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Characterizations of the solution set for non-essentially quasiconvex programming

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Abstract Characterizations of the solution set in terms of subdifferentials play an important role in research of mathematical programming. Previous characterizations are based on necessary and sufficient optimality conditions and invariance properties of subdifferentials. Recently, characterizations of the solution set for essentially quasiconvex programming in terms of Greenberg-Pierskalla subdifferential are studied by the authors. Unfortunately, there are some examples such that these characterizations do not hold for non-essentially quasiconvex programming. As far as we know, characterizations of the solution set for non-essentially quasiconvex programming have not been studied yet.

In this paper, we study characterizations of the solution set in terms of subdifferentials for non-essentially quasiconvex programming. For this purpose, we use Martínez-Legaz subdifferential which is introduced by Martínez-Legaz as a special case of c-subdifferential by Moreau. We derive necessary and sufficient optimality conditions for quasiconvex programming by means of Martínez-Legaz subdifferential, and, as a consequence, investigate characterizations of the solution set in terms of Martínez-Legaz subdifferential. In addition, we compare our results with previous ones. We show an invariance property of Greenberg-Pierskalla subdifferential as a consequence of an invariance property of Martínez-Legaz subdifferential. We give characterizations of the solu-

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tion set for essentially quasiconvex programming in terms of Martínez-Legaz subdifferential.

Keywords quasiconvex programming · solution set · subdifferential · optimality condition

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1 Introduction

In this paper, we study the following mathematical programming problem (P):

$$(P) \begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in F, \end{cases}$$

where f is a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, \infty]$, and F is a convex subset of $\mathbb{R}^n$. Let $N_F(x)$ be the normal cone of F at $x \in F$, and S be the solution set of (P). Characterizations of the solution set in terms of subdifferentials play an important role in research of mathematical programming. In [15], Mangasarian introduces characterizations of the solution set for convex programming in terms of the subdifferential, $\partial f(x)$. Mangasarian's characterizations are based on the optimality condition in terms of the subdifferential, $0 \in \partial f(x) + N_F(x)$, and an invariance property of the subdifferential. Especially, the following characterization is closely related to the optimality condition: let $\bar{x} \in S$, then

$$x \in S \iff \exists v \in \partial f(x) \text{ such that } \langle v, x - \bar{x} \rangle \leq 0.$$

Motivated by Mangasarian's results, many researchers introduce characterizations of the solution set for mathematical programming, see, for example, [2, 9–13, 25, 33–35]. Recently, in [31], characterizations of the solution set for essentially quasiconvex programming in terms of Greenberg-Pierskalla subdifferential, $\partial^{GP} f(x)$, are studied by the authors. Greenberg-Pierskalla subdifferential in [6] plays important roles in quasiconvex analysis and surrogate duality. Characterizations in [31] are based on a necessary and sufficient optimality condition in terms of Greenberg-Pierskalla subdifferential, $0 \in \partial^{GP} f(x) + N_F(x)$, and an invariance property of Greenberg-Pierskalla subdifferential. Especially, the following characterization is closely related to the necessary and sufficient optimality condition: let $\bar{x} \in S$, then

$$x \in S \iff \exists v \in \partial^{GP} f(x) \text{ such that } \langle v, x - \bar{x} \rangle \leq 0.$$

Unfortunately, characterizations in [31] are valid for only essentially quasiconvex programming. There are some examples such that characterizations in [31] do not hold for non-essentially quasiconvex programming. As far as we know, characterizations of the solution set for non-essentially quasiconvex programming have not been studied yet.

In this paper, we study characterizations of the solution set in terms of subdifferentials for non-essentially quasiconvex programming. For this purpose, we use Martínez-Legaz subdifferential which is introduced in [18] as a special case of c-subdifferential by Moreau in [20]. We derive necessary and sufficient optimality conditions for quasiconvex programming by means of Martínez-Legaz subdifferential. Using the optimality condition, we investigate characterizations of the solution set in terms of Martínez-Legaz subdifferential. In addition, we compare our results with previous ones. We show an invariance property of Greenberg-Pierskalla subdifferential as a consequence of an invariance property of Martínez-Legaz subdifferential. We give characterizations of the solution set for essentially quasiconvex programming in terms of Martínez-Legaz subdifferential. Furthermore, we explain a relation between Martínez-Legaz subdifferential and other types of subdifferentials used in the literature.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we study characterizations of the solution set for non-essentially quasiconvex programming in terms of Martínez-Legaz subdifferential. In Section 4, we compare our results with those existing in the literature.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and $\langle v, x\rangle$ denote the inner product of two vectors v and x. For a set $A \subset \mathbb{R}^n$, we denote the relative interior generated by A, by ri$A$. The indicator function δ_A is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ \infty & \textit{otherwise}. \end{cases}$$

