DC programming and its Lagrange-type duality

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Introduction

Mathematical programming is a method to find the minimum value or a point which gives the minimum value under certain condition. A function to minimize is said to be an objective function, and a given condition is said to be a constraint. In this paper, we treat the following mathematical programming problem:

$$\text{(P)} \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0,\ \forall i = 1, \cdots, m, \end{array}$$

where $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Functions g_i are said to be constraint functions. Mathematical programming problem is said to be a linear programming problem, a convex programming problem when the objective function and the constraint functions are affine functions, convex functions, respectively.

In order to calculate the minimum value of mathematical programming problems, we usually use duality problems. Especially in a convex programming problem, the following Lagrange duality problem is famous:

$$\max_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$$

In the duality problem, $\inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}$ is a convex programming problem which has no constraint. Therefore it is comparatively easy to solve the problem by using subdifferential. Moreover, under some assumption, the optimal value of the primal convex programming problem is equal to the optimal value of its Lagrange duality problem. This assumption is said to be a constraint qualification. The Slater constraint qualification is the most famous. In 2008, M. A. Goberna, V. Jeyakumar and M. A. Lopez show that the Farkas Minkowski property (FM, in short) is a necessary and sufficient constraint qualification that the optimal value of the primal convex programming problem is equal to its Lagrange duality problem([4]).

The main content of this paper is to consider Lagrange-type duality in DC programming problems. The function which is represented difference of two convex functions is said to be a difference of convex functions (DC function, in short). A mathematical programming problem whose the objective function and the constraint functions are DC function is said to be a DC programming problem. Every function whose second partial derivatives are continuous everywhere is a DC function. This fact shows every function in C^2 is a DC function. Therefore the class of DC functions is quite wide. DC programming problems are represented as the following form:

$$(\mathrm{P}) \quad \begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & f_i(x) - g_i(x) \leq 0, \ \forall i = 1, \cdots, m. \end{array}$$

J. E. Martinez-Legaz and M. Volle give constraint qualifications that the optimal value of primal DC programming problem is equal to the optimal value of its Lagrange-type duality problem([8]). However, these constraint qualifications are not necessary and sufficient constraint qualifications. In this paper, we provide new constraint qualifications for Lagrange-type duality.

The outline of the paper is as follows: In Section 1, we introduce definitions and preliminary results of convex analysis, convex programming problems and DC functions. In Section 2, we introduce previous constraint qualifications for Lagrange-type duality in DC programming problem and canonical DC programming problems. Moreover, we give a new constraint qualification for a Lagrange-type duality theorem in a DC programming problem. In section 3, we give another Lagrange-type duality theorem by using the following fact: *Maximum function of finitely many DC functions is also DC function.* Finally, we compare two Lagrange-type duality theorems which we provided.

Contents

Chapter 1

Preliminaries

In this chapter, we introduce some notation and preliminaries.

1.1 Convex analysis

The n-dimensional real Euclidean space will be denoted by $\mathbb{R}^n$. The inner product of two vectors x and y in $\mathbb{R}^n$ will be denoted by $\langle x, y \rangle$.

Definition 1.1. Let A is a subset of $\mathbb{R}^n$.

(i) A is said to be a convex set if $(1-\alpha)x + \alpha y \in A$ for all $x, y \in A$ and $\alpha \in (0, 1)$.

(ii) A is said to be a cone if A is nonempty and $\lambda x \in A$ for all $x \in A$ and $\lambda \geq 0$.

For a set $A \subseteq \mathbb{R}^n$, we define the closure, convex hull and conical hull of A by

$$\operatorname{cl} A = \{x \in \mathbb{R}^n \mid \exists \{x_k\} \subseteq A \text{ s.t. } x_k \to x \, (k \to \infty)\},$$

$$\operatorname{co} A = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists m \in \mathbb{N}, \exists x_i \in A, \exists \alpha_i > 0 \, (i = 1, \ldots, m) \\ \text{s.t. } x = \sum_{i=1}^{m} \alpha_i x_i \text{ and } \sum_{i=1}^{m} \alpha_i = 1 \end{array} \right\} \text{ and}$$

$$\operatorname{cone} A = \{x \in \mathbb{R}^m \mid \exists y \in A, \exists \alpha \geq 0 \text{ s.t. } x = \alpha y\} \cup \{0\},$$

respectively.

Theorem 1.1. For $x_1, \ldots, x_m \in \mathbb{R}^n$, $\operatorname{cone} \operatorname{co}\{x_1, \ldots, x_m\}$ is a closed set.

The following separation theorem is a very important theorem in convex analysis.

Theorem 1.2. (separation theorem) For a nonempty convex set $A \subseteq \mathbb{R}^n$ and $x \notin \operatorname{cl} A$, the following holds:

$$\exists a \in \mathbb{R}^n \setminus \{0\}, \exists \alpha \in \mathbb{R} \text{ s.t. } \forall y \in A, \langle a, x \rangle < \alpha \leq \langle a, y \rangle .$$

In this article, we consider the extended real $\mathbb{R} \cup \{+\infty, -\infty\}$, where $+\infty$ and $-\infty$ satisfy the following conditions:

(i) For all $x \in \mathbb{R}$, $x + (+\infty) = (+\infty) + x = +\infty$.

(ii) For all $x \in \mathbb{R}$, $x + (-\infty) = (-\infty) + x = -\infty$.

(iii) For all $t > 0$, $t \cdot (+\infty) = +\infty$.

(iv) For all $t < 0$, $t \cdot (+\infty) = -\infty$.

(v) $(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$.

(vi) For all $x \in \mathbb{R}$, $-\infty < x < +\infty$.

(vii) $(+\infty) - (+\infty) = +\infty$.

(viii) $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$.

