AN INFINITE FAMILY OF PAIRS OF IMAGINARY QUADRATIC FIELDS WITH BOTH CLASS NUMBERS DIVISIBLE BY FIVE

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Abstract. We construct a new infinite family of pairs of imaginary quadratic fields with both class numbers divisible by five. Let n be a positive integer that satisfy $n \equiv \pm 3 \pmod{500}$ and $n \not\equiv 0 \pmod 3$. We prove that 5 divides the class numbers of both $\mathbb{Q}(\sqrt{2-F_n})$ and $\mathbb{Q}(\sqrt{5(2-F_n)})$, where F_n is the nth Fibonacci number.

1. Introduction

Some infinite families of quadratic fields with class numbers divisible by a fixed integer N were given by Nagell [15], Ankeny and Chowla [1], Yamamoto [19], Weinberger [18], Gross and Rohrich [5], Ichimura [6] and Louboutin [13]. In the case $N = 5$, some results are known due to Parry [16], Mestre [14], Sase [17] and Byeon [3]. One of the authors [10], by using the Fibonacci numbers F_n, gave an infinite family of imaginary quadratic fields with class numbers divisible by five: the $\mathbb{Q}(\sqrt{-F_n})$ with $n \equiv 25 \pmod{50}$.

Recently, Komatsu [11], [12] and Ito [9] (resp. Iizuka, Konomi and Nakano [7]) gave infinite families of pairs of quadratic fields with both class numbers divisible by 3 (resp. 3, 5 or 7). In the present article, by using the Fibonacci numbers F_n, we will give an infinite family of pairs of imaginary quadratic fields with both class numbers divisible by 5.

Theorem. *For $n \in \mathcal{N} := \{n \in \mathbb{N} \mid n \equiv \pm 3 \pmod{500}, n \not\equiv 0 \pmod 3\}$, the class numbers of both $\mathbb{Q}(\sqrt{2-F_n})$ and $\mathbb{Q}(\sqrt{5(2-F_n)})$ are divisible by 5. Moreover, the set of pairs $\{(\mathbb{Q}(\sqrt{2-F_n}), \mathbb{Q}(\sqrt{5(2-F_n)})) \mid n \in \mathcal{N}\}$ is infinite.*

For an algebraic extension K/k, denote the norm map and the trace map of K/k by $N_{K/k}$ and $\mathrm{Tr}_{K/k}$, respectively. For simplicity, we denote N_K and Tr_K if the base field is $k = \mathbb{Q}$. For a prime number p and an integer m, we denote the greatest exponent μ of p such that $p^\mu \mid m$ by $v_p(m)$.

2. Certain parametric quartic polynomial

Let $k = \mathbb{Q}(\sqrt{5})$. For an algebraic integer $\alpha \in k$, we consider the polynomial

$$f(X) = f_\alpha(X) := X^4 - TX^3 + (N+2)X^2 - TX + 1 \in \mathbb{Z}[X], \tag{2.1}$$

where $T := \mathrm{Tr}_k(\alpha)$ and $N := N_k(\alpha)$. The discriminant of $f(X)$ is $\mathrm{disc}(f) = d_1^2 d_2$ with $d_1 := T^2 - 4N$ and $d_2 := (N+4)^2 - 4T^2$. Let L be the minimal splitting field of $f(X)$ over $\mathbb{Q}$. All four complex roots of $f(X)$ are units of L and can be denoted by $\varepsilon, \varepsilon^{-1}, \eta, \eta^{-1}$, $|\varepsilon| \geq |\varepsilon^{-1}|$, $|\eta| \geq |\eta^{-1}|$, $\alpha = \varepsilon + \varepsilon^{-1}$, $\overline{\alpha} = \eta + \eta^{-1}$, where $\overline{\alpha}$ denotes the Galois conjugate of α ([2, Lemmas 2.2 and 2.3]). We assume $\alpha \notin \mathbb{Z}$, $\alpha^2 - 4 \notin \mathbb{Z}^2$,

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$d_2 \in 5\mathbb{Q}^2$ and $\alpha^2 - 4 > 0$. The assumptions $\alpha \notin \mathbb{Z}$ and $\alpha^2 - 4 \notin \mathbb{Z}^2$ imply that the polynomial $f(X)$ is $\mathbb{Q}$-irreducible, and we have $\mathrm{Gal}(L/\mathbb{Q}) \simeq C_4$ from $d_2 \in 5\mathbb{Q}^2$ ([2, Proposition 2.1]). Furthermore, we have $\varepsilon, \eta \in \mathbb{R}$ by the assumption $\alpha^2 - 4 > 0$, $d_2 > 0$ and the factorization

$$f(X) = (X^2 - \alpha X + 1)(X^2 - \overline{\alpha} X + 1) = (X - \varepsilon)(X - \varepsilon^{-1})(X - \eta)(X - \eta^{-1}) \tag{2.2}$$