Throughout the paper, let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, \infty]$. We define the domain of f by $\mathrm{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$. The epigraph of f is denoted by $\mathrm{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}$. A function f is said to be convex if for each x, $y \in \mathbb{R}^n$, and $\alpha \in (0, 1)$, $f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y)$. The subdifferential of f at x is defined as $\partial f(x) = \{v \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle v, y - x\rangle\}$. Additionally, the normal cone of A at $x \in A$ is defined as $N_A(x) = \{v \in \mathbb{R}^n \mid \forall y \in A, \langle v, y - x\rangle \leqq 0\}$. It is clear that $N_A(x) = \partial \delta_A(x)$. Fenchel conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as $f^*(v) = \sup_{x \in \mathbb{R}^n}\{\langle v, x\rangle - f(x)\}$. A function f is said to be quasiconvex if for each x, $y \in \mathbb{R}^n$ and $\alpha \in (0, 1)$, $f((1-\alpha)x + \alpha y) \leq \max\{f(x), f(y)\}$. Define level sets of f as

$$L(f, \leq, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\},$$
$$L(f, <, \alpha) = \{x \in \mathbb{R}^n \mid f(x) < \alpha\}$$

for any $\alpha \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\alpha \in \mathbb{R}$, $L(f, \leq, \alpha)$ is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$, $L(f, <, \alpha)$ is a convex set. Any convex function is quasiconvex, but the converse is not true in general. A quasiconvex function f is said to be essentially quasiconvex if each local minimizer $x \in \mathbb{R}^n$ of f in $\mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$. It is clear that any convex functions are essentially quasiconvex. We can easily see that there exists a function such that it is quasiconvex and not essentially quasiconvex. Some types of characterizations of essentially quasiconvexity have been introduced, see [1,3,9,10,31] for more details.

In quasiconvex analysis, various types of subdifferentials have been investigated, see [4–7,14,16–24,26–32]. Especially, in [6], Greenberg and Pierskalla introduce Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) = \{v \in \mathbb{R}^n \mid \langle v, x\rangle \geq \langle v, x_0\rangle \text{ implies } f(x) \geq f(x_0)\}.$$

Greenberg-Pierskalla subdifferential is closely related to surrogate duality, and plays an important role in inexact subgradient methods, see [6,8,18].

Throughout the paper, we study the following quasiconvex programming problem (P):

$$(P)\begin{cases}\text{minimize } f(x),\\ \text{subject to } x \in F,\end{cases}$$

where f is a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and F is a convex subset of $\mathbb{R}^n$. Let S be the solution set of (P), that is, $S = \{x \in F \mid f(x) = \min_{y\in F} f(y)\}$.

In [31], we introduce the following necessary and sufficient optimality condition for essentially quasiconvex programming.

Theorem 2.1 *[31] Let f be an upper semicontinuous (usc) essentially quasiconvex function, F a convex subset of $\mathbb{R}^n$, and $x \in F$. Then, the following statements are equivalent:*

(i) $f(x) = \min\limits_{y\in F} f(y)$,
(ii) $0 \in \partial^{GP} f(x) + N_F(x)$.

In [31], we study the following characterizations of the solution set for essentially quasiconvex programming.

Theorem 2.2 *[31] Let f be an usc essentially quasiconvex function, F a convex subset of $\mathbb{R}^n$, $\bar{x} \in S = \{x \in F \mid f(x) = \min_{y\in F} f(y)\}$ and $x_0 \in \mathrm{ri}S$. Then the following statements hold:*

(i) $\partial^{GP} f(\bar{x}) \supset \partial^{GP} f(x_0)$,
(ii) $\partial^{GP} f(x)$ *is constant on* $x \in \mathrm{ri}S$,
(iii) the following sets are equal:
  (a) $S = \{x \in F \mid f(x) = \min_{y\in F} f(y)\}$,
  (b) $S_1 = \{x \in F \mid \exists v \in \partial^{GP} f(\bar{x}) \cap \partial^{GP} f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\}$,
  (c) $S_2 = \{x \in F \mid \exists v \in \partial^{GP} f(\bar{x}) \cap \partial^{GP} f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle \leq 0\}$,
  (d) $S_3 = \{x \in F \mid \partial^{GP} f(x_0) \subset \partial^{GP} f(x), \exists v \in \partial^{GP} f(x_0) \text{ s.t. } \langle v, x - x_0\rangle = 0\}$,
  (e) $S_4 = \{x \in F \mid \partial^{GP} f(x_0) \subset \partial^{GP} f(x), \exists v \in \partial^{GP} f(x_0) \text{ s.t. } \langle v, x - x_0\rangle \leq 0\}$,
  (f) $S_5 = \{x \in F \mid \exists v \in \partial^{GP} f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\}$,
  (g) $S_6 = \{x \in F \mid \exists v \in \partial^{GP} f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle \leq 0\}$.