Definition 1.2. Let $\{\alpha_k\}$ be a sequence of $\mathbb{R} \cup \{+\infty, -\infty\}$ and $\overline{\alpha}$, $\underline{\alpha} \in \mathbb{R} \cup \{+\infty, -\infty\}$. $\overline{\alpha}$ is said to be the limit superior if the following condition holds:

(i) For each $\alpha \in \mathbb{R}$, where satisfy $\alpha > \overline{\alpha}$, the cardinality of $\{k \in \mathbb{N} \mid \alpha_k > \alpha\}$ is finite.

(ii) For each $\alpha \in \mathbb{R}$, where satisfy $\alpha < \overline{\alpha}$, the cardinality of $\{k \in \mathbb{N} \mid \alpha_k > \alpha\}$ is infinite.

Then we denote $\overline{\alpha} = \limsup_{k\to\infty} \alpha_k$. In the same way, $\underline{\alpha}$ is said to be the limit inferior if the following condition holds:

(i) For each $\alpha \in \mathbb{R}$, where satisfy $\alpha < \underline{\alpha}$, the cardinality of $\{k \in \mathbb{N} \mid \alpha_k < \alpha\}$ is finite.

(ii) For each $\alpha \in \mathbb{R}$, where satisfy $\alpha > \underline{\alpha}$, the cardinality of $\{k \in \mathbb{N} \mid \alpha_k < \alpha\}$ is infinite.

Then we denote $\underline{\alpha} = \liminf_{k\to\infty} \alpha_k$.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$. We say that f is upper semi continuous if

$$f(x) \geq \limsup_{k\to\infty} f(x_k)$$

for each $x \in \mathbb{R}^n$ and $\{x_k\} \subseteq \mathbb{R}^n$ converges to x. In the same way, we say that f is lower semi continuous if

$$f(x) \leq \liminf_{k\to\infty} f(x_k)$$

for each $x \in \mathbb{R}^n$ and $\{x_k\} \subseteq \mathbb{R}^n$ converges to x. For $\alpha \in \mathbb{R}$,

$$\{f \le \alpha\} = \{x \in \mathbb{R}^n \mid f(x) \le \alpha\}$$

is said to be the level set of f at α. It is well-known that f is lower semi continuous if and only if $\{f \le \alpha\}$ are closed sets for all $\alpha \in \mathbb{R}$. For an extended real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$, the domain and the epigraph of f are defined by

$$\begin{aligned} \operatorname{dom} f &= \{x \in \mathbb{R}^n \mid f(x) < +\infty\}, \\ \operatorname{epi} f &= \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \operatorname{dom} f, f(x) \le r\}. \end{aligned}$$

Definition 1.3. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$.

(i) f is said to be a convex function if $\operatorname{epi} f$ is a convex set.

(ii) f is said to be a closed function if $\operatorname{epi} f$ is a closed set.

(iii) f is said to be a proper function if $\operatorname{dom} f \ne \emptyset$ and $f(x) > -\infty$ for each $x \in \mathbb{R}^n$.

Remark 1.1. It is well known that a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function if and only if

$$f((1-\alpha)x + \alpha y) \le (1-\alpha)f(x) + \alpha f(y), \ \forall x, y \in \mathbb{R}^n, \ \forall \alpha \in (0, 1).$$

Definition 1.4. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ is said to be a concave function if $-f$ is a convex function.

Theorem 1.3. Let $f, f_i : \mathbb{R}^n \to \mathbb{R}$ be convex functions for each $i = 1, \dots, m$ and $\alpha > 0$. Then the following functions are convex functions:

(i) $\sum_{i=1}^{m} f_i$.

(ii) αf.

(iii) $\max_{i=1,\dots,m} f_i$.

Theorem 1.4. ([12]) Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be real-valued convex functions for each $i = 1, \dots, m$. Then,

$$\operatorname{epi}(\max_{i=1,\cdots,m} f_i)^* = \operatorname{co}\Big(\bigcup_{i=1}^{m} \operatorname{epi} f_i^*\Big). \tag{1.1}$$

Definition 1.5. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then the conjugate function of f is defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - f(x)\}, \ \forall y \in \mathbb{R}^n.$$

It is well-known that f^* is a lower semi continuous proper convex function if f is a proper convex function. Moreover $f^{**} = f$ if f is lower semi continuous. The indicator function of $A \subseteq \mathbb{R}^n$ is denoted by δ_A, i.e.

$$\delta_A(x) = \begin{cases} 0 & (x \in A) \\ +\infty & (x \notin A). \end{cases}$$

Next, we view the relationship between a convex function and its gradient.

Definition 1.6. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $x \in \operatorname{dom} f$.

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t} \quad (i = 1, \ldots, m)$$

is said to be a partial differential coefficient of f at x, where e_i is the ith unit vector, and

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)$$

is said to be the gradient of f at x. Moreover, f is said to be differentiable at x if

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - \langle \nabla f(x), h \rangle}{\| h \|} = 0.$$

Theorem 1.5. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function, where $\operatorname{dom} f$ is an open convex set. If f is differentiable on $\operatorname{dom} f$, then the following statements are equivalent:

(i) f is a convex function.

(ii) $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x)$ for each $x, y \in \operatorname{dom} f$ with $x \neq y$.

Definition 1.7. For each $x \in \operatorname{dom} f$, the subdifferential of the function f at x is defined by

$$\partial f(x) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x \rangle + f(x) \leq f(y), \forall y \in \mathbb{R}^n\}.$$

The function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$. Moreover, We say that f is subdifferentiable if $\partial f(x) \neq \emptyset$ for each $x \in \operatorname{dom} f$.

Theorem 1.6. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, $x \in \operatorname{int} \operatorname{dom} f$. Then the following statements are equivalent:

(i) f is differentiable at x.

(ii) $\exists a \in \mathbb{R}^n$ s.t. $\partial f(x) = \{a\}$.

Remark 1.2. In (ii) of Theorem 1.6, if f is differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$. From this fact, the subdifferential of a convex function is an extension of its gradient.

The following theorem shows that the minimum value of a convex function can be characterized by its subdifferential.

