BIHARMONIC HYPERSURFACES WITH BOUNDED MEAN CURVATURE

SHUN MAETA

ABSTRACT. We consider a complete biharmonic hypersurface with nowhere zero mean curvature vector field $\phi : (M^m, g) \to (S^{m+1}, h)$ in a sphere. If the squared norm of the second fundamental form B is bounded from above by m, and $\int_M H^{-p} dv_g < \infty$, for some $0 < p < \infty$, then the mean curvature is constant.

1. INTRODUCTION

The problem of biharmonic maps was suggested in 1964 by J. Eells and J. H. Sampson (cf. [6]). Biharmonic maps are generalizations of harmonic maps. As well known, harmonic maps have been applied into various fields in differential geometry. However there are non-existence results for harmonic maps. Therefore a generalization of harmonic maps is an important subject.

G. Y. Jiang [8] considered a biharmonic submanifold, and gave some examples of non-minimal biharmonic submanifolds in S^n as follows: (i) $S^{n-1}(\frac{1}{\sqrt{2}})$ and (ii) $S^{n-p}(\frac{1}{\sqrt{2}}) \times S^{p-1}(\frac{1}{\sqrt{2}})$, $(n-p \neq p-1)$.

There are many studies of biharmonic submanifolds in spheres. Interestingly, their studies suggest the following BMO conjecture which was introduced by Balmus, Montaldo and Oniciuc (cf. [2]).

Conjecture 1 (BMO conjecture)**.** *Any biharmonic submanifold in spheres has constant mean curvature.*

2010 *Mathematics Subject Classification.* primary 53C43, secondary 58E20, 53C40.

Key words and phrases. biharmonic hypersurfaces, constant mean curvature hypersurfaces, spheres.

Supported in part by the Grant-in-Aid for Young Scientists(B), No.15K17542, Japan Society for the Promotion of Science.

On the other hand, since there is no assumption of *completeness* for submanifolds in BMO conjecture, in a sense it is a problem in *local* differential geometry. The author reformulated BMO conjecture into a problem in *global* differential geometry (cf. [13]).

Conjecture 2. *Any* complete *biharmonic submanifold in spheres has constant mean curvature.*

Remark 1.1. *Interestingly, Z.-P. Wang and Y.-L. Ou treated a biharmonic Riemannian submersion from a sphere and got non-existence results* (*cf.* [16])*.*

There are affirmative partial answers to BMO conjecture, if M is one of the following:

(i) A compact hypersurface with nowhere zero mean curvature vector field and $|B|^2 \geq m$ or $|B|^2 \leq m$, where $|B|^2$ is the squared norm of the second fundamental form (cf. [4], [1]).

(ii) An orientable Dupin hypersurface (cf. [1]).

(iii) A compact submanifold with $|\mathbf{H}| \geq 1$ (cf. [3], see also [13]).

(iv) A complete submanifold with $|\mathbf{H}| \geq 1$ and the Ricci curvature of M is bounded from below (cf. [13]).

In [13], the author showed the following.

Theorem 1.2 ([13])**.** *Let $\phi : (M^m, g) \to (S^{m+1}, h)$ be a complete biharmonic hypersurface in a sphere. If the mean curvature $H \geq 1$, and*

$$\int_M \left(H^2 - 1\right)^p dv_g < \infty,$$

for some $0 < p < \infty$, then H is 1.

Here we remark that the author obtained some affirmative partial answers to BMO conjecture under more general situation. Since we gave an affirmative partial answer to BMO conjecture under the assumption $H \geq 1$ in Theorem 1.2, in this paper, we consider $0 < H \leq 1$.

Before proving our main theorem, we show the following theorem.

Theorem 1.3. *Let $\phi : (M^m, g) \to (N^{m+1}, h)$ be a complete non-positive biminimal hypersurface. Assume that the mean curvature H satisfies $0 < H \leq 1$. We also assume that $|B|^2 \leq \mathrm{Ric}^N(\xi, \xi)$, where B is the second fundamental form of M in N, Ric^N is the Ricci curvature of N, and ξ is the unit normal vector field on M. If*

$$\int_M H^{-p} dv_g < \infty,$$

for some $0 < p < \infty$, then H is constant.

