How to Find the Expectation of Number of Female Cattles in a Reclaimed Farm

(Part 1)

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開拓地におけるめす家畜の数の推定

（その１）

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In a reclaimed farm it is one of the problems of concern to estimate the number of female cattles after some years. Yet it is not easy to calculate its expectation, for there are many conditions which affect the number. They are as follows :

1. A female cattle becomes able to be fertilized A years after its birth ;
2. It will give birth to the next generation b years after the fertilization ;
3. It will become able to be fertilized again B years after the delivery ;
4. Assume that all female cattles can be fertilized and give birth to the next generation continuously as many times as they can ;
5. The probability of birth of female youngsters is p, and let it be constant to all cattles;
6. It becomes sterile D years after its birth :
7. We count neither the number of male cattles nor that of sterile females, because they are generally disposed.

If we do not distinguish males and females the problem belongs to the "discontinuous Markov process (chain)," and is called "birth-and-death process" or "Furry process." They have already established good theories by using functionel equations for this process, but as for the number of females I could not find any solution. Its theory seems rather simple, I say, but its calculation is quite tedious. As I have not enough space to show all the evaluations, I can only offer a table here for the beginning part of it.

In this calculation I assumed that $A=2$, $b=1$, $B=1$, and $D=9$.
The notation Sx,y denotes a status that there are x females at the end of y-th year from the birth of the original female. As for $y>6$ the table becomes very tedious, I omitted using of Sx,y. I classified them into ①, ②, ③, and ④ instead. The expectation for $y \leqq 2$ is naturally one, and as for $3 \leqq y \leqq 9$ please see the bottom of the table.

