(a) - h, b - t_E(b) + h] \times [c + t_S(c) - h, d - t_N(d) + h]$$

where $t_W(a) = 3(4-a)/16$, $t_E(b) = b/8$, $t_S(c) = 5(4-c)/32$, $t_N(d) = 3d/32$, and h is the biggest solution of the following quadratic function:

$$(b - t_E(b) - a - t_W(a) + 2h)(d - t_N(d) - c - t_S(c) + 2h) = 1.$$

Since $\mathcal{A}$ does not include any singleton, every fixed set $\bar{A}$ of T is non-singleton. We can see that the only fixed set is $[4 - k/6, k/4] \times [4 - k/5, k/3]$ where $k = (171 - 3\sqrt{249})/20$. This example shows a model of residence movement against natural threats from north, south, east, and west. The orbit $\{T^n([0,1] \times [0,1])\}$ is given in Figure 3.1.

Example 3.5.8. Let X be a nonempty compact convex subset of a normed space E and for any $\varepsilon > 0$, define

$$\mathcal{A}_\varepsilon = \{A \subset \mathcal{C}_X \mid \text{there exists } x \in X \text{ such that } B(x,\varepsilon) \subset A\},$$

where $B(x,\varepsilon) = \{y \in X \mid \|y - x\| \le \varepsilon\}$. Then we can check that $\mathcal{A}_\varepsilon$ is closed with respect to the Hausdorff metric H and (3.2). If T is a continuous self-mapping on $\mathcal{A}_\varepsilon$, then there exists a fixed set $\bar{A} \in \mathcal{A}_\varepsilon$. Clearly, $\bar{A}$ is a non-singleton set.

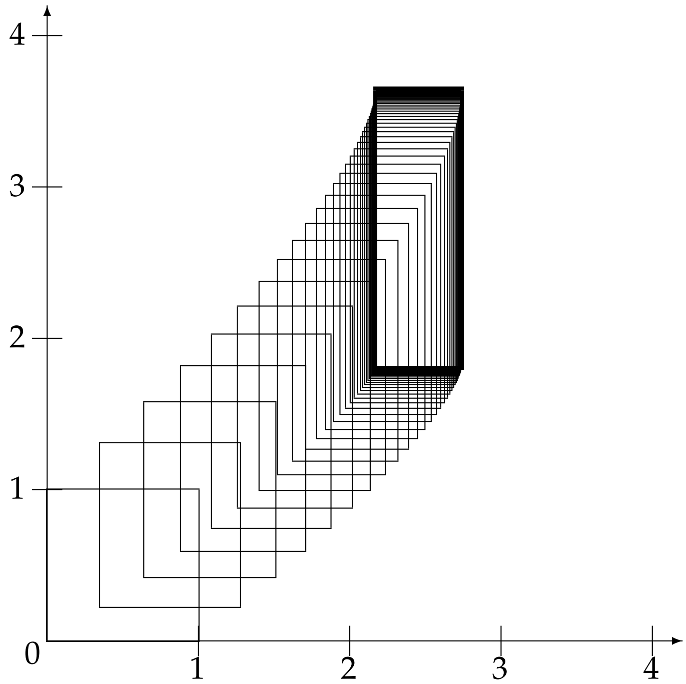

Figure 3.1: The orbit $\{T^n([0,1]\times[0,1])\}$ in Example 3.5.7

Bibliography

[1] Brouwer L. E. J., Über Abbildungen von Mannigfaltigkeiten Mathematische Annalen 1911;71:97–115

[2] Banach S., Sur les operetions dans les ensembles abstraits et leur application aux equations integrales. Fund. Math. 1922;3:133–181

[3] Kakutani S. A generalization of Brouwer's fixed point theorem. Duke Mathematical Journal 1941;8(3):457–459.

[4] Radström H. An embedding theorem for spaces of convex sets, Proceeding American Mathematical Society. 1952;3(1);165–169

[5] Boyd D. W. , Wong J. S. W., On nonlinear contractions. Proc. Amer. Math. Soc. 1969;20:458–464

[6] Luenberger D. G., Optimization by vector space methods. Series in Decision and Control. New York-London-Sydney-Toronto: John Wiley and Sons, Inc.; 1969.

[7] Meir A., Keeler E., A theorem on contraction mappings. J. Math. Anal. Appl. 1969;28:326–329

[8] Nadler S. B. Jr., Multi-valued contraction mappings. Pacific J. Math. 1969;30;475–488.

[9] Wong C. S., Characterizations of certain maps of contractive type. Pacific J. Math. 1972;68(1):26–42

[10] Borwein J. M., Optimization with respect to partial orderings [PhD dissertation]. Oxford University; 1974.

[11] Subrahmanyam P. V., Remarks on some fixed point theorems related to Banach's contraction. J. Math. Phys. Sci. 1974;8:445–457

[12] Robinson S. M., Regularity and stability for convex multivalued functions. Math. Oper. Res. 1976;1:130–143.

[13] Urbański R., A generalization of the Minkowski-Rådström-Hörmander theorem. Bull. Pol. Acad. Sci. Math. 1976;24:709–715.

[14] Borwein J. M., Multivalued convexity and optimization: a unified approach to inequality and equality constraints. Math. Program. 1977;13:183–199.

[15] Crouzeix J-P., Contribution à l'étude des fonctions quasiconvexes [Contribution to the study of quasi-convex functions. PhD dissertation]. University of Clermont-Ferrand II; 1977. French.