Theorem 1.7. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function and $x \in \operatorname{dom} f$. Then the following statements are equivalent:

(i) $f(x) = \min_{x \in \mathbb{R}^n} f(x)$.

(ii) $0 \in \partial f(x)$.

About the subdifferential of the sum of two convex functions, the following results hold:

Theorem 1.8. Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous convex functions with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Then the following statements hold:

(i) $\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x)$ for each $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

(ii) If $\operatorname{epi} f^* + \operatorname{epi} g^*$ is a closed set, then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x)$$

for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

In the proof of Theorem 1.8, the following Fenchel duality is used.

Theorem 1.9. (Fenchel duality) Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semi continuous proper convex functions with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. If $\operatorname{epi} f^* + \operatorname{epi} g^*$ is a closed set, then

$$\inf_{x \in \mathbb{R}^n} \{f(x) + g(x)\} = \max_{x^* \in \mathbb{R}^n} \{-f^*(x^*) - g^*(-x^*)\}.$$

If $x \in \operatorname{dom} f$, then $f(x) + f^*(y) \geq \langle y, x \rangle$ (the *Young-Fenchel inequality*) holds for each $y \in \mathbb{R}^n$ and

$$f(x) + f^*(y) = \langle y, x \rangle \Leftrightarrow y \in \partial f(x).$$

Next, we define the tangent cone and the normal cone.

Definition 1.8. For a set $A \subseteq \mathbb{R}^n$,

$$A^* = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0, \forall x \in A\}$$

is said to be the polar cone of A.

Definition 1.9. Let $S \subseteq \mathbb{R}^n$ and $\bar{x} \in S$.

$$T_S(\bar{x}) = \left\{ y \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\{\alpha_k\} \subseteq [0, +\infty),\ \exists\{x_k\} \subseteq S \\ \text{s.t. } \lim\limits_{k\to+\infty} \alpha_k(x_k - \bar{x}) = y,\ \lim\limits_{k\to+\infty} x_k = \bar{x} \end{array} \right\}$$

is said to be the tangent cone of S at $\bar{x}$. Moreover,

$$N_S(\bar{x}) = (T_S(\bar{x}))^*$$

is said to be the normal cone of S at $\bar{x}$, i.e.

$$N_S(\bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y\rangle \le 0,\ \forall y \in T_S(\bar{x})\}.$$

Theorem 1.10. Let $S \subseteq \mathbb{R}^n$ be a convex set and $\bar{x} \in S$. Then the following statements hold:

(i) $N_S(\bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, x - \bar{x}\rangle \le 0,\ \forall x \in S\}$.

(ii) $\partial\delta_S(\bar{x}) = N_S(\bar{x})$.

(iii) $N_S(\bar{x}) = \{x^* \in \mathbb{R}^n \mid \delta_S^*(x^*) = \langle x^*, \bar{x}\rangle\}$.

Theorem 1.11. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, $S \subseteq \mathbb{R}^n$ be a nonempty set and $\bar{x} \in S$. If $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is a closed set, then the following statements are equivalent:

(i) $f(\bar{x}) = \min\limits_{x\in S} f(x)$.

(ii) $0 \in \partial f(\bar{x}) + N_S(\bar{x})$.

Definition 1.10. For two extended real-valued functions f, $g : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, the infimal convolution of f and g is defined by

$$(f \oplus g)(x) = \inf_{x_1+x_2=x} \{f(x_1) + g(x_2)\},\ \forall x \in \mathbb{R}^n.$$

From Theorem 1.8, for extended real-valued convex functions $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \dots, m$, if $\bigcap\limits_{i=1}^{m} \operatorname{dom} f_i \neq \emptyset$, then

$$\partial(f_1 + \cdots + f_m)(x) = \partial f_1(x) + \cdots + \partial f_m(x) \tag{1.2}$$

for all $x \in \bigcap\limits_{i=1}^{m} \operatorname{dom} f_i$. Moreover the following statements hold:

Theorem 1.12. ([9]) Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ for each $i = 1, \dots, m$. If $\bigcap\limits_{i=1}^{m} \operatorname{dom} f_i \neq \emptyset$, then the following statements holds:

(i) For each $y \in \partial(f_1 + \cdots + f_m)(x)$, there exists $y_i \in \partial f_i(x)\,(i = 1, \ldots, m)$ such that

$$(f_1 + \cdots + f_m)^*(y) = f_1^*(y_1) + \cdots + f_m^*(y_m). \tag{1.3}$$

(ii) For each $y \in \partial(f_1 + \cdots + f_m)(x)$,

$$(f_1 + \cdots + f_m)^*(y) = (f_1^* \oplus \cdots \oplus f_m^*)(y), \tag{1.4}$$

and the infimal convolution is attained for all y.

(iii) If (1.4) hold, then

$$\mathrm{epi}(f_1 + \cdots + f_m)^* = \mathrm{epi}\, f_1^* + \cdots + \mathrm{epi}\, f_m^*. \tag{1.5}$$

1.2 Convex programming problem

In this section, we consider the following convex programming problem:

$$\begin{array}{cl} \text{minimize} & f(x) \\ \text{s.t.} & g_i(x) \leq 0,\ i \in I, \end{array}$$

where I is an index set and $f, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are convex functions for each $i \in I$. In order to calculate the minimum value of this convex programming problem, Lagrange duality is effective:

$$\max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i \in I} \lambda_i g_i(x)\}.$$

In this duality problem, $\inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i \in I} \lambda_i g_i(x)\}$ is a convex programming problem which has no constraint. For example, we use Theorem 1.7, $\inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i \in I} \lambda_i g_i(x)\}$ is comparatively easy to solve. Moreover, under some assumption, the following statement holds:

$$\inf_{g_i(x) \leq 0} f(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i \in I} \lambda_i g_i(x)\}.$$

This assumption is said to be a constraint qualification. The following condition is a well-known constraint qualification, called the Slater constraint qualification.