Unfortunately, characterizations in Theorem 2.2 are not always valid for non-essentially quasiconvex programming, see the following example.

Example 2.1 Let $F = [1, 3]$, and f a real-valued function on $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & x \in (-\infty, 1], \\ 1 & x \in [1, 2], \\ x - 1 & x \in [2, \infty). \end{cases}$$

Then, f is usc quasiconvex, not essentially quasiconvex, and $S = [1, 2]$. Let $\bar{x} \in S$ and $x_0 \in \mathrm{ri}S$. Then, we can check that the following equations hold:

$$\begin{aligned} \partial^{GP} f(x) &= (0, \infty) \quad \forall x \in F, \\ S_1 &= S_5 = \{\bar{x}\}, \\ S_2 &= S_6 = [1, \bar{x}], \\ S_3 &= \{x_0\}, \\ S_4 &= [1, x_0]. \end{aligned}$$

Hence, characterizations in Theorem 2.2 do not hold.

3 Characterizations of solution set for non-essentially quasiconvex programming

In this section, we study characterizations of the solution set for non-essentially quasiconvex programming. For the purpose, we use the following subdifferential. Martínez-Legaz subdifferential of f at $x \in \mathbb{R}^n$ is defined as follows:

$$\partial^M f(x) := \{(v, t) \in \mathbb{R}^{n+1} \mid \inf\{f(y) \mid \langle v, y \rangle \geq t\} \geq f(x), \langle v, x \rangle \geq t\}.$$

Martínez-Legaz subdifferential is introduced by Martínez-Legaz [18] as a special case of c-subdifferential in Moreau's generalized conjugation [20], in detail, see Section 4.

In the following theorem, we show a necessary and sufficient optimality condition for quasiconvex programming problem with a convex set constraint.

Theorem 3.1 *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, F a convex subset of $\mathbb{R}^n$, and $x \in F$. Then, the following statements are equivalent:*

(i) $f(x) = \min_{y \in F} f(y)$,

(ii) $0 \in \partial^M f(x) + \mathrm{epi}\delta_F^*$.

Proof Assume that $f(x) = \min_{y \in F} f(y)$. If $f(x) = \min_{y \in \mathbb{R}^n} f(y)$, then $(0, 0) \in \partial^M f(x)$ since $\langle 0, x \rangle \geq 0$ and

$$\inf\{f(y) \mid \langle 0, y \rangle \geq 0\} = \inf_{y \in \mathbb{R}^n} f(y) = f(x).$$

Clearly, $(0, 0) \in \mathrm{epi}\delta_F^*$, hence (ii) holds. Assume that $f(x) = \min_{y \in F} f(y) > \inf_{y \in \mathbb{R}^n} f(y)$. By usc quasiconvexity of f, $L(f, <, f(x))$ is a nonempty, open and convex set. Since x is a minimizer of f in F, $F \cap L(f, <, f(x))$ is empty. By the separation theorem, there exist $v \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$ such that for each $z \in F$ and $y \in L(f, <, f(x))$,

$$\langle v, z \rangle \geq t > \langle v, y \rangle .$$

Since $x \in F$, $\langle v, x \rangle \geq t$. By using the above separation inequality, for all $y \in \mathbb{R}^n$,

$$f(y) < f(x) \Longrightarrow \langle v, y \rangle < t.$$

Hence,

$$\langle v, x \rangle \geq t \Longrightarrow f(y) \geq f(x),$$

that is,

$$\inf\{f(y) \mid \langle v, y \rangle \geq t\} \geq f(x).$$

This means that $(v, t) \in \partial^M f(x)$. Also, for each $z \in F$, $\langle -v, z \rangle \leq -t$, that is, $(-v, -t) \in \mathrm{epi}\delta_F^*$. This shows that (ii) holds.

Conversely, assume that $0 \in \partial^M f(x) + \mathrm{epi}\delta_F^*$. Then, there exist $(v, t) \in \partial^M f(x)$ such that $-(v, t) \in \mathrm{epi}\delta_F^*$. By the definition of Martínez-Legaz subdifferential,

$$\inf\{f(y) \mid \langle v, y \rangle \geq t\} \geq f(x).$$

Since $-(v, t) \in \mathrm{epi}\delta_F^*$, $\langle -v, y \rangle \leq -t$ for each $y \in F$. Hence

$$F \subset \{y \mid \langle v, y \rangle \geq t\}.$$

This shows that

$$\inf_{y \in F} f(y) \geq \inf\{f(y) \mid \langle v, y \rangle \geq t\} \geq f(x),$$

that is, (i) holds. This completes the proof.