(y=2)	(y=3)		(y=4)		(y=5)		(y = 6)		(y = 7)					(y = 8)			(y = 9)		
									Expectation					Number of cases	Expectatlon		Number of cases	Expectation	
		Exp.		Exp.		Exp.		Exp.	①	②	③	④	Σ	① ② ③ ④	Σ (① ～ ④)	ΣΣ	① ② ③ ④	Σ (① ～ ④)	ΣΣ
$S_{1,2}$ ↗ ↘	$S_{2,3}$	$2p$	$S_{3,4}$ ↗	$3p^2$	$S_{4,5}$ ↗ ↘	$4p^3$	↗ $S_{6,6}$	$6p^5$	$9p^8$	$24p^7(1-p)$	$21p^6(1-p)^2$	$6p^5(1-p)^3$	$4p^3+5p^4$	5 5 5 5	$6p^5+7p^6$	$4p^3+9p^4$	35 35 35 35	$6p^5+13p^6$	$4p^3+13p^4+2p^5$
							→$S_{5,6}$	$10p^4(1-p)$	$16p^7(1-p)$	$42p^6(1-p)^2$	$36p^5(1-p)^3$	$10p^4(1-p)^4$		5 5 5 5	$10p^4+4p^5-14p^6$		30 30 30 30	$10p^4+14p^5-24p^6$	
							↘ $S_{4,6}$	$4p^3(1-p)^2$	$7p^6(1-p)^2$	$18p^5(1-p)^3$	$15p^4(1-p)^4$	$4p^3(1-p)^5$		5 5 5 5	$4p^3-p^4-10p^5+7p^6$		25 25 25 25	$4p^3+3p^4-18p^5+11p^6$	
					$S_{3,5}$	$3p^2(1-p)$	↗ $S_{5,6}$	$5p^4(1-p)$	$8p^7(1-p)$	$21p^6(1-p)^2$	$18p^5(1-p)^3$	$5p^4(1-p)^4$	$3p^2+2p^3-5p^4$	4 4 4 4	$5p^4+p^5-6p^6$	$3p^2+5p^3-8p^4$	24 24 24 24	$5p^4+6p^5-11p^6$	$3p^2+8p^3-9p^4-2p^5$
							→$S_{4,6}$	$8p^3(1-p)^2$	$14p^6(1-p)^2$	$36p^5(1-p)^3$	$30p^4(1-p)^4$	$8p^3(1-p)^5$		4 4 4 4	$8p^3-4p^4-16p^5+12p^6$		20 20 20 20	$8p^3+4p^4-32p^5+20p^6$	
							↘ $S_{3,6}$	$3p^2(1-p)^3$	$6p^5(1-p)^3$	$15p^4(1-p)^4$	$12p^3(1-p)^5$	$3p^2(1-p)^6$		4 4 4 4	$3p^2-3p^3-9p^4+15p^5-6p^6$		16 16 16 16	$3p^2-18p^4+24p^5-9p^6$	
			↘ $S_{2,4}$	$2p(1-p)$	$S_{3,5}$ ↗ ↘	$3p^2(1-p)$	↗ $S_{5,6}$	$5p^4(1-p)$	$7p^6(1-p)$	$12p^5(1-p)^2$	$5p^4(1-p)^3$	/	$3p^2+p^3-4p^4$	4 4 4 /	$5p^4-5p^6$	$3p^2+4p^3-7p^4$	24 24 24 /	$5p^4+5p^5-10p^6$	$3p^2+7p^3-8p^4-2p^5$
							→$S_{4,6}$	$8p^3(1-p)^2$	$12p^5(1-p)^2$	$20p^4(1-p)^3$	$8p^3(1-p)^4$	/		4 4 4 /	$8p^3-6p^4-12p^5+10p^6$		20 20 20 /	$8p^3+2p^4-28p^5+18p^6$	
							↘ $S_{3,6}$	$3p^2(1-p)^3$	$5p^4(1-p)^3$	$8p^3(1-p)^4$	$3p^2(1-p)^5$	/		4 4 4 /	$3p^2-4p^3-6p^4+12p^5-5p^6$		16 16 16 /	$3p^2-p^3-15p^4+21p^5-8p^6$	
					$S_{2,5}$	$2p(1-p)^2$	$S_{4,6}$ ↗ ↘	$4p^3(1-p)$	$6p^5(1-p)$	$10p^4(1-p)^2$	$4p^3(1-p)^3$	/	$3p^2-3p^4$	4 4 4 /	$4p^3+p^4-5p^5$	$3p^2+3p^3-6p^4$	20 20 20 /	$4p^3+5p^4-9p^5$	$3p^2+6p^3-8p^4-p^5$
							$S_{3,6}$	$3p^2(1-p)^2$	$5p^4(1-p)^2$	$8p^3(1-p)^3$	$3p^2(1-p)^4$	/		4 4 4 /	$3p^2-p^3-7p^4+5p^5$		16 16 16 /	$3p^2+2p^3-13p^4+8p^5$	
	$S_{1,3}$	$1-p$	$S_{2,4}$ ↗ ↘	$2p(1-p)$	$S_{3,5}$ ↗ ↘	$3p^2(1-p)$	↗ $S_{4,6}$	$4p^3(1-p)^2$	$6p^5(1-p)^2$	$10p^4(1-p)^3$	$4p^3(1-p)^4$	/	$-6p^3+4p^4$	3 3 3 /	$4p^3-4p^4-4p^5+4p^6$	$2p+2p^2-10p^3+6p^4$	15 15 15 /	$4p^3-12p^5+8p^6$	$2p+4p^2-12p^3+4p^4+2p^5$
							→$S_{3,6}$	$6p^2(1-p)^3$	$10p^4(1-p)^3$	$16p^3(1-p)^4$	$6p^2(1-p)^5$	/		3 3 3 /	$6p^2-10p^3-6p^4+18p^5-8p^6$		12 12 12 /	$6p^2-4p^3-24p^4+36p^5-14p^6$	
							↘ $S_{2,6}$	$2p(1-p)^4$	$4p^3(1-p)^4$	$6p^2(1-p)^5$	$2p(1-p)^6$	/		3 3 3 /	$2p-4p^2-4p^3+16p^4-14p^5+4p^6$		9 9 9 /	$2p-2p^2-12p^3+28p^4-22p^5+6p^6$	
					$S_{2,5}$	$2p(1-p)^2$	$S_{3,6}$ ↗ ↘	$3p^2)1-p)^2$	$5p^4(1-p)^2$	$8p^3(1-p)^3$	$3p^2(1-p)^4$	/	$2p-p^2-4p^3+3p^4$	3 3 3 /	$3p^2-2p^3-5p^4+4p^5$	$2p+p^2-8p^3+5p^4$	12 12 12 /	$3p^2+p^3-11p^4+7p^5$	$2p+3p^2-11p^3+5p^4+p^5$
							$S_{2,6}$	$2p(1-p)^3$	$4p^3(1-p)^3$	$6p^2(1-p)^4$	$2p(1-p)^5$	/		3 3 3 /	$2p-2p^2-6p^3+10p^4-4p^5$		9 9 9 /	$2p-12p^3+16p^4-6p^5$	
			$S_{1,4}$	$(1-p)^2$	$S_{2,5}$	$2p(1-p)^2$	$S_{3,6}$ ↗ ↘	$3p^2(1-p)^2$	$4p^3(1-p)^2$	$3p^2(1-p)^3$	/	/	$2p-2p^2-2p^3+2p^4$	3 3 / /	$3p^2-3p^3-3p^4+3p^5$	$2p-6p^3+4p^4$	12 12 / /	$3p^2-9p^4+6p^5$	$2p+2p^2-9p^3+4p^4+p^5$
							$S_{2,6}$	$2p(1-p)^3$	$3p^2(1-p)^3$	$2p(1-p)^4$	/	/		3 3 / /	$2p-3p^2-3p^3+7p^4-3p^5$		9 9 / /	$2p-p^2-9p^3+13p^4-5p^5$	
					$S_{1,5}$	$(1-p)^3$	$S_{2,6}$ ↗ ↘	$2p(1-p)^3$	$3p^2(1-p)^3$	$2p(1-p)^4$	/	/	$1-p-3p^2+5p^3-2p^4$	2 2 / /	$2p-4p^2+4p^4-2p^5$	$1-6p^2+8p^3-3p^4$	6 6 / /	$2p-2p^2-6p^3+10p^4-4p^5$	$1+p-8p^2+8p^3-p^4-p^5$
							$S_{1,6}$	$(1-p)^4$	$2p(1-p)^4$	$(1-p)^5$	/	/		2 2 / /	$1-2p-2p^2+8p^3-7p^1+2p^5$		4 4 / /	$1-p-6p^2+14p^3-11p^4+3p^5$	
otal	$1+p$		$1+2p$		$1+3p$		$1+4p+p^2$		$1+5p+3p^2$					$1+6p+6p^2$			$1+7p+10p^2+p^3$		

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摘　　　要

開拓地などにおいて新生の**めす**家畜を買入れたとき，それが何年後に何頭になるかということは，その所有者にとって重大な関心事である．しかしながらたとえ病気にならないまでも，生れてから種付けまでの年数，めすの生れる割合，生後何年で死亡又は繁殖不能になるかといった要素がある．その上生れた**めす**も同様の経過をたどる．**めす**と**おす**を区別しない場合は，不連続のマルコフ過程の特別なものであって，増殖過程乃至ファーリー過程として既に研究されているが，**めす**だけの数を考えるものはその理論が簡単らしくみえるのにかかわらず，その計算はすこぶる複雑となる．ここに紙面の許す限りの範囲内で計算結果を表示した．普通（例えば**うし**）の場合なら，**めす**の生れる確率pを½として計算すれば十分であろう．