Remark 1.4. *If we assume $\int_M H^{-p} dv_g < \infty$ and $\int_M H^{-(p+\varepsilon)} dv_g < \infty$, for some $\varepsilon > 0$ and $0 < p < \infty$, then we don't need $H \leq 1$.*

By applying Theorem 1.3, we can show our main theorem:

Theorem 1.5. *Let $\phi : (M^m, g) \to (S^{m+1}, h)$ be a complete biharmonic hypersurface with nowhere zero mean curvature vector field in a sphere. If $|B|^2 \leq m$, and*

$$\int_M H^{-p} dv_g < \infty,$$

for some $0 < p < \infty$, then H is constant.

Theorem 1.5 is an affirmative partial answer to BMO conjecture.

In this paper, we assume that the mean curvature vector field is nowhere zero. The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. In section 3, we prove Theorem 1.3 and Theorem 1.5.

2. Preliminaries

In this section, we shall give the definitions of biharmonic hypersurfaces and biminimal hypersurfaces.

The problem of biharmonic maps was suggested in 1964 by J. Eells and J. H. Sampson (cf. [6], [5]). Biharmonic maps are critical points of the bi-energy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g,$$

on the space of smooth maps $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds (M^m, g) and (N^n, h). ∇ and ∇^N denote the Levi-Civita connections on (M, g) and (N, h), respectively. $\overline{\nabla}$ denotes the induced connection on $\phi^{-1}TN$. In 1986, G. Y. Jiang [8] derived the first and the second variational formulas of the bi-energy and studied biharmonic maps. The Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) = -\Delta^\phi \tau(\phi) - \sum_{i=1}^m R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) = 0, \tag{1}$$

where $\{e_i\}_{i=1}^m$ is an orthonormal frame field on M, $\Delta^\phi := \sum_{i=1}^m \left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} - \overline{\nabla}_{\nabla_{e_i}e_i}\right)$, $\tau(\phi) = \operatorname{Trace} \nabla d\phi$ is the *tension field* and R^N is the Riemannian curvature tensor of (N, h) given by $R^N(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z -$

$\nabla^N_{[X,Y]}Z$ for $X,\ Y,\ Z \in \mathfrak{X}(N)$. $\tau_2(\phi)$ is called the *bi-tension field* of ϕ. A map $\phi : (M, g) \to (N, h)$ is called a *biharmonic map* if $\tau_2(\phi) = 0$.

Let M be an m-dimensional immersed submanifold in (N^{m+1}, h), $\phi : (M^m, g) \to (N^{m+1}, h)$ its immersion and g its induced Riemannian metric. The Gauss and Weingarten formulas are given by

$$\nabla^N_X Y = \nabla_X Y + B(X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{2}$$

$$\nabla^N_X \xi = -A_\xi X, \quad X \in \mathfrak{X}(M),\ \xi \in \mathfrak{X}(M)^\perp, \tag{3}$$

where B is the second fundamental form of M in N, A_ξ is the shape operator for a unit normal vector field ξ on M. It is well known that B and A are related by

$$\langle B(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle. \tag{4}$$

For any $x \in M$, let $\{e_1, \cdots, e_m, \xi\}$ be an orthonormal basis of N at x such that $\{e_1, \cdots, e_m\}$ is an orthonormal basis of T_xM. The mean curvature vector field $\mathbf{H}$ of M at x is given by

$$\mathbf{H}(x) = \frac{1}{m}\sum_{i=1}^m B(e_i, e_i).$$

If an isometric immersion $\phi : (M^m, g) \to (N^{m+1}, h)$ is biharmonic, then M is called a *biharmonic hypersurface* in N. In this case, we remark that the tension field $\tau(\phi)$ of ϕ is written as $\tau(\phi) = m\mathbf{H}$. The necessary and sufficient condition for M in N to be biharmonic is the following:

$$\Delta^\phi \mathbf{H} + \sum_{i=1}^m R^N(\mathbf{H}, d\phi(e_i))d\phi(e_i) = 0. \tag{5}$$

From (5), the necessary and sufficient condition for $\phi : (M^m, g) \to (N^{m+1}, h)$ to be a biharmonic hypersurface is as follows (cf. [14]):

$$\Delta H - H\,|A|^2 + H\,\mathrm{Ric}^N(\xi, \xi) = 0, \tag{6}$$

$$2A(\operatorname{grad} H) + \frac{1}{2}m \operatorname{grad} H^2 - 2H(\mathrm{Ric}^N(\xi))^T = 0. \tag{7}$$

Remark 2.1. *Biharmonic hypersurfaces satisfy an overdetermined problem* (*see* [9]).