([2, Lemma 2.7]). Set $\widetilde{L} = L(\zeta_5)$ where ζ_5 is a primitive fifth root of unity. Since $\mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \supset \mathrm{Gal}(\widetilde{L}/k) \simeq C_2 \times C_2$ and $\mathrm{Gal}(\widetilde{L}/\mathbb{Q})/\mathrm{Gal}(\widetilde{L}/\mathbb{Q}(\zeta_5)) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \simeq C_4$, we have $\mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \simeq C_2 \times C_4$. Therefore, $\mathrm{Gal}(\widetilde{L}/\mathbb{Q})$ has three subgroups of order 4. One of them is isomorphic to $C_2 \times C_2$ that corresponds to the subfield k, the others are isomorphic to C_4. Let us denote them by $\langle \tau \rangle$ ($\simeq C_4$) and $\langle \tau' \rangle$ ($\simeq C_4$) for some automorphisms $\tau, \tau' \in \mathrm{Gal}(\widetilde{L}/\mathbb{Q})$ of order 4. Note that $\zeta_5^\tau \neq \zeta_5, \zeta_5^4$, because τ acts trivial on $k = \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$ if $\zeta_5^\tau = \zeta_5$ or ζ_5^4. Likewise, we have $\zeta_5^{\tau'} \neq \zeta_5, \zeta_5^4$. We may assume that $\zeta_5^\tau = \zeta_5^2$ and $\zeta_5^{\tau'} = \zeta_5^3$.

Lemma 1. *The actions of τ and τ' on the roots $\varepsilon, \varepsilon^{-1}, \eta$ and η^{-1} of $f(X)$ are as follows:*

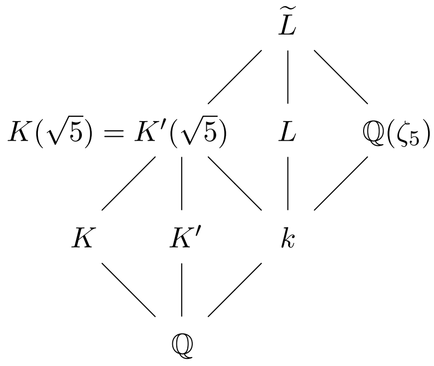
$$\begin{array}{rl} \tau : & \varepsilon \mapsto \eta \mapsto \varepsilon^{-1} \mapsto \eta^{-1} \mapsto \varepsilon \\ \tau' : & \varepsilon \mapsto \eta^{-1} \mapsto \varepsilon^{-1} \mapsto \eta \mapsto \varepsilon \end{array}$$

Proof. If $\varepsilon^\tau = \varepsilon^{-1}$, then we have $\alpha^\tau = (\varepsilon + \varepsilon^{-1})^\tau = \alpha$. This is a contradiction since the restriction of τ to L is a generator of $\mathrm{Gal}(L/\mathbb{Q})$ ($\simeq C_4$). Therefore, we have $\varepsilon^\tau \neq \varepsilon^{-1}$, and hence $\varepsilon^\tau = \eta$ or η^{-1}. Similarly we have $\varepsilon^{\tau'} = \eta$ or η^{-1}. Without loss of generality, we can assume that $\varepsilon^\tau = \eta$ and $\varepsilon^{\tau'} = \eta^{-1}$. Next, we will prove $\eta^\tau = \varepsilon^{-1}$. We get $\eta^\tau \neq \eta^{-1}$ by the same argument as the proof of $\varepsilon^\tau \neq \varepsilon^{-1}$. If $\eta^\tau = \varepsilon$, then we have $(\varepsilon + \eta)^\tau = \varepsilon + \eta$, $(\varepsilon\eta)^\tau = \varepsilon\eta$, $(\varepsilon^{-1} + \eta^{-1})^\tau = \varepsilon^{-1} + \eta^{-1}$, $(\varepsilon^{-1}\eta^{-1})^\tau = \varepsilon^{-1}\eta^{-1}$, and hence $\varepsilon + \eta, \varepsilon\eta, \varepsilon^{-1} + \eta^{-1}, \varepsilon^{-1}\eta^{-1} \in \mathbb{Q}$. Noting (2.2), therefore, $f(X)$ is factored in $\mathbb{Q}[X]$ as

$$f(X) = (X^2 - (\varepsilon + \eta)X + \varepsilon\eta)(X^2 - (\varepsilon^{-1} + \eta^{-1})X + \varepsilon^{-1}\eta^{-1}).$$

However, this contradicts the assumption that $f(X)$ is irreducible over $\mathbb{Q}$. We conclude $\eta^\tau \neq \varepsilon$ and hence $\eta^\tau = \varepsilon^{-1}$. Similarly we can get $(\eta^{-1})^{\tau'} = \varepsilon^{-1}$. The proof is complete. □

Let K and K' denote the subfields of $\widetilde{L}$ correspond to $\langle \tau \rangle$ and $\langle \tau' \rangle$, respectively.