Next, we show characterizations of the solution set for quasiconvex programming in terms of Martínez-Legaz subdifferential.

Theorem 3.2 *Let f be an usc quasiconvex function, F a nonempty convex subset of $\mathbb{R}^n$, and $\bar{x} \in S = \{x \in F \mid f(x) = \min_{y \in F} f(y)\}$. Then, the following sets are equal:*

(i) $S = \{x \in F \mid f(x) = \min_{y \in F} f(y)\}$,

(ii) $S_2' = \{x \in F \mid \partial^M f(\bar{x}) \cap \partial^M f(x) \neq \emptyset\}$,

(iii) $S_6' = \{x \in F \mid \exists (v, t) \in \partial^M f(x) \text{ s.t. } \langle v, \bar{x} \rangle \geq t\}$.

Proof Let $x \in S$. By Theorem 3.1, $0 \in \partial^M f(x) + \mathrm{epi}\delta_F^*$. Hence, there exists $(v, t) \in \partial^M f(x)$ such that $(v, t) \in -\mathrm{epi}\delta_F^*$. Since $\bar{x} \in S \subset F$ and $(v, t) \in -\mathrm{epi}\delta_F^*$, $\langle v, \bar{x} \rangle \geq t$. By the definition of Martínez-Legaz subdifferential,

$$\inf\{f(y) \mid \langle v, y \rangle \geq t\} \geq f(x) = f(\bar{x}).$$

This shows that $(v, t) \in \partial^M f(\bar{x})$, that is, $x \in S_2'$.

Let $x \in S_2'$, then there exists $(v,t) \in \partial^M f(\bar{x}) \cap \partial^M f(x)$. Since $(v,t) \in \partial^M f(\bar{x})$, $\langle v, \bar{x} \rangle \geq t$, that is, $x \in S_6'$.

Let $x \in S_6'$, then there exists $(v,t) \in \partial^M f(x)$ such that $\langle v, \bar{x} \rangle \geq t$. Hence

$$\min_{y \in F} f(y) = f(\bar{x}) \geq \inf\{f(y) \mid \langle v, y \rangle \geq t\} \geq f(x).$$

This shows that $x \in S$ and completes the proof.

Remark 3.1 S_2' and S_6' in Theorem 3.2 are similar to S_2 and S_6 in Theorem 2.2 since we can easily see that

$$S_2' = \{x \in F \mid \exists (v,t) \in \partial^M f(\bar{x}) \cap \partial^M f(x) \text{ s.t. } \langle v, \bar{x} \rangle \geq t\}$$

by the definition of Martínez-Legaz subdifferential.

Characterizations in Theorem 2.2 are not always valid for non-essentially quasiconvex programming. On the other hand, in Theorem 3.2, we show characterizations of the solution set for non-essentially quasiconvex programming, see the following example.

Example 3.1 Let $F = [1,3]$, and f a real-valued function on $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & x \in (-\infty, 1], \\ 1 & x \in [1,2], \\ x-1 & x \in [2, \infty). \end{cases}$$

F and f are the same in Example 2.1. Hence f is usc quasiconvex, not essentially quasiconvex, and $S = [1,2]$.

We can check that for each $x \in F$,

$$\partial^M f(x) = \begin{cases} \{(v,t) \in \mathbb{R}^2 \mid v > 0, vx \geq t \geq v\}, & x \in [1,2], \\ \{(v, vx) \in \mathbb{R}^2 \mid v > 0\}. & x \in (2,3]. \end{cases} \tag{1}$$

Actually, let $x \in [1,2]$ and $(v_0, t_0) \in \partial^M f(x)$, then $v_0 x \geq t_0$ and

$$\inf\{f(y) \mid v_0 y \geq t_0\} \geq f(x) = 1.$$

We show that $v_0 > 0$. If $v_0 < 0$, then $\inf\{f(y) \mid v_0 y \geq t_0\} = -\infty$. If $v_0 = 0$ and $t_0 \leq 0$ then $\inf\{f(y) \mid v_0 y \geq t_0\} = -\infty$. Also if $v_0 = 0$ and $t_0 > 0$, $0 = v_0 x \geq t_0 > 0$. Hence $v_0 > 0$. It is clear that $v_0 x \geq v_0$ since $x \geq 1$. If $t_0 < v_0$, then $\frac{t_0}{v_0} < 1$, $v_0 \frac{t_0}{v_0} = t_0$, and

$$\inf\{f(y) \mid v_0 y \geq t_0\} \leq f\left(\frac{t_0}{v_0}\right) < 1 = f(x).$$