Definition 1.11. Let $g_i : \mathbb{R}^n \to \mathbb{R}$ be convex function for all $i = 1, \ldots, m$ and $C \subseteq \mathbb{R}^n$. The system $\{g_i \leq 0,\ i = 1, \ldots, m\}$ is said to satisfy Slater constraint qualification on C if

$$\exists x_0 \in C \text{ s.t. } g_i(x_0) < 0\,(\forall i = 1, \ldots, m).$$

Theorem 1.13. Let $g_i : \mathbb{R}^n \to \mathbb{R}$ be convex function for all $i = 1, \ldots, m$, $C \subseteq \mathbb{R}^n$, $S = \{x \in C \mid g_i(x) \leq 0, \forall i = 1, \ldots, m\}$ and the system $\{g_i \leq 0, i = 1, \ldots, m\}$ satisfies the Slater constraint qualification on C. Then the following condition holds for all lower semi continuous proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$:

$$\inf_{x \in S} f(x) = \max_{\lambda_i \geq 0} \inf_{x \in C} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$$

The Slater constraint qualification is easy to check. However, the Slater constraint qualification is not a necessary condition that the optimal value of primal convex programming problem is equal to the optimal value of Lagrange duality problem. The following Farkas-minkowski is a necessary and sufficient constraint qualification that the optimal value of primal convex programming problem is equal to its Lagrange duality problem.

Definition 1.12. ([4]) Let I be a inedex set, $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semi continuous convex function for all $i \in I$, $C \subseteq \mathbb{R}^n$ be a closed convex set. We say that the system $\{g_i \leq 0, i \in I\}$ is Farkas-Minkowski (FM, in short) if

$$\operatorname{cone} \operatorname{co} \bigcup_{i \in I} \operatorname{epi} g_i^* + \operatorname{epi} \delta_C^* \text{ is closed set.}$$

Theorem 1.14. Let $g_i : \mathbb{R}^n \to \mathbb{R}$ be a convex function for each $i = 1, \ldots, m$ and C be a closed convex set. If $\{g_i \leq 0, i = 1, \ldots, m\}$ satisfies the Slater constraint qualification on C, then $\{g_i \leq 0, i \in I\}$ is FM, i.e.

$$\operatorname{cone} \operatorname{co} \bigcup_{i=1}^{m} \operatorname{epi} g_i^* + \operatorname{epi} \delta_C^* \text{ is closed set.}$$

Theorem 1.15. (Goberna M. A., Jeyakumar V., Lppez M. A., [4]) Let I be a index set, $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semi continuous convex functions for all $i \in I$ and C be a closed convex set. Moreover, each g_i is continuous at least at one point of $S = \{x \in C \mid g_i(x) \leq 0, i \in I\}$. Then the following statements are equivalent:

(i) The system $\{g_i \leq 0, i \in I\}$ is FM.

(ii) For every lower semi continuous proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $A \cap \operatorname{dom} f \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is a closed set,

$$\inf_{x \in S} f(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in C} \{f(x) + \sum_{i \in I} \lambda_i g_i(x)\}.$$

(iii) For every $v \in \mathbb{R}^n$,

$$\inf_{x \in S} \langle v, x \rangle = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in C} \{\langle v, x \rangle + \sum_{i \in I} \lambda_i g_i(x)\}.$$

Theorem 1.15 shows that FM is a necessary and sufficient condition that the optimal value of the primal convex programming problem is equal to the optimal value of the Lagrange duality problem.

1.3 DC function

In this section, we examine a DC function and its property.

Definition 1.13. A function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be a difference of convex function (DC function, in short) if there exists two convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \to \mathbb{R}$ such that $h = f - g$.

Theorem 1.16. ([7]) The following statements hold:

(i) Every function $f : \mathbb{R}^n \to \mathbb{R}$ whose second partial derivatives are continuous everywhere is a DC function, that is, every function in C^2 is a DC function.

(ii) Let C be a compact convex subset of C. Then, every continuous function on C is the limit of a sequence of DC functions which converges uniformly on C, i.e., for all continuous function $c : C \to \mathbb{R}$ and for all ε, there exists a DC function $f : C \to \mathbb{R}$ such that $|c(x) - f(x)| \leq \varepsilon$ for all $x \in C$.

Theorem 1.16 inspires us that the class of DC functions is quite wide. To prove (i) of Theorem 1.16, we provide the following definition and theorem without proof.

Definition 1.14. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is a locally DC function if

$$\forall x \in \mathbb{R}^n,\ \exists \varepsilon > 0 \text{ s.t. } f \text{ is a DC function on } B(x, \varepsilon).$$

Theorem 1.17. (Hartman, [13]) Every locally DC function $f : \mathbb{R}^n \to \mathbb{R}$ is a DC function.

Proof of (i) of Theorem 1.16. The elements of the Hessian $\nabla^2 f$ of f are bounded on every closed neighborhood $B(x_0, \varepsilon)$. Therefore, for sufficient large $\mu > 0$, the function $f(x) + \mu\|x\|^2$ is a convex function on $B(x_0, \varepsilon)$ since its Hessian $\nabla^2 f(x) + 2\mu E$ is positive semidefinite on $B(x_0, \varepsilon)$ for sufficiently large μ. Then

$$f(x) = (f(x) + \mu\|x\|^2) - \mu\|x\|^2$$

is obviously DC on $B(x_0, \varepsilon)$. From Theorem 1.17, f is a DC function. □

Theorem 1.18. Let $h, h_i : \mathbb{R}^n \to \mathbb{R}(i = 1, \ldots, m)$ be DC functions. Then the following functions are DC functions.

(i) $\sum_{i=1}^{m} \lambda_i h_i$, where $\lambda_i \in \mathbb{R}$ for each $i = 1, \ldots, m$.

(ii) $\max_{i=1,\dots,m} h_i$ and $\min_{i=1,\dots,m} h_i$.

(iii) $|h|$.

(iv) $\prod_{i=1}^{m} h_i$.