Lemma 2. *We have*

$$(\varepsilon - \varepsilon^{-1})(\eta - \eta^{-1}) = \begin{cases} \sqrt{d_2} & \text{if } N > 0, \\ -\sqrt{d_2} & \text{if } N < 0, \end{cases}$$

$$\varepsilon\eta+\varepsilon^{-1}\eta^{-1}=\begin{cases}\dfrac{N+\sqrt{d_2}}{2} & \text{if } N>0,\\ \dfrac{N-\sqrt{d_2}}{2} & \text{if } N<0\end{cases} \quad \text{and} \quad \varepsilon\eta^{-1}+\varepsilon^{-1}\eta=\begin{cases}\dfrac{N-\sqrt{d_2}}{2} & \text{if } N>0,\\ \dfrac{N+\sqrt{d_2}}{2} & \text{if } N<0.\end{cases}$$

Proof. Put $\lambda := (\varepsilon-\varepsilon^{-1})(\eta-\eta^{-1})$. By using $N=\alpha\overline{\alpha}=(\varepsilon+\varepsilon^{-1})(\eta+\eta^{-1})$, we have $\varepsilon\eta+\varepsilon^{-1}\eta^{-1}=(N+\lambda)/2$ and $\varepsilon\eta^{-1}+\varepsilon^{-1}\eta=(N-\lambda)/2$. By direct calculation, we get $\lambda^2=d_2$. Recall that $\varepsilon,\eta\in\mathbb{R}$. Since $\alpha=\varepsilon+\varepsilon^{-1}$ (resp. $\overline{\alpha}=\eta+\eta^{-1}$) is positive if and only if ε (resp. η) is positive, and $|\varepsilon|\geq|\varepsilon^{-1}|$ and $|\eta|\geq|\eta^{-1}|$, we have

$$N=\alpha\overline{\alpha}>0 \iff \varepsilon\eta>0 \iff \lambda=(\varepsilon-\varepsilon^{-1})(\eta-\eta^{-1})>0.$$

The proof is complete. □

Lemma 3. *Let i,j be integers which are not divisible by 5. If $\varepsilon^i\eta^j\in L^5$, then we have $\varepsilon,\eta\in L^5$.*

Proof. We put $\mathrm{Gal}(\widetilde{L}/k)\simeq\langle\sigma\rangle\times\langle\sigma'\rangle$ ($\simeq C_2\times C_2$), where $\varepsilon^\sigma=\varepsilon^{-1}$, $\eta^\sigma=\eta$, $\varepsilon^{\sigma'}=\varepsilon$ and $\eta^{\sigma'}=\eta^{-1}$. If $\varepsilon^i\eta^j\in L^5$, then so are $(\varepsilon^i\eta^j)^\sigma=\varepsilon^{-i}\eta^j$, their ratio ε^{2i} and their product η^{2j}. Since $\gcd(2i,5)=\gcd(2j,5)=1$, we conclude that both ε and η are fifth powers in L. □

3. Fibonacci and Lucas sequences

Let (F_n) and (L_n) be the Fibonacci and Lucas sequences, respectively, defined by $F_1=1$, $F_2=1$, $F_{n+2}=F_{n+1}+F_n$ ($n\in\mathbb{Z}$) and $L_1=1$, $L_2=3$, $L_{n+2}=L_{n+1}+L_n$ ($n\in\mathbb{Z}$). Assertions (1) and (2) in the following lemma follow from the explicit formulae for

$$F_n=\frac{\omega^n-\overline{\omega}^n}{\omega-\overline{\omega}} \quad \text{and} \quad L_n=\omega^n+\overline{\omega}^n,$$

where $\omega=(1+\sqrt{5})/2$ and $\overline{\omega}=(1-\sqrt{5})/2$. We can prove (3) by direct calculation.

Lemma 4. *For any $n\in\mathbb{Z}$, we have the following.*

(1) $L_n^2=5F_n^2+(-1)^n4$.
(2) $5F_{2n-1}+L_{2n-1}+(-1)^n4=2L_n^2$ *and* $5F_{2n-1}+L_{2n-1}-(-1)^n4=10F_n^2$.
(3) (F_n) *mod* 5^3 *is* 500*-periodic and* $F_n\equiv 2 \pmod{5^3}$ *if* $n\equiv\pm3 \pmod{500}$.

From now on, we assume that n (>3) is an odd integer and consider the polynomial (2.1) for $\alpha=(L_n+(F_n-2)\sqrt{5})/2$. By (2.1) and Lemma 4 (1), we get