This is a contradiction. Hence $v_0 x \geq t_0 \geq v_0$, that is,

$$(v_0, t_0) \in \{(v,t) \in \mathbb{R}^2 \mid v > 0, vx \geq t \geq v\}.$$

Conversely, let and $(v_0, t_0) \in \mathbb{R}^2$ with $v_0 > 0$, and $v_0 x \geq t \geq v_0$, then

$$\begin{aligned}\inf\{f(y) \mid v_0 y \geq t_0\} &\geq \inf\{f(y) \mid v_0 y \geq v_0\}\\ &= \inf\{f(y) \mid y \geq 1\}\\ &= f(x).\end{aligned}$$

This shows that $(v_0, t_0) \in \partial^M f(x)$. The other inclusions are similar and omitted.

Let $\bar{x} = 1 \in S$ and $x \in F$. If $x \in (2, 3]$, then $x \notin S'_2$. Actually, if $x \in S'_2$, then there exists $(v, t) \in \partial^M f(\bar{x}) \cap \partial^M f(x)$. By Equation (1), $v\bar{x} \geq t \geq v$ and $t = vx$. Since $v > 0$, $\bar{x} = 1 \geq x$. This is a contradiction. Hence by Theorem 3.2, $x \notin S$. Of course, we can check that $f(x) > 1 = \inf_{y \in F} f(y)$. If $x \in [1, 2]$, then $(1, 1) \in \partial^M f(x)$. Since $(1, 1) \in \partial^M f(\bar{x})$, $(1, 1) \in \partial^M f(\bar{x}) \cap \partial^M f(x)$, that is, $x \in S'_2$. Hence by Theorem 3.2, $x \in S$. Actually, $f(x) = 1 = \min_{y \in F} f(y)$.

Similarly, we can check whether $x \in S$ or not by using a characterization $S = S'_6$.

4 Comparisons

In this section, we compare our results with previous ones. We show an invariance property of Martínez-Legaz subdifferential and prove an invariance property of Greenberg-Pierskalla subdifferential as a corollary. We give characterizations of the solution set for essentially quasiconvex programming in terms of Martínez-Legaz subdifferential. Furthermore, we explain a relation between Martínez-Legaz subdifferential and previous subdifferentials.

As seen in Theorem 2.2 (ii), Greenberg-Pierskalla subdifferential are constant on $\mathrm{ri}S$. Characterizations of the solution set, Theorem 2.2 (iii), are consequences of this invariance property. Motivated by this results, we study invariance properties of Martínez-Legaz subdifferential for essentially quasiconvex functions.

Theorem 4.1 *Let f be an usc essentially quasiconvex function, F a convex subset of $\mathbb{R}^n$, $x \in S = \{z \in F \mid f(z) = \min_{y \in F} f(y)\}$, and $x_0 \in \mathrm{ri}S$. Assume that $\inf_{y \in F} f(y) > \inf_{y \in \mathbb{R}^n} f(y)$. Then, the following statements hold:*

(i) $\partial^M f(x_0) \subset -\mathrm{epi}\delta^*_S$,
(ii) $\partial^M f(x_0) \subset \partial^M f(x)$.

Proof (i) Let $(v, t) \in \partial^M f(x_0)$. By the definition of Martínez-Legaz subdifferential, $\langle v, x_0 \rangle \geq t$. Assume that there exists $y \in S$ such that $\langle v, y \rangle < t$. Since $x_0 \in \mathrm{ri}S$, $z = x_0 + \varepsilon(x_0 - y) \in S$ for sufficiently small $\varepsilon > 0$. Then,

$$\begin{aligned}\langle v, z \rangle &= (1 + \varepsilon)\langle v, x_0 \rangle - \varepsilon \langle v, y \rangle\\ &= \langle v, x_0 \rangle + \varepsilon(\langle v, x_0 \rangle - \langle v, y \rangle)\\ &> \langle v, x_0 \rangle\\ &\geq t.\end{aligned}$$

Since $\inf_{y\in F} f(y) > \inf_{y\in\mathbb{R}^n} f(y)$, z is not a global minimizer of f in $\mathbb{R}^n$. By essential quasiconvexity of f, there exists $\bar{z} \in L(f, <, f(z))$ such that $\langle v, \bar{z}\rangle > \langle v, x_0\rangle \geq t$. However, since $(v,t) \in \partial^M f(x_0)$,

$$f(x_0) = f(z) > f(\bar{z}) \geq \inf\{f(y) \mid \langle v, y\rangle \geq t\} \geq f(x_0).$$

This is a contradiction. Hence, $(v,t) \in -\mathrm{epi}\delta_S^*$.