[16] Ferro F., Minimax type theorems for n-valued functions. Ann. Mat. Pura Appl. 1982;32:113–130.

[17] Jahn J., Mathematical vector optimization in partially ordered linear spaces. Frankfurt: Peter Lang Verlag; 1986.

[18] Jahn J., Sachs E. W., Generalized quasiconvex mappings and vector optimization. SIAM J. Control Optim. 1986;24(2):306–322.

[19] Luc D.T., Connectedness of the efficient point sets in quasiconcave vector maximization. J. Optim. Theory Appl. 1987;122:346–354.

[20] Nikodem K., On concave and midpoint concave set-valued functions. Glas. Mat. Ser. III. 1987;22(42):69–76.

[21] Luc D. T., Theory of vector optimization. Berlin: Springer; 1989.

[22] Mizoguchi N., Takahashi W., Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Apple. 1989;141;177–188.

[23] Nikodem K., K-convex and K-concave set-valued functions [Habilitation dissertation]. Zeszyty Nauk. Politech. Łódź. Mat. 1989;559(Rozprawy Nauk. 114):1–75.

[24] Luc D. T., Cristobal V., A saddlepoint theorem for set-valued maps. Nonlinear Anal. 1992;18:1–7.

[25] Tanaka T., Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions. J. Optim. Theory Appl. 1994;81(2):355–377.

[26] Kuroiwa D., Convexity for set-valued maps and optimization [PhD dissertation]. Niigata University; 1996.

[27] Kuroiwa D., Convexity for set-valued maps. Appl. Math. Lett. 1996;9:97–101.

[28] Henrikson J., Completeness and total boundedness of the Hausdorff metric, MIT Undergrad. J. Math. 1999;1;69–79.

[29] Takahasi W., Nonlinear Functional Analysis Fixed Point Theory and its Applications, Yokohama Publishers, 2000.

[30] Cauty R., Solution du problème de point fixe de Schauder. Fund. Math. 2001;170:231–246.

[31] Lim T-C., On characterizations of Meir-Keeler contractive maps. Nonlinear Anal. 2001;46:113–120

[32] Benoist J., Popovici N., Characterizations of convex and quasiconvex set-valued maps. Math. Methods Oper. Res. 2003;57:427–435.

[33] Göpfert A., Riahi H., Tammer C., Zălinescu C., Variational methods in partially ordered spaces. New York: Springer; 2003.

[34] Rus I. A., Picard operators and applications. Sci. Math. 2003;58:191–219

[35] Aliprantis C. D., Tourky R., Cones and duality. Graduate Studies in Mathematics, Vol. 84. Providence, RI: American Mathematical Society; 2007.

[36] Gorokhovik V. V., Representations of affine multifunctions by affine selections. Set-Valued Anal. 2008;16:185–198.

[37] Suzuki T., A sufficient and necessary condition for convergence of the successive approximations to a unique fixed point. Proc. Amer. Math. Soc. 2008;136:4089–4093

[38] Suzuki T., Subrahmanyam's fixed point theorem. Nonlinear Anal. 2009;71:1678–1683

[39] Brooks R., Schmitt K., Warner B., Fixed set theorems for discrete dynamics and nonlinear boundary-value problems, Electron. J. Differential Equations 2011 2011;No. 56; 15 pp.

[40] Jahn J., Vector optimization: theory, applications, and extensions. 2nd ed. Berlin Heidelberg: Springer; 2011.

[41] Khan A. A., Tammer C., Zălinescu C., Set-valued optimization: an introduction with applications. Berlin: Springer; 2015.

[42] Kuroiwa D., Popovici N., Rocca M., A characterization of cone-convexity for set-valued functions by cone-quasiconvexity. Set-Valued Var. Anal. 2015;23(2):295–304.

[43] Kuroiwa D., Generalized minimality in set optimization. In: Hamel AH, Heyde F, Löhne A, Rudloff B, Schrage C, editors. Set optimization and applications - the state of the art: From set relations to set-valued risk measures. Springer Proceedings in Mathematics & Statistics, Vol. 151. Springer; 2015. p. 293–311.

[44] Kuroiwa D., Popovici N., Rocca M., A characterization of cone-convex vector-valued functions. Carpathian J. Math. 2016;32(1):79–85.

[45] Fakhar M., Soltani Z., Zafarani J., Fixed set theorems with application to the fixed point theory, J. Nonlinear Convex Anal. 2016;17(3);569–578.

[46] Crespi G. P., Kuroiwa D., Rocca M., Quasiconvexity of set-valued maps assures well-posedness of robust vector optimization. Ann. Oper. Res. 2017;251(1-2):89–104.

[47] Debnath P., Choudhury B. S., Neog M., Fixed set of set valued mappings with set valued domain in terms of start set on a metric space with a graph, Fixed point Theory Appl. 2017;5;8 pp.

[48] Seto K., Kuroiwa D., A convergence theorem of the Picard iteration whose mapping has multiple fixed points, Advances in Fixed Point Theory, 2015;5(4):387–395.

[49] Seto K., Kuroiwa D., Popovici N., A systematization of convexity and quasiconvexity concepts for set-valued maps, defined by l-type and u-type preorder relations, Optimization, Submittied.

[50] Seto K., Kuroiwa D., A fixed set theorem for set-to-set maps, Accepted to Applied Analysis and Opitmization.