$$f(X)=f_\alpha(X)=X^4-L_nX^3+(5F_n-4)X^2-L_nX+1,$$

and all four roots of $f(X)$ are given by $\varepsilon,\varepsilon^{-1},\eta,\eta^{-1}$ which satisfy $\alpha=\varepsilon+\varepsilon^{-1}$, $\overline{\alpha}=\eta+\eta^{-1}$. Moreover, we see from $d_1=T^2-4N=5(F_n-2)^2$ and $d_2=(N+4)^2-4T^2=5(F_n-2)^2$ that the discriminant of $f(X)$ is $\mathrm{disc}(f)=d_1^2d_2=5^3(F_n-2)^6$. Furthermore, since $\alpha\notin\mathbb{Z}$, $\alpha^2-4\notin\mathbb{Z}^2$, $d_2\in5\mathbb{Q}^2$ and $\alpha^2-4>0$, the polynomial $f(X)$ is $\mathbb{Q}$-irreducible, all the roots $\varepsilon,\varepsilon^{-1},\eta,\eta^{-1}$ are real, and $\mathrm{Gal}(L/\mathbb{Q})\simeq C_4$ (see §2). Next, we will prove that the three quadratic fields contained in $\widetilde{L}$ are $\mathbb{Q}(\sqrt{2-F_n}),\mathbb{Q}(\sqrt{5(2-F_n)})$ and $k=\mathbb{Q}(\sqrt{5})$.

Lemma 5. *Put $\alpha=(L_n+(F_n-2)\sqrt{5})/2$ for an odd integer $n>3$ and $\zeta=\zeta_5$. For the roots ε,η of $f_\alpha(X)$, we have the following.*

(1) $\xi_1:=(\varepsilon+\varepsilon^{-1})(\zeta+\zeta^{-1})+(\eta+\eta^{-1})(\zeta^2+\zeta^{-2})=\{-L_n+5(F_n-2)\}/2.$

(2) $\xi_2 := (\varepsilon-\varepsilon^{-1})^2(\zeta-\zeta^{-1})^2+(\eta-\eta^{-1})^2(\zeta^2-\zeta^{-2})^2 = -5(F_n-2)(5F_n+L_n)/2$.
(3) $\xi_3 := (\varepsilon-\varepsilon^{-1})(\eta-\eta^{-1})(\zeta-\zeta^{-1})(\zeta^2-\zeta^{-2}) = -5(F_n-2)$.

Proof. Set $c=\zeta+\zeta^{-1}=(-1+\sqrt{5})/2$. Noting that $\alpha=\varepsilon+\varepsilon^{-1}$ and $\overline{\alpha}=\eta+\eta^{-1}$, we have $\xi_1=\alpha c+\overline{\alpha}(c^2-2)$ and $\xi_2=(\alpha^2-4)(c^2-4)+(\overline{\alpha}^2-4)(c-2)$. The assertions (1) and (2) follow by using Lemma 4 (1). From Lemma 2 and $N>0$, we have $(\varepsilon-\varepsilon^{-1})(\eta-\eta^{-1})=\sqrt{d_2}=(F_n-2)\sqrt{5}$. On the other hand, we have $(\zeta-\zeta^{-1})(\zeta^2-\zeta^{-2})=c^3-4c=-\sqrt{5}$. Hence we get the assertion (3). □

Lemma 6. *Under the same situation as in Lemma 5, we have the following.*

(1) $\mathrm{Tr}_{\widetilde{L}/K}(\varepsilon\zeta)=\{-L_n+5(F_n-2)+2\xi\}/4$, *where* $\xi:=(\varepsilon-\varepsilon^{-1})(\zeta-\zeta^{-1})+(\eta-\eta^{-1})(\zeta^2-\zeta^{-2})$ *is such that*

$$\xi^2=\begin{cases}-5^2(F_n-2)F^2_{\frac{n+1}{2}} & \text{if } n\equiv 1 \pmod 4,\\ -5(F_n-2)L^2_{\frac{n+1}{2}} & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

(2) $\mathrm{Tr}_{\widetilde{L}/K'}(\varepsilon\zeta)=\{-L_n+5(F_n-2)+2\xi'\}/4$, *where* $\xi':=(\varepsilon-\varepsilon^{-1})(\zeta-\zeta^{-1})-(\eta-\eta^{-1})(\zeta^2-\zeta^{-2})$ *is such that*

$$\xi'^2=\begin{cases}-5(F_n-2)L^2_{\frac{n+1}{2}} & \text{if } n\equiv 1 \pmod 4,\\ -5^2(F_n-2)F^2_{\frac{n+1}{2}} & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Proof. We prove only the assertion (1). By Lemma 1, we have

$$\gamma:=\mathrm{Tr}_{\widetilde{L}/K}(\varepsilon\zeta)=\varepsilon\zeta+\eta\zeta^2+\varepsilon^{-1}\zeta^{-1}+\eta^{-1}\zeta^{-2}$$

and

$$\gamma^{\tau'}=\eta^{-1}\zeta^2+\varepsilon\zeta^{-1}+\eta\zeta^{-2}+\varepsilon^{-1}\zeta.$$

Now $\gamma=\{(\gamma+\gamma^{\tau'})+(\gamma-\gamma^{\tau'})\}/2$ with

$$\gamma+\gamma^{\tau'}=\xi_1=\frac{-L_n+5(F_n-2)}{2}$$

and

$$\begin{aligned}(\gamma-\gamma^{\tau'})^2&=\{(\varepsilon-\varepsilon^{-1})(\zeta-\zeta^{-1})+(\eta-\eta^{-1})(\zeta^2-\zeta^{-2})\}^2\\ &=\xi_2+2\xi_3=-\frac{5(F_n-2)(5F_n+L_n+4)}{2}\end{aligned}$$

by Lemma 5. Therefore, we get the desired result, by Lemma 4 (2). □

By Lemma 6, we get the following proposition immediately.