Proof. (i) For each $i = 1, \dots, m$, there exist convex functions $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$ such that $h_i = f_i - g_i$. From this,

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i h_i &= \sum_{i=1}^{m} \lambda_i (f_i - g_i) \\ &= \sum_{\lambda_i \geq 0} \lambda_i f_i - \sum_{\lambda_i < 0} \lambda_i g_i - \left(\sum_{\lambda_i \geq 0} \lambda_i g_i - \sum_{\lambda_i < 0} \lambda_i f_i \right).\end{aligned}$$

Both of $\sum_{\lambda_i \geq 0} \lambda_i f_i - \sum_{\lambda_i < 0} \lambda_i g_i$ and $\sum_{\lambda_i \geq 0} \lambda_i g_i - \sum_{\lambda_i < 0} \lambda_i f_i$ are convex functions. Therefore $\sum_{i=1}^{m} \lambda_i h_i$ is a DC function.

(ii) For each $i = 1, \dots, m$, there exist convex functions $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$ such that $h_i = f_i - g_i$. From this,

$$\begin{aligned}\max_{i=1,\dots,m} h_i = \max_{i=1,\dots,m} \{f_i - g_i\} &= \max_{i=1,\dots,m} \{f_i + \sum_{j \neq i} g_j - \sum_{j=1}^{m} g_j\} \\ &= \max_{i=1,\dots,m} \{f_i + \sum_{j \neq i} g_j\} - \sum_{j=1}^{m} g_j.\end{aligned}$$

Both of $\max_{i=1,\dots,m} \{f_i + \sum_{j \neq i} g_j\}$ and $\sum_{j=1}^{m} g_j$ are convex functions. Therefore $\max_{i=1,\dots,m} h_i$ is a DC function. $\min_{i=1,\dots,m} h_i$ is same.

(iii)There exist convex functions $f, g : \mathbb{R}^n \to \mathbb{R}$ such that $h = f - g$. From this,

$$|h| = 2 \max\{f, g\} - (f + g).$$

Both of $2 \max\{f, g\}$ and $f + g$ are convex functions. Therefore $|h|$ is a DC function.

(iv) We prove the case of $m = 2$. As noted Lemma 1.2 after this proof, for each $i = 1, 2$, there exist nonnegative convex functions $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$ such that $h_i = f_i - g_i$. From this,

$$h_1 h_2 = (f_1 - g_1)(f_2 - g_2) = \frac{1}{2}[(f_1 + f_2)^2 + (g_1 + g_2)^2] - \frac{1}{2}[(f_1 + g_2)^2 + (f_2 + g_1)^2].$$

As noted Lemma 1.3 after this proof, both of $\frac{1}{2}[(f_1 + f_2)^2 + (g_1 + g_2)^2]$ and $\frac{1}{2}[(f_1 + g_2)^2 + (f_2 + g_1)^2]$ are convex functions. Therefore $h_1 h_2$ is a DC function. $\square$

Lemma 1.1. Let $h : \mathbb{R}^n \to \mathbb{R}$ is a DC function. Then there exist a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a nonnegative convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $h = f - g$. Also, there exists nonnegative convex function $f : \mathbb{R}^n \to \mathbb{R}$ and convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $h = f - g$.

Proof. From h is a DC function, there exist convex functions $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ such that $h = h_1 - h_2$. Since h_2 is a convex function, there exist $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $\langle a, \cdot \rangle - b \leq h_2$. Moreover,

$$h = (h_1 - \langle a, \cdot \rangle - b) - (h_2 - \langle a, \cdot \rangle - b).$$

From this, we put $f = h_1 - \langle a, \cdot \rangle - b$ and $g = h_2 - \langle a, \cdot \rangle - b$, then f and g are convex functions and g is nonnegative. □

Lemma 1.2. Let $h : \mathbb{R}^n \to \mathbb{R}$ is a DC function. Then there exist two nonnegative convex functions $f, g : \mathbb{R}^n \to \mathbb{R}$ such that $h = f - g$.

Proof. It is clear that

$$h = \max\{h, 0\} + \min\{h, 0\}.$$

From (ii) of Theorem 1.18, both of $\max\{h, 0\}$ and $\min\{h, 0\}$ are DC functions. Moreover, $\max\{h, 0\}$ is nonnegative and $\min\{h, 0\}$ is nonpositive. Therefore there exist convex functions $f_1, f_2, g_1, g_2 : \mathbb{R}^n \to \mathbb{R}$ such that $\max\{h, 0\} = f_1 - g_1 \geq 0$ and $\min\{h, 0\} = f_2 - g_2 \leq 0$. Without loss of generality, g_1 and f_2 are nonnegative, from Lemma 1.1. Moreover,

$$h = \max\{h, 0\} + \min\{h, 0\} = (f_1 - g_1) + (f_2 - g_2) = (f_1 - g_1) - (g_2 - f_2).$$

Since $f_1 - g_1$, $g_2 - f_2$, g_1 and f_2 are nonnegative, f_1 and g_2 are nonnegative. Therefore

$$h = (f_1 + f_2) - (g_1 + g_2),$$

and $f_1 + f_2$ and $g_1 + g_2$ are nonnegative convex functions. □

Lemma 1.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative convex function. Then g^2 is a convex function.

Chapter 2

DC programming problem

In this chapter, we study the following DC programming problem (P):

$$(\text{P})\quad \begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & f_i(x) - g_i(x) \leq 0,\ \forall i = 1, 2, \ldots, m, \end{array}$$

where $f_i, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are lower semi continuous proper convex function for each $i = 0, 1, \ldots, m$, and assume that the set of all feasible solutions $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \leq 0, \forall i = 1, 2, \ldots, m\}$ is nonempty. At first, we introduce previous Lagrange-type duality results for DC programming and canonical programming. Secondly, we show a Lagrange-type duality theorem for a DC programming problem, which is a generalization of previous ones when all constraint functions are real-valued. Finally, we apply this result to a DC programming problem with reverse convex constraints.