(ii) Let $(v,t) \in \partial^M f(x_0)$. By the statement (i), $(v,t) \in \partial^M f(x_0) \subset -\mathrm{epi}\delta_S^*$. Since $(v,t) \in -\mathrm{epi}\delta_S^*$, $\langle v, x\rangle \geq t$. This shows that $(v,t) \in \partial^M f(x)$ since

$$\inf\{f(y) \mid \langle v, y\rangle \geq t\} \geq f(x_0) = f(x).$$

This completes the proof.

In the following theorem, we show an invariance property of Martínez-Legaz subdifferential on $\mathrm{ri}S$.

Theorem 4.2 *Let f be an usc essentially quasiconvex, F a convex subset of $\mathbb{R}^n$, and $S = \{x \in F \mid f(x) = \min_{y\in F} f(y)\}$. Assume that $\inf_{y\in F} f(y) > \inf_{y\in\mathbb{R}^n} f(y)$. Then, $\partial^M f$ is constant on $\mathrm{ri}S$.*

Proof Let x, $y \in \mathrm{ri}S$. By Theorem 4.1, $\partial^M f(x) \subset \partial^M f(y)$, and $\partial^M f(x) \supset \partial^M f(y)$. This completes the proof.

Remark 4.1 In Theorem 4.2, essential quasiconvexity is necessary, see the following function in Example 3.1:

$$f(x) = \begin{cases} x & x \in (-\infty, 1], \\ 1 & x \in [1,2], \\ x-1 & x \in [2, \infty). \end{cases}$$

We can check that f is usc quasiconvex, not essentially quasiconvex, and for each $x \in F = [1,3]$,

$$\partial^M f(x) = \begin{cases} \{(v,t) \in \mathbb{R}^2 \mid v > 0, vx \geq t \geq v\}, & x \in [1,2], \\ \{(v, vx) \in \mathbb{R}^2 \mid v > 0\}. & x \in (2,3]. \end{cases}$$

Clearly, $\partial^M f$ is not constant on $\mathrm{ri}S = (1,2)$.

Also, the assumption "$\inf_{y\in F} f(y) > \inf_{y\in\mathbb{R}^n} f(y)$" is important in Theorem 4.2. Actually, let $f(x) \equiv 0$ and $F = [0,1]$. Then f is an usc essentially quasiconvex function on $\mathbb{R}$, $\inf_{y\in F} f(y) = \inf_{y\in\mathbb{R}^n} f(y)$, $S = F$, and

$$\partial^M f(x) = \{(v,t) \mid v \in \mathbb{R}^n, \langle v, x\rangle \geq t\}.$$

Let x, $y \in \mathrm{ri}S$ satisfying $x \neq y$. Then, $\langle x-y, x\rangle > \langle x-y, y\rangle$. Hence

$$(x-y, \langle x-y, x\rangle) \in \partial^M f(x), \text{ and } (x-y, \langle x-y, x\rangle) \notin \partial^M f(y).$$

This shows that Martínez-Legaz subdifferential of f is not constant on $\mathrm{ri}S$.

We show a relation between Martínez-Legaz subdifferential and Greenberg-Pierskalla subdifferential in the following theorem.

Theorem 4.3 *Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and $x_0 \in \mathbb{R}^n$. Then,*

$$\partial^{GP} f(x_0) = \{v \in \mathbb{R}^n \mid (v, \langle v, x_0\rangle) \in \partial^M f(x_0)\}.$$

Proof Let $v \in \partial^{GP} f(x_0)$. Then

$$\langle v, x\rangle \geq \langle v, x_0\rangle \text{ implies } f(x) \geq f(x_0).$$

Hence

$$\inf\{f(x) \mid \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0),$$

that is, $(v, \langle v, x_0\rangle) \in \partial^M f(x_0)$.

Conversely, let $v \in \mathbb{R}^n$ with $(v, \langle v, x_0\rangle) \in \partial^M f(x_0)$. By the definition of Martínez-Legaz subdifferential,

$$\inf\{f(x) \mid \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0),$$

that is, $v \in \partial^{GP} f(x_0)$.

We show an invariance property of Greenberg-Pierskalla subdifferential as a consequence of our results.

Corollary 4.1 *Let f be an usc essentially quasiconvex, F a convex subset of $\mathbb{R}^n$, and $S = \{x \in F \mid f(x) = \min_{y\in F} f(y)\}$. Then, $\partial^{GP} f$ is constant on* $\mathrm{ri}S$.