Proposition 1. *We have*

$$(K,K')=\begin{cases}(\mathbb{Q}(\sqrt{2-F_n}),\mathbb{Q}(\sqrt{5(2-F_n)})) & \text{if } n\equiv 1 \pmod 4,\\ (\mathbb{Q}(\sqrt{5(2-F_n)}),\mathbb{Q}(\sqrt{2-F_n})) & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

4. Certain parametric quintic polynomial

For an element $\gamma \in L$, we define

$$\begin{aligned} g_{\gamma,\tau}(X) &:= X^5 - 10N_L(\gamma)X^3 - 5N_L(\gamma)N_k\mathrm{Tr}_{L/k}(\gamma)X^2 \\ &\quad + 5N_L(\gamma)\{N_L(\gamma) - N_k\mathrm{Tr}_{L/k}(\gamma^{1+\tau})\}X - N_L(\gamma)N_k\mathrm{Tr}_{L/k}(\gamma^{2+\tau}) \in \mathbb{Q}[X], \\ g_{\gamma,\tau'}(X) &:= X^5 - 10N_L(\gamma)X^3 - 5N_L(\gamma)N_k\mathrm{Tr}_{L/k}(\gamma)X^2 \\ &\quad + 5N_L(\gamma)\{N_L(\gamma) - N_k\mathrm{Tr}_{L/k}(\gamma^{1+\tau'})\}X - N_L(\gamma)N_k\mathrm{Tr}_{L/k}(\gamma^{2+\tau'}) \in \mathbb{Q}[X]. \end{aligned}$$

Define subsets $\mathcal{M}_\tau$ and $\mathcal{M}_{\tau'}$ of $\widetilde{L} = L(\zeta_5)$ by

$$\mathcal{M}_\tau := \{\gamma \in \widetilde{L}^\times \mid \gamma^{3+4\tau+2\tau^2+\tau^3} \notin \widetilde{L}^5\},$$
$$\mathcal{M}_{\tau'} := \{\gamma \in \widetilde{L}^\times \mid \gamma^{3+4\tau'+2\tau'^2+\tau'^3} \notin \widetilde{L}^5\}.$$

Proposition 2 ([8, Example 3.3], [4, Chapter 5, Examples (2), p.253])**.** *Let the notation be as above. Assume $\gamma \in \mathcal{M}_\tau \cap L$ (resp. $\gamma \in \mathcal{M}_{\tau'} \cap L$). Then the minimal splitting field of $g_{\gamma,\tau}$ (resp. $g_{\gamma,\tau'}$) over $\mathbb{Q}$ is a D_5-extension containing K (resp. K').*

Recall $d_2 \in 5\mathbb{Q}^2$. Let t be the positive integer so that $d_2 = 5t^2$, and denote $\alpha = (T + b\sqrt{5})/2$ ($b \in \mathbb{Z}$). Now we calculate the coefficients of $g_{\gamma,\tau}(X)$ and $g_{\gamma,\tau'}(X)$ in the case $\gamma = \varepsilon, \eta$.

Lemma 7. *For $\gamma = \varepsilon, \eta$, we have the following.*

(1) $N_L(\gamma) = 1$.
(2) $N_k\mathrm{Tr}_{L/k}(\gamma) = N$.
(3) $N_k\mathrm{Tr}_{L/k}(\gamma^{1+\tau}) = N_k\mathrm{Tr}_{L/k}(\gamma^{1+\tau'}) = T^2 - 2N - 4$.
(4) $N_k\mathrm{Tr}_{L/k}(\gamma^{2+\tau}) = \begin{cases} \{N(T^2-2N) - 5btT\}/2 - 3N & \text{if } N > 0, \\ \{N(T^2-2N) + 5btT\}/2 - 3N & \text{if } N < 0, \end{cases}$

$N_k\mathrm{Tr}_{L/k}(\gamma^{2+\tau'}) = \begin{cases} \{N(T^2-2N) + 5btT\}/2 - 3N & \text{if } N > 0, \\ \{N(T^2-2N) - 5btT\}/2 - 3N & \text{if } N < 0. \end{cases}$