2.1 Lagrange-type duality theorem in DC programming problem

In this section, we give previous Lagrange-type duality results for (P). At first, we give a duality theorem with no constraint DC programming.

Theorem 2.1. (Toland duality) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous proper convex function, $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then

$$\inf_{x \in \mathbb{R}^n} \{f(x) - g(x)\} = \inf_{x^* \in \mathbb{R}^n} \{g^*(x^*) - f^*(x^*)\}.$$

Theorem 2.2 and Theorem 2.3 give duality for the DC programming problem (P). We adopt the conventions $0 \cdot (+\infty) = +\infty$ and $0 \cdot (-\infty) = 0$ in Theorem 2.2 and Theorem 2.3.

Theorem 2.2. (J.-E. Martinez-Legaz, M. Volle, [8]) Let $f_i : \mathbb{R}^n \to \mathbb{R}$, be a convex for each $i = 0, 1, , \ldots, m$, let $g_0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ satisfies $g_0 = g_0^{**}$, and let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ be subdifferentiable on S for each $i = 1, \ldots, m$. If

$$\text{for each } (x_1^*, \ldots, x_m^*) \in \prod_{i=1}^{m} \{g_i^* - f_i^* \leq 0\}, \text{ there exists } \bar{x} \in \mathrm{dom} f_0$$
$$\text{such that } f_i(\bar{x}) - \langle \bar{x}, x_i^* \rangle + g_i^*(x_i^*) < 0 \text{ for each } i = 1, \cdots, m,$$

then

$$\inf_{f_i(x)-g_i(x)\leq 0} \{f_0(x) - g_0(x)\} = \inf_{x^* \in \mathrm{dom} g_0^*} \inf_{g_i^*(x_i^*) - f_i^*(x_i^*) \leq 0} \max_{\lambda \in \mathbb{R}_+^m}$$
$$\left\{ g_0^*(x^*) + \sum_{i=1}^{m} \lambda_i g_i^*(x_i^*) - \left(f_0 + \sum_{i=1}^{m} \lambda_i f_i \right)^* \left(x^* + \sum_{i=1}^{m} \lambda_i x_i^* \right) \right\}.$$

Theorem 2.3. (J.-E. Martinez-Legaz, M. Volle, [8]) Let $f_i : \mathbb{R}^n \to \mathbb{R}$, be a convex for each $i = 0, 1, \ldots, m$, let $g_0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ satisfies $g_0 = g_0^{**}$, and let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ be subdifferentiable on S for each $i = 1, \ldots, m$. If

for each $(x_1^*, \ldots, x_m^*) \in \Omega = \{(x_1^*, \ldots, x_m^*) \in \mathbb{R}^{nm} \mid \partial g_1^*(x_1^*) \cap \cdots \cap \partial g_m^*(x_m^*) \neq \emptyset\}$, there exists $x_0 \in \mathrm{dom} f_0$ such that $f_i(x_0) - \langle x_0, x_i^* \rangle + g_i^*(x_i^*) < 0$ for each $i = 1, \ldots, m$,

then

$$\inf_{f_i(x)-g_i(x)\leq 0} \{f_0(x) - g_0(x)\} = \inf_{(x^*, x_1^*, \cdots, x_m^*) \in \mathrm{dom} g_0^* \times \Omega} \max_{\lambda \in \mathbb{R}_+^m}$$
$$\left\{ g_0^*(x^*) + \sum_{i=1}^{m} \lambda_i g_i^*(x_i^*) - \left(f_0 + \sum_{i=1}^{m} \lambda_i f_i \right)^* \left(x^* + \sum_{i=1}^{m} \lambda_i x_i^* \right) \right\}.$$

Remark 2.1. The right-hand side of Theorem 2.2 and Theorem 2.3 can transform to a formulation of Lagrange-type duality. Indeed,

$$\begin{aligned}
& g_0^*(x^*) + \sum_{i=1}^{m} \lambda_i g_i^*(x_i^*) - \left(f_0 + \sum_{i=1}^{m} \lambda_i f_i \right)^* \left(x^* + \sum_{i=1}^{m} \lambda_i x_i^* \right) \\
&= g_0^*(x^*) + \sum_{i=1}^{m} \lambda_i g_i^*(x_i^*) - \sup_{x \in \mathbb{R}^n} \left\{ \left\langle x, x^* + \sum_{i=1}^{m} \lambda_i x_i^* \right\rangle - \left(f_0 + \sum_{i=1}^{m} \lambda_i f_i \right)(x) \right\} \\
&= \inf_{x \in \mathbb{R}^n} \left\{ f_0(x) - \langle x, x^* \rangle + g_0^*(x^*) + \sum_{i=1}^{m} \lambda_i (f_i(x) - \langle x, x_i^* \rangle + g_i^*(x_i^*)) \right\}.
\end{aligned}$$

Next, we consider the following canonical DC programming problem (Q):

$$(\mathrm{Q}) \quad \begin{array}{ll} \text{minimize} & \langle a, x \rangle \\ \text{subject to} & f(x) \leq 0,\ g(x) \geq 0, \end{array}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are convex functions and $a \in \mathbb{R}^n$. It is well-known that the DC programming problem (P), where primal function and constraint functions are real-valued functions, can be transformed into canonical DC programming as follows.