If an isometric immersion $\phi : (M^m, g) \to (N^{m+1}, h)$ satisfies

$$\Delta H - H\,|A|^2 + H\,\mathrm{Ric}^N(\xi, \xi) = \lambda H \quad (\text{for some } \lambda \in \mathbb{R}), \tag{8}$$

then M is called a *biminimal hypersurface*. Biminimal hypersurfaces were introduced by E. Loubeau and S. Montaldo (cf. [10]). We call an biminimal hypersurface *free biminimal* if it satisfies the biminimal condition for $\lambda = 0$. If M is a biminimal hypersurface with $\lambda \leq 0$ in N, then M is called a *non-positive biminimal hypersurface* in N.

Remark 2.2. *We remark that* **every biharmonic hypersurface is free biminimal**.

3. Proof of Theorem 1.3 and Theorem 1.5

In this section, we will prove our main theorem. To prove our main theorem, we will use Petersen-Wylie's Yau-Naber Liouville theorem (cf. [15]). Liouville type theorem is a strong tool for biharmonic submanifolds (cf. [12], [11]).

Theorem 3.1 ([15]). *Let (M, g) be a manifold with finite h-volume: $\int_M e^{-h} dv_g < \infty$. If u is a smooth function in $L^2(e^{-h} dv_g)$ which is bounded below such that $\Delta_h u \geq 0$, $(\Delta_h = \Delta - \nabla_{\nabla h})$, then u is constant.*

We prove Theorem 1.3.

Proof of Theorem 1.3. Let $\varepsilon > 0$ be small enough. One can easily compute

(9)
$$\begin{aligned}\Delta H^{-\varepsilon} &= \varepsilon(\varepsilon+1)H^{-(\varepsilon+2)}|\nabla H|^2 - \varepsilon H^{-(\varepsilon+1)}\Delta H\\ &= \varepsilon(\varepsilon+1)H^{-(\varepsilon+2)}|\nabla H|^2 - \varepsilon H^{-\varepsilon}|A|^2 + \varepsilon H^{-\varepsilon}\mathrm{Ric}^N(\xi,\xi) - \lambda\varepsilon H^{-\varepsilon},\end{aligned}$$

and

$$\nabla_{\nabla h} H^{-\varepsilon} = -\varepsilon H^{-(\varepsilon+1)}\langle \nabla h, \nabla H\rangle, \tag{10}$$

where the second line of (9), we used (8). Thus we have

$$\begin{aligned}\Delta_h H^{-\varepsilon} =& \varepsilon(\varepsilon+1)H^{-(\varepsilon+2)}|\nabla H|^2 - \varepsilon H^{-\varepsilon}|A|^2 + \varepsilon H^{-\varepsilon}\mathrm{Ric}^N(\xi,\xi)\\ &- \lambda\varepsilon H^{-\varepsilon} + \varepsilon H^{-(\varepsilon+1)}\langle \nabla h, \nabla H\rangle.\end{aligned} \tag{11}$$

Set $h = \log H^{(p-1)}$. Since we have

$$\nabla h = (p-1)\frac{\nabla H}{H},$$

one can obtain that

$$\begin{aligned}&\varepsilon H^{-(\varepsilon+2)}\left\{(\varepsilon+1)|\nabla H|^2 + H\langle \nabla h, \nabla H\rangle\right\}\\ &\qquad= \varepsilon(\varepsilon+p)H^{-(\varepsilon+2)}|\nabla H|^2 \geq 0.\end{aligned}$$

On the other hand, by assumption,

$$\begin{aligned}\varepsilon H^{-\varepsilon}(- |A|^2 + \mathrm{Ric}^N(\xi,\xi) - \lambda) \\ \geq \varepsilon H^{-\varepsilon}(-|A|^2 + \mathrm{Ric}^N(\xi,\xi)) \geq 0,\end{aligned}$$

where we used $|B|^2 = |A|^2$. Therefore we obtain $\Delta_h H^{-\varepsilon} \geq 0$.

Since $h = \log H^{(p-1)}$, by assumption, we have

$$\int_M e^{-h} dv_g = \int_M H^{-(p-1)} dv_g \leq \int_M H^{-p} dv_g < \infty.$$

On the other hand, one can get that

$$\int_M H^{-\varepsilon} e^{-h} dv_g = \int_M H^{-(p-1+\varepsilon)} dv_g \leq \int_M H^{-p} dv_g < \infty.$$

Applying Petersen-Wylie's Yau-Naber Liouville theorem, we obtain $H^{-\varepsilon}$ is constant. Therefore H is constant. □

Applying Theorem 1.3, one can prove our main theorem (Theorem 1.5).