Proof. Let $\overline{\tau} = \tau|_L$ be the restriction of τ to L. Then $\overline{\tau}$ is a generator of the cyclic quartic Galois group $\mathrm{Gal}(L/\mathbb{Q})$, and $\overline{\tau}^2$ is the generator of $\mathrm{Gal}(L/k)$. We can show the assertions (1), (2) and (3) by these facts and $\alpha = \varepsilon + \varepsilon^{-1}$ and $\overline{\alpha} = \eta + \eta^{-1}$ are roots of $X^2 - TX + N$. Therefore, we will give a proof of the assertion (4) only for $N_k\mathrm{Tr}_{L/k}(\varepsilon^{2+\tau})$ in the case $N > 0$ (we can prove the other assertions similarly). In this case, we see from Lemmas 1 and 2 that

$$\begin{aligned} N_k\mathrm{Tr}_{L/k}(\varepsilon^{2+\tau}) &= N_k\mathrm{Tr}_{L/k}(\varepsilon^2\eta) = N_k(\varepsilon^2\eta + \varepsilon^{-2}\eta^{-1}) \\ &= (\varepsilon^2\eta + \varepsilon^{-2}\eta^{-1})(\eta^2\varepsilon^{-1} + \eta^{-2}\varepsilon) \\ &= \overline{\alpha}^2(\varepsilon\eta + \varepsilon^{-1}\eta^{-1}) + \alpha^2(\varepsilon\eta^{-1} + \varepsilon^{-1}\eta) - 3N \\ &= \frac{N + \sqrt{d_2}}{2}\,\overline{\alpha}^2 + \frac{N - \sqrt{d_2}}{2}\,\alpha^2 - 3N \\ &= \frac{N}{2}(\alpha^2 + \overline{\alpha}^2) - \frac{t\sqrt{5}}{2}(\alpha^2 - \overline{\alpha}^2) - 3N. \end{aligned}$$

Since $\alpha - \overline{\alpha} = b\sqrt{5}$, we have $\alpha^2 + \overline{\alpha}^2 = T^2 - 2N$, $\alpha^2 - \overline{\alpha}^2 = bT\sqrt{5}$. Thus we get the assertion. □

Lemma 8. *Put* $\alpha = (L_n + (F_n - 2)\sqrt{5})/2$ *for an odd integer* $n > 3$. *For the roots* $\gamma = \varepsilon, \eta$ *of* $f_\alpha(X)$, *we have the following.*

(1) $N_L(\gamma) = 1$.
(2) $N_k \mathrm{Tr}_{L/k}(\gamma) = 5F_n - 6$.
(3) $N_k \mathrm{Tr}_{L/k}(\gamma^{1+\tau}) = N_k \mathrm{Tr}_{L/k}(\gamma^{1+\tau'}) = 5F_n^2 - 10F_n + 4$.
(4) $N_k \mathrm{Tr}_{L/k}(\gamma^{2+\tau}) = 5(F_n - 2)\{(F_n - 2)(5F_n - L_n + 4) + 10\}/2 + 4$,
$N_k \mathrm{Tr}_{L/k}(\gamma^{2+\tau'}) = 5(F_n - 2)\{(F_n - 2)(5F_n + L_n + 4) + 10\}/2 + 4$.

Proof. The assertions (1), (2) and (3) follow from Lemma 7 and Lemma 4 (1). We will prove the assertion (4). Since $N = 5F_n - 6 > 0$, we have from Lemma 7 (4) and Lemma 4 (1) that

$$\begin{aligned} N_k \mathrm{Tr}_{L/k}(\gamma^{2+\tau}) &= \frac{1}{2}\{(5F_n - 6)(L_n^2 - 10F_n + 12) - 5L_n(F_n - 2)^2\} - 15F_n + 18 \\ &= \frac{1}{2}\{(5F_n - 6)(5F_n^2 - 10F_n + 8) - 5L_n(F_n - 2)^2 - 30F_n + 28\} + 4 \\ &= \frac{5(F_n - 2)}{2}\{(F_n - 2)(5F_n - L_n + 4) + 10\} + 4. \end{aligned}$$

We can prove the equality for $N_k \mathrm{Tr}_{L/k}(\gamma^{2+\tau'})$ similarly. □

Lemma 9. *Put* $\alpha = (L_n + (F_n - 2)\sqrt{5})/2$ *for an odd integer* $n > 3$. *If* $n \not\equiv 0 \pmod 3$, *then for the roots* $\gamma = \varepsilon, \eta$ *of* $f_\alpha(X)$ *and for any integers* i, j *which are not divisible by* 5, *we have* $\varepsilon^i \eta^j \notin L^5$.

Proof. For any $x \in L^5$, since $L \subset \mathbb{R}$, there exists only one $y \in L$ satisfying $x = y^5$, we denote it by $\sqrt[5]{x}$. Suppose to the contrary that $\varepsilon^i \eta^j \in L^5$. Then we have $\varepsilon, \eta \in L^5$ by Lemma 3. Recall that $\overline{\tau}^2$ is the generator of $\mathrm{Gal}(L/k)$, where $\overline{\tau} = \tau|_L$. For $y = \sqrt[5]{\varepsilon} \in L$, we have

$$(y^{\overline{\tau}^2})^5 = (y^5)^{\overline{\tau}^2} = \varepsilon^{\overline{\tau}^2} = \varepsilon^{-1}.$$