Proof If $\inf_{y\in F} f(y) = \inf_{y\in\mathbb{R}^n} f(y)$, then we can prove easily that $\partial^{GP} f(x) = \mathbb{R}^n$ for each $x \in S$.

Assume that $\inf_{y\in F} f(y) > \inf_{y\in\mathbb{R}^n} f(y)$. Let $x, y \in \mathrm{ri}S$, and $v_0 \in \partial^{GP} f(x)$. By Theorem 4.2 and Theorem 4.3,

$$v_0 \in \partial^{GP} f(x) = \{v \mid (v, \langle v, x\rangle) \in \partial^M f(x)\} = \{v \mid (v, \langle v, x\rangle) \in \partial^M f(y)\}.$$

Hence $\langle v_0, y\rangle \geq \langle v_0, x\rangle$, and

$$\inf\{f(z) \mid \langle v_0, z\rangle \geq \langle v_0, y\rangle\} \geq \inf\{f(z) \mid \langle v_0, z\rangle \geq \langle v_0, x\rangle\} \geq f(y).$$

This shows that $v_0 \in \partial^{GP} f(y)$. Similarly, we can show that $\partial^{GP} f(y) \subset \partial^{GP} f(x)$. This completes the proof.

For essentially quasiconvex programming, we show characterizations of the solution set in terms of Martínez-Legaz subdifferential as corollaries of Theorem 2.2 and our results in this paper.

Corollary 4.2 *Let f be an usc essentially quasiconvex function, F a nonempty convex subset of $\mathbb{R}^n$, $\bar{x} \in S = \{x \in F \mid f(x) = \min_{y\in F} f(y)\}$, and $x_0 \in \mathrm{ri}S$. Assume that $\inf_{y\in F} f(y) > \inf_{y\in\mathbb{R}^n} f(y)$. Then the following sets are equal to S:*

(i) $S_1' = \{x \in F \mid \exists (v, t) \in \partial^M f(\bar{x}) \cap \partial^M f(x) \text{ s.t. } \langle v, \bar{x}\rangle = t\}$,
(ii) $S_3' = \{x \in F \mid \partial^M f(x_0) \subset \partial^M f(x), \exists (v, t) \in \partial^M f(x_0) \text{ s.t. } \langle v, x_0\rangle = t\}$,
(iii) $S_4' = \{x \in F \mid \emptyset \neq \partial^M f(x_0) \subset \partial^M f(x)\}$,
(iv) $S_5' = \{x \in F \mid \exists (v, t) \in \partial^M f(x) \text{ s.t. } \langle v, \bar{x}\rangle = t\}$.

Proof By Theorem 3.2, we can show that

$$S'_1 \subset S'_5 \subset S'_6 = S, \text{ and } S'_3 \subset S'_4.$$

Hence we only show that $S \subset S'_1$ and $S'_4 \subset S \subset S'_3$.

Let $x \in S$. By Theorem 2.2, $x \in S_1$, that is, there exists $v \in \partial^{GP} f(\bar{x}) \cap \partial^{GP} f(x)$ s.t. $\langle v, x - \bar{x} \rangle = 0$. By Theorem 4.3, $(v, \langle v, x \rangle) \in \partial^M f(x) \cap \partial^M f(\bar{x})$ since $\langle v, x \rangle = \langle v, \bar{x} \rangle$. This shows that $x \in S'_1$.

Let $x \in S'_4$, then there exists $(v, t) \in \partial^M f(x_0) \subset \partial^M f(x)$. Since $\langle v, x_0 \rangle \geq t$,

$$f(x_0) \geq \inf\{f(y) \mid \langle v, y \rangle \geq t\} \geq f(x),$$

that is, $x \in S$.

Let $x \in S$. Then by Theorem 2.2 and Theorem 4.1, $x \in S_3$ and $\partial^M f(x_0) \subset \partial^M f(x)$. Hence there exists $v \in \partial^{GP} f(x_0)$ such that $\langle v, x - x_0 \rangle = 0$. By Theorem 4.3, $(v, \langle v, x_0 \rangle) \in \partial^M f(x_0)$, that is, $x \in S'_3$. This completes the proof.

Remark 4.2 S'_1, S'_3, S'_4, and S'_5 in Corollary 4.2 are similar to S_1, S_3, S_4, and S_5 in Theorem 2.2 since we can easily see that

$$S'_4 = \{x \in F \mid \partial^M f(x_0) \subset \partial^M f(x), \exists (v, t) \in \partial^M f(x_0) \text{ s.t. } \langle v, x_0 \rangle \geq t\}.$$

For non-essential quasiconvex programming, characterizations in Corollary 4.2 do not always hold. Actually, let $F = [1, 3]$, f be the function in Example 3.1, $\bar{x} = 2 \in S = [1, 2]$, then we can check that

$$S'_1 = S'_3 = S'_4 = S'_5 = \{\bar{x}\}.$$

On the other hand, let $\bar{x} = 1 \in S$, then we can check that $S'_1 = S'_3 = S'_4 = S'_5 = S$.