Proof of Theorem 1.5. Since $N = S^{m+1}$, $Ric^N(\xi,\xi) = m$. By assumption, one can obtain $|B|^2 \leq m = \mathrm{Ric}^N(\xi,\xi)$. Since $mH^2 \leq |B|^2$, $H \leq 1$ is automatically satisfied. Note that biharmonic hypersurfaces are non-negative biminimal. Applying Theorem 1.3, we obtain H is constant. □

4. Appendix

We can apply our method to p-biharmonic submanifolds (cf. [7]). If an isometric immersion $\phi : (M, g) \to (N, h)$ satisfies

$$\Delta^{\phi}(|\mathbf{H}|^{p-2}\mathbf{H}) + R^N(|\mathbf{H}|^{p-2}\mathbf{H}, d\phi(e_i))d\phi(e_i) = 0,$$

then M is called a *p-biharmonic submanifold.* For p-biharmonic submanifolds, it is easy to see that we can get same (similar) results as in the results of biharmonic submanifolds in many cases. (For example, Corollary 3.6, 3.9 in [12], and so on.) In fact, the same argument as in Proof of Theorem 1.3 shows the following result.

Proposition 4.1. *Let $\phi : (M^m, g) \to (N^{m+1}, h)$ be a complete p-biharmonic hypersurface. Assume that the mean curvature H satisfies*

$0 < H \leq 1$. *We also assume that* $|B|^2 \leq \mathrm{Ric}^N(\xi,\xi)$. *If*

$$\int_M H^{-q} dv_g < \infty,$$

for some $0 < q < \infty$, *then* H *is constant.*

Proof. Set $u = H^{p-1}$. We have only to consider $\Delta u^{-\varepsilon}$ and $h = \log u^{\frac{q}{p-1}-\varepsilon}$. □

Therefore we give one problem.

Problem 1. Does any (complete) p-biharmonic submanifold in spheres have constant mean curvature?

References

[1] A. Balmus, S. Montaldo and C. Oniciuc, *New results toward the classification of biharmonic submanifolds in* S^n, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. **20** (2012), no. 2, 89–114.

[2] A. Balmus, S. Montaldo and C. Oniciuc, *Classification results for biharmonic submanifolds in spheres*, Israel J. Math. **168** (2008), 201–220.

[3] A. Balmus and C. Oniciuc, *Biharmonic submanifolds with parallel mean curvature vector field in spheres*, J. Math. Anal. Appl. **386** (2012), 619–630.

[4] J. H. Chen, *Compact 2-harmonic hypersurfaces in* S^{n+1}*(1)*, Acta Math. Sinica **36** (1993), 49–56.

[5] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CBMS, **50** Amer. Math. Soc, (1983).

[6] J. Eells and J. H. Sampson, *Harmonic Mappings of Riemannian Manifolds*, Amer. J Math. **86** (1964), 109–160.

[7] P. Hornung and R. Moser, *Intrinsically p-biharmonic maps*, Calc. Ver. **51** (2014), 597–620.

[8] G. Y. Jiang, 2*-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. **7**A (1986), 388–402; the English translation, Note Mat. **28** (2008), 209–232.

[9] N. Koiso and H. Urakawa, *Biharmonic submanfolds in a Riemannian manifold*, arXiv:1408.5494v1[math.DG].

[10] E. Loubeau and S. Montaldo, *Biminimal immersion*, Proc. Edinb. Math. Soc. **51** (2008), 421–437.

[11] Y. Luo, *Liouville-type Theorems on complete manifolds and non-existence of bi-harmonic maps*, J. Geom. Anal. (to appear).

[12] S. Maeta, *Properly immersed submanifolds in complete Riemannian manifolds*, Adv. Math. **253** (2014), 139–151.

[13] S. Maeta, Biharmonic submanifolds in manifolds with bounded curvature, arXiv:1405.5947 [mathDG].

[14] Y.-L. Ou, *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific J. Math, **248** (1) (2010), 217-232.

[15] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, Geom. Topo. **14** (2010), 2277–2300.
[16] Z.-P. Wang and Y.-L. Ou, *Biharmonic Riemannian submersions from 3-manifolds*, Math. Z. **269** (2011), 917–925.

Division of Mathematics, Shimane University, Nishikawatsu 1060 Matsue, 690-8504, Japan.
E-mail address: shun.maeta@gmail.com *or* maeta@riko.shimane-u.ac.jp