The optimal value of (P) is

$$\inf\{f_0(x) - g_0(x) \mid f_i(x) - g_i(x) \leq 0, \forall i = 1, \ldots, m\}.$$

By using an additional variable x_{n+1}, this optimal value is rewritten as follows:

$$\inf\{x_{n+1} \mid f_0(x) - g_0(x) - x_{n+1} \leq 0, \ f_i(x) - g_i(x) \leq, \forall i = 1, \ldots, m\}.$$

We define a function $h : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$h(x, x_{n+1}) = \max\{f_0(x) - g_0(x) - x_{n+1}, \ f_i(x) - g_i(x) \, (i = 1, \ldots, m)\}.$$

From Theorem 1.18, h is a DC function, therefore there exists convex functions $f, g : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $h = f - g$. It is clear that

$$f_0(x) - g_0(x) - x_{n+1} \leq 0 \text{ and } f_i(x) - g_i(x) \leq 0, \forall i = 1, \ldots, m$$

if and only if

$$h(x, x_{n+1}) \leq 0.$$

Therefore

$$\begin{aligned}
&\inf\{x_{n+1} \mid f_0(x) - g_0(x) - x_{n+1} \leq 0, \ f_i(x) - g_i(x) \leq 0, \forall i = 1, \ldots, m\} \\
&= \inf\{x_{n+1} \mid h(x, x_{n+1}) \leq 0\} \\
&= \inf\{x_{n+1} \mid f(x, x_{n+1}) - g(x, x_{n+1}) \leq 0\} \\
&= \inf\{x_{n+1} \mid f(x, x_{n+1}) \leq x_{n+2}, \ x_{n+2} \leq g(x, x_{n+1}), \ x_{n+2} \in \mathbb{R}\} \\
&= \inf\{x_{n+1} \mid f(x, x_{n+1}) - x_{n+2} \leq 0, \ 0 \leq g(x, x_{n+1}) - x_{n+2}, \ x_{n+2} \in \mathbb{R}\}.
\end{aligned}$$

Using another additional variable x_{n+2}, we put functions $F, G : \mathbb{R}^{n+2} \to \mathbb{R}$ and $a \in \mathbb{R}^{n+2}$ by

$$\begin{aligned}
F(x, x_{n+1}, x_{n+2}) &= f(x, x_{n+1}) - x_{n+2}, \\
G(x, x_{n+1}, x_{n+2}) &= g(x, x_{n+1}) - x_{n+2} \text{ and} \\
a &= (0, \ldots, 0, 1, 0).
\end{aligned}$$

Then

$$\begin{aligned}
&\inf\{x_{n+1} \mid f(x, x_{n+1}) - x_{n+2} \leq 0, \ 0 \leq g(x, x_{n+1}) - x_{n+2}, \ x_{n+2} \in \mathbb{R}\} \\
&= \inf\{\langle a, x \rangle \mid x \in \mathbb{R}^{n+2}, \ F(z) \leq 0, \ G(z) \geq 0\}.
\end{aligned}$$

Theorem 2.4 and Theorem 2.5 give duality in the canonical DC programming problem (Q).

Theorem 2.4. (Y. Fujiwara, D. Kuroiwa, [3]) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, and let $a \in \mathbb{R}^n$, let $S = \{x \in \mathbb{R}^n \mid f(x) \leq 0,\ g(x) \geq 0\}$ be nonempty and let $\bigcup_{x \in S} \partial g(x) \subseteq A$. If for each $z \in A \cap \operatorname{dom} g^*$, $\{f \leq 0\} \cap \{\langle -z, \cdot \rangle + g^*(z) \leq 0\}$ is nonempty and $\operatorname{cone}\operatorname{co}(\operatorname{epi} f^* \cup \{-z\} \times [-g^*(z), +\infty) \cup \{0\} \times [0, +\infty))$ is closed, then

$$\inf_{f(x) \leq 0, g(x) \geq 0} \langle a, x \rangle = \inf_{y \in A} \sup_{\lambda, \mu \geq 0} \inf_{x \in \mathbb{R}^n} \{\langle a, x \rangle + \lambda f(x) + \mu(\langle -y, x \rangle + g^*(y))\}$$

holds and the supremum on λ, $\mu \geq 0$ being attained for all $y \in A$.

Theorem 2.5. (Y. Fujiwara, D. Kuroiwa, [3]) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, and let $a \in \mathbb{R}^n$, $S = \{x \in \mathbb{R}^n \mid f(x) \leq 0,\ g(x) \geq 0\}$ is nonempty, $Y' = \{y \in \mathbb{R}^n \mid \{f \leq 0\} \cap \{\langle y, \cdot \rangle > g^*(y)\} \neq \emptyset\}$ and $\bigcup_{x \in S} \partial g(x) \subseteq A \subseteq Y'$. If $\operatorname{cone}\operatorname{epi} f^* + \{0\} \times [0, +\infty)$ is closed, then

$$\inf_{f(x) \leq 0, g(x) \geq 0} \langle a, x \rangle = \inf_{y \in A} \sup_{\lambda, \mu \geq 0} \inf_{x \in \mathbb{R}^n} \{\langle a, x \rangle + \lambda f(x) + \mu(\langle -y, x \rangle + g^*(y))\}$$

holds and the supremum on λ, $\mu \geq 0$ being attained for all $y \in A$.

In [3], Theorem 2.4 and Theorem 2.5 proved another way. In fact, Theorem 2.5 can prove by using Theorem 2.4.

Proof of Theorem 2.5. We assume the assumption of Theorem 2.5, i.e. assume that

$$\bigcup_{x \in S} \partial g(x) \subseteq A \subseteq Y'$$

hold and

$$\operatorname{cone}\operatorname{epi} f^* + \{0\} \times [0, +\infty) \text{ is closed.}$$

For each $z \in A$,

$$\exists x_0 \in \{f \leq 0\} \text{ s.t. } -\langle x_0, z \rangle + g^*(z) < 0.$$