In Corollary 4.2, we need the assumption "$\inf_{y \in F} f(y) > \inf_{y \in \mathbb{R}^n} f(y)$". On the other hand, in Theorem 2.2, we characterize the solution set without the assumption. Hence, for essentially quasiconvex programming, characterizations in terms of Greenberg-Pierskalla subdifferential is more suitable than characterizations in terms of Martínez-Legaz subdifferential. Of course, for non-essentially quasiconvex programming, characterizations in terms of Martínez-Legaz subdifferential is useful, and characterizations in terms of Greenberg-Pierskalla subdifferential do not always hold.

Finally, we explain a relation between Martínez-Legaz subdifferential and previous subdifferentials. In many cases, conjugates and subdifferentials concern special cases of Moreau's generalized conjugation in [20]. In [18], Martínez-Legaz summarizes the quasiconvex conjugate duality by using the notion of Moreau's generalized conjugation. We can regard Martínez-Legaz subdifferential as a special case of Moreau's generalized subdifferential. We recall the essentials of Moreau's conjugation theory in [18,20].

Let X and Y be arbitrary sets and c a function from $X \times Y$ to $\overline{\mathbb{R}}$. For a function f from X to $\overline{\mathbb{R}}$, c-conjugate of f, denoted by f^c, is a function from Y to $\overline{\mathbb{R}}$ as follows:

$$f^c(y) = \sup_{x \in X} \{c(x, y) \dot{+} - f(x)\},$$

where $\dot{+}$ is a natural extension to $\overline{\mathbb{R}}$ of the ordinary addition, such that

$$\infty\dot{+}(-\infty) = (-\infty)\dot{+}\infty = -\infty.$$

Then, $y_0 \in Y$ is said to be a c-subgradient of f at $x_0 \in \mathrm{dom} f$ if $c(x_0, y_0) \in \mathbb{R}$ and for all $x \in X$,

$$f(x) - f(x_0) \geq c(x, y_0) - c(x_0, y_0).$$

The set of all c-subgradients of f at x_0 is called c-subdifferential of f at x_0, denoted by $\partial_c f(x_0)$. If c is the inner product in the n-dimensional Euclidean space $\mathbb{R}^n$, then $f^c = f^*$ and $\partial_c f(x) = \partial f(x)$ for each $x \in \mathbb{R}^n$. In this sense, Fenchel conjugate and the subdifferential are special cases of Moreau's generalized conjugation.

Let $\bar{c}$ be a function from $\mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R})$ to $\overline{\mathbb{R}}$ as follows:

$$\bar{c}(x, (v, t)) = \begin{cases} 0 & \langle v, x \rangle \geq t, \\ -\infty & \textit{otherwise}. \end{cases}$$

Then,

$$f^{\bar{c}}(v, t) = -\inf\{f(x) \mid \langle v, x \rangle \geq t\}.$$

In [18], Martínez-Legaz investigates this conjugate by using the notion of H-duality and Moreau's generalized conjugation. By Proposition 4.3 in [18],

$$\partial_{\bar{c}} f(x_0) = \left\{ (v, t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; v \in \partial^{GP} f(x_0), t \in \bigcap_{x \in L(f, <, f(x_0))} (\langle v, x \rangle, \langle v, x_0 \rangle] \right\}.$$

We can check easily that $\partial_{\bar{c}} f(x_0) = \partial^M f(x_0)$. Hence, Martínez-Legaz subdifferential is a special case of c-subdifferential, and satisfies properties of c-subdifferential in [18, 20].

5 Conclusion

In this paper, we study characterizations of the solution set for non-essentially quasiconvex programming. We show a necessary and sufficient optimality condition for quasiconvex programming by Martínez-Legaz subdifferential. As a consequence, we investigate characterizations of the solution set in terms of Martínez-Legaz subdifferential. Also, we compare our results with previous ones. We show an invariance property of Martínez-Legaz subdifferential and prove an invariance property of Greenberg-Pierskalla subdifferential as a consequence of our results. We give characterizations of the solution set for essentially quasiconvex programming in terms of Martínez-Legaz subdifferential. Furthermore, we explain a relation between Martínez-Legaz subdifferential and previous subdifferentials, especially c-subdifferential.

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