This equality yields $(\sqrt[5]{\varepsilon})^{\overline{\tau}^2} = \sqrt[5]{\varepsilon^{-1}}$. Therefore, we have

$$\beta := \mathrm{Tr}_{L/k}(\sqrt[5]{\varepsilon}) = \sqrt[5]{\varepsilon} + (\sqrt[5]{\varepsilon})^{\overline{\tau}^2} = \sqrt[5]{\varepsilon} + \sqrt[5]{\varepsilon^{-1}} \in k.$$

By direct calculation, we get $\beta^5 - 5\beta^3 + 5\beta = \varepsilon + \varepsilon^{-1} = \alpha = (L_n + (F_n - 2)\sqrt{5})/2$, and $h(\beta) = 0$, where

$$\begin{aligned} h(X) &:= \left(X^5 - 5X^3 + 5X - \frac{L_n}{2}\right)^2 - \frac{5(F_n - 2)^2}{4} \\ &= X^{10} - 10X^8 + 35X^6 - L_n X^5 - 50X^4 + 5L_n X^3 + 25X^2 - 5L_n X + 5F_n - 6. \end{aligned}$$

On the one hand, $h(X)$ is reducible over $\mathbb{Q}$ because it has a root $\beta \in k = \mathbb{Q}(\sqrt{5})$. On the other hand, since

$$F_n \equiv L_n \equiv 1 \pmod 2 \quad \text{if } n \equiv 1, 2 \pmod 3, \tag{4.1}$$

we have that

$$h(X) \equiv X^{10} + X^6 + X^5 + X^3 + X^2 + X + 1 \pmod 2$$

is irreducible over $\mathbb{F}_2$. Hence $h(X)$ is $\mathbb{Q}$-irreducible. Since we obtain a contradiction, the proof is complete. □

5. Proof of our theorem

In this section, we will prove our main theorem in §1. The keys of the proof are Proposition 2 and the following proposition.

Proposition 3 ([17, Proposition 2]). *Let p $(\neq 2)$ and q be prime numbers. Suppose that the polynomial*

$$\varphi(X) = X^p + \sum_{j=0}^{p-2} a_j X^j, \quad a_j \in \mathbb{Z}$$

is irreducible over $\mathbb{Q}$ and satisfies the condition

$$v_q(a_j) < p - j \quad \text{for some } j,\ 0 \leq j \leq p-2. \tag{5.1}$$

Let θ be a root of $\varphi(X)$.

(1) *If q is different from p, then q is totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$ if and only if*

$$0 < \frac{v_q(a_0)}{p} \leq \frac{v_q(a_j)}{p-j} \quad \text{for every } j,\ 1 \leq j \leq p-2.$$

(2) *If neither*

$$0 < \frac{v_p(a_0)}{p} \leq \frac{v_p(a_j)}{p-j} \quad \text{for every } j,\ 1 \leq j \leq p-2 \tag{5.2}$$

nor

$$v_p(\varphi^{(j)}(-a_0)) < p - j \quad \text{for some } j,\ 0 \leq j \leq p-1 \tag{5.3}$$

holds, then p is not totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$, where $\varphi^{(j)}(X)$ is the j-th differential of $\varphi(X)$.

Proof of Theorem. Let n be in $\mathcal{N}$. First, we will show $\varepsilon \in \mathcal{M}_\tau \cap L$ and $\varepsilon \in \mathcal{M}_{\tau'} \cap L$, where $\mathcal{M}_\tau$ and $\mathcal{M}_{\tau'}$ are defined in §4. By Lemma 1, we have

$$\varepsilon^{3+4\tau+2\tau^2+\tau^3} = \varepsilon^3\eta^4\varepsilon^{-2}\eta^{-1} = \varepsilon\eta^3. \tag{5.4}$$

If $\varepsilon\eta^3 \in \widetilde{L}^5$, then we have $\varepsilon^2\eta^6 = N_{\widetilde{L}/L}(\varepsilon\eta^3) \in L^5$, which contradicts Lemma 9. Hence we have $\varepsilon\eta^3 \notin \widetilde{L}^5$. From (5.4), therefore, we get $\varepsilon \in \mathcal{M} \cap L$. Similarly, we can see that

$$\varepsilon^{3+4\tau'+2\tau'^2+\tau'^3} = \varepsilon^3\eta^{-4}\varepsilon^{-2}\eta = \varepsilon\eta^{-3} \notin \widetilde{L}^5,$$

and so $\varepsilon \in \mathcal{M}_{\tau'} \cap L$. Let $g_{\varepsilon,\tau}(X)$ and $g_{\varepsilon,\tau'}(X)$ be the polynomials defined in §4. From Lemma 8, we have

$$\begin{aligned} g_{\varepsilon,\tau}(X) &= X^5 - 10X^3 - 5(5F_n - 6)X^2 - 5(5F_n^2 - 10F_n + 3)X \\ &\quad - \frac{5(F_n - 2)}{2}\{(F_n - 2)(5F_n - L_n + 4) + 10\} - 4, \\ g_{\varepsilon,\tau'}(X) &= X^5 - 10X^3 - 5(5F_n - 6)X^2 - 5(5F_n^2 - 10F_n + 3)X \\ &\quad - \frac{5(F_n - 2)}{2}\{(F_n - 2)(5F_n + L_n + 4) + 10\} - 4. \end{aligned}$$

By Proposition 2, the minimal splitting fields $\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau})$ of $g_{\varepsilon,\tau}(X)$ and $\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau'})$ of $g_{\varepsilon,\tau'}(X)$ are D_5-extensions containing K and K', respectively, and the quadratic fields K and K' are given by Proposition 1. Therefore, it is enough to prove that both C_5-extensions $\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau})/K$ and $\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau'})/K'$ are unramified. We will prove only for $\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau})/K$ (we can prove similarly for $\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau'})/K'$).

Let θ be a root of $g_{\varepsilon,\tau}(X)$ and consider the quintic extension $\mathbb{Q}(\theta)/\mathbb{Q}$. For a prime number q, a prime ideal of K above q is ramified in $\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau})/K$ if and only if q is totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$ because $[\mathrm{Spl}_{\mathbb{Q}}(g_{\varepsilon,\tau}) : K] = 5$ and $[K : \mathbb{Q}] = 2$. Hence we prove that no prime number q is totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$ by using Proposition 3. We denote the coefficient of X^j of $g_{\varepsilon,\tau}(X)$ by a_j. First, $g_{\varepsilon,\tau}(X)$ satisfies the condition (5.1) because $v_q(a_3) < 5-3 = 2$ for any prime number q. From Proposition 3 (1), we see that no prime $q \neq 5$ is totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$ since $v_q(a_3) = 0$ if $q \neq 2$ and $v_2(a_2) = 0$ by (4.1). We will show, therefore, that 5 is not totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$. Since $a_0 \equiv -4 \pmod 5$ is not divisible by 5, (5.2) does not hold. Furthermore, by the assumption $n \equiv \pm 3 \pmod{500}$ and Lemma 4 (3), we have $F_n - 2 \equiv 0 \pmod{5^3}$, $-a_0 \equiv 4 \pmod{5^5}$, $5F_n - 6 = 5(F_n - 2) + 4 \equiv 4 \pmod{5^4}$, $5F_n^2 - 10F_n + 3 = 5F_n(F_n - 2) + 3 \equiv 3 \pmod{5^4}$, and hence

$$\begin{aligned}
g_{\varepsilon,\tau}(-a_0) &\equiv 4^5 - 10\cdot 4^3 - 5\cdot 4\cdot 4^2 - 5\cdot 3\cdot 4 - 4 = 0 \pmod{5^5},\\
g^{(1)}_{\varepsilon,\tau}(-a_0) &\equiv 5\cdot 4^4 - 30\cdot 4^2 - 10\cdot 4\cdot 4 - 5\cdot 3 = 625 \equiv 0 \pmod{5^4},\\
g^{(2)}_{\varepsilon,\tau}(-a_0) &\equiv 20\cdot 4^3 - 60\cdot 4 - 10\cdot 4 = 1000 \equiv 0 \pmod{5^3},\\
g^{(3)}_{\varepsilon,\tau}(-a_0) &\equiv 60\cdot 4^2 - 60 = 900 \equiv 0 \pmod{5^2},\\
g^{(4)}_{\varepsilon,\tau}(-a_0) &\equiv 120\cdot 4 \equiv 0 \pmod 5.
\end{aligned}$$

Then (5.3) does not hold. Hence 5 is not totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$.

Finally, we prove that the set $\{(\mathbb{Q}(\sqrt{2-F_n}), \mathbb{Q}(\sqrt{5(2-F_n)})) \mid n \in \mathcal{N}\}$ is infinite. For an integer m, let $s(m)$ denote the square free integer satisfying $m = s(m)A^2$ for some $A \in \mathbb{N}$, and assume that $\{(\mathbb{Q}(\sqrt{2-F_n}), \mathbb{Q}(\sqrt{5(2-F_n)})) \mid n \in \mathcal{N}\}$ is finite. Then the set $\{s(F_n - 2) \mid n \in \mathcal{N}\}$ is finite. Since $\mathcal{N}$ is infinite, there exists $k \geq 1$ such that $\mathcal{N}_k := \{n \in \mathcal{N} \mid s(F_n - 2) = k\}$ is infinite. For any integer $n \in \mathcal{N}_k$, let $F_n - 2 = kA_n^2$. Then by Lemma 4 (1), we have

$$L_n^2 = 5F_n^2 - 4 = 5(kA_n^2 + 2)^2 - 4 = 5k^2A_n^4 + 20kA_n^2 + 16.$$

This implies that infinitely many pairs (A_n, L_n) are integer solutions of the equation

$$Y^2 = 5k^2X^4 + 20kX^2 + 16.$$

However, the equation has only finitely many integer solutions by Siegel's theorem. This is a contradiction. Hence the proof is complete. □

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