Therefore the system $\{-\langle x, z \rangle + g^*(z) \leq 0,\ x \in \mathbb{R}^n\}$ satisfy Slater condition on $\{f \leq 0\}$. From Theorem 1.14,

$$\operatorname{cone}\operatorname{epi}(-\langle \cdot, z \rangle + g^*(z))^* + \operatorname{epi} \delta^*_{\{f \leq 0\}}$$

is a closed set. For each $v \in \mathbb{R}^n$,

$$\inf_{f(x) \leq 0, -\langle x, z \rangle + g^*(z) \leq 0} \langle x, v \rangle = \max_{\lambda \geq 0} \inf_{x \in \{f \leq 0\}} \{\langle x, v \rangle + \lambda(-\langle x, z \rangle + g^*(z))\}$$

hold, from Theorem 1.15. From $\operatorname{cone}\operatorname{epi} f^* + \{0\} \times [0, +\infty)$ is a closed set, by using Theorem 1.14,

$$\begin{aligned}
&\inf_{x \in \{f \le 0\}} \{\langle x, v\rangle + \lambda(-\langle x, z\rangle + g^*(z))\} \\
&= \inf_{f(x) \le 0} \{\langle x, v\rangle + \lambda(-\langle x, z\rangle + g^*(z))\} \\
&= \max_{\mu \ge 0} \inf_{x \in \mathbb{R}^n} \{\langle x, v\rangle + \lambda(-\langle x, z\rangle + g^*(z)) + \mu f(x)\}
\end{aligned}$$

for each $\lambda \ge 0$. Therefore, for each $v \in \mathbb{R}^n$,

$$\begin{aligned}
&\inf_{f(x) \le 0, -\langle x, z\rangle + g^*(z) \le 0} \langle x, v\rangle \\
&= \max_{\lambda \ge 0} \max_{\mu \ge 0} \inf_{x \in \{f \le 0\}} \{\langle x, v\rangle + \lambda(-\langle x, z\rangle + g^*(z)) + \mu f(x)\} \\
&= \max_{\lambda \ge 0, \mu \ge 0} \inf_{x \in \{f \le 0\}} \{\langle x, v\rangle + \lambda(-\langle x, z\rangle + g^*(z)) + \mu f(x)\}.
\end{aligned}$$

From Theorem 1.15,

$$\operatorname{cone}\operatorname{co}(\operatorname{epi} f^* \cup \operatorname{epi}((-\langle \cdot, z\rangle + g^*(z))^*)) + \{0\} \times [0, +\infty)$$

is a closed set. Moreover,

$$\begin{aligned}
&\operatorname{cone}\operatorname{co}(\operatorname{epi} f^* \cup \operatorname{epi}((-\langle \cdot, z\rangle + g^*(z))^*)) + \{0\} \times [0, +\infty) \\
&= \operatorname{cone}\operatorname{co}(\operatorname{epi} f^* \cup \{-z\} \times [-g^*(z), +\infty) \cup \{0\} \times [0, +\infty)).
\end{aligned}$$

Therefore the assumption of Theorem 2.4 hold, and we can use Theorem 2.4.

□

It is well-known that canonical DC programming problems are special cases of DC programming problems, so Theorems 2.2 and 2.3 have broader application areas than Theorems 2.4 and 2.5. However, the assumptions of Theorems 2.2 and 2.3 are stronger than Theorems 2.4 and 2.5 whenever the DC programming problem is canonical. In this paper, we give a Lagrange-type duality result for a general DC programming problem, which is a generalization of Theorems 2.2, 2.3, 2.4, and 2.5.

We consider the following subproblems $(\mathrm{P}(y_0,(y_i)_{i=1}^m))$ of (P):

$$(\mathrm{P}(y_0,(y_i)_{i=1}^m)) \quad \begin{array}{ll} \text{minimize} & f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) \\ \text{subject to} & f_i(x) - \langle x, y_i\rangle + g_i^*(y_i) \le 0, \forall i = 1, \ldots, m, \end{array}$$

where $(y_0, (y_i)_{i=1}^m) = (y_0, y_1, \ldots, y_m) \in \mathbb{R}^{n(m+1)}$. It is clear that all subproblems $(\mathrm{P}(y_0,(y_i)_{i=1}^m))$ are convex programming. Let $\mathrm{Val}(\mathrm{P})$ and $\mathrm{Val}(\mathrm{P}(y_0,(y_i)_{i=1}^m))$ be the minimum values of (P) and $(\mathrm{P}(y_0,(y_i)_{i=1}^m))$, respectively.

Lemma 2.1. Let $f_i, g_i : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be lower semi continuous proper convex functions for each $i = 1, \ldots, m$, $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \leq 0, \forall i = 1, \ldots, m\}$, $(y_i)_{i=1}^m \in \mathbb{R}^{nm}$ and $S(y_i)_{i=1}^m = \{x \in \mathbb{R}^n \mid f_i(x) - \langle x, y_i\rangle + g_i^*(y_i) \leq 0, \ \forall i = 1, \ldots, m\}$. Then

$$S(y_i)_{i=1}^m \subseteq S,$$

furthermore if g_i is subdifferentiable on S for each $i = 1, \ldots, m$ and $\bigcup_{x\in S}\left(\prod_{i=1}^m \partial g_i(x)\right) \subseteq D \subseteq \mathbb{R}^{nm}$, then

$$\bigcup_{(y_i)_{i=1}^m \in D} S(y_i)_{i=1}^m = S.$$

Proof. For each $x \in S(y_i)_{i=1}^m$,

$$0 \geq f_i(x) - \langle x, y_i\rangle + g_i^*(y_i) \geq f_i(x) - g_i(x)$$

by using the Young-Fenchel duality for each $i = 1, \ldots, m$. So $S(y_i)_{i=1}^m \subseteq S$. Let $z \in S$. since $\partial g_i(z)$ is a nonempty set for each $i = 1, \ldots, m$, there exists $y_i \in \partial g_i(z)$ and

$$0 \geq f_i(z) - g_i(z) = f_i(z) - \langle z, y_i\rangle + g_i^*(y_i).$$

Therefore $z \in S(y_i)_{i=1}^m$. Also we have $(y_i)_{i=1}^m \in \bigcup_{x\in S}\left(\prod_{i=1}^m \partial g_i(x)\right)$. So $z \in \bigcup_{(y_i)_{i=1}^m \in D} S(y_i)_{i=1}^m$. Therefore $S \subseteq \bigcup_{(y_i)_{i=1}^m \in D} S(y_i)_{i=1}^m$. It is clear that the opposite inclusion holds. □

Lemma 2.2 needs to the proof of Theorem 2.6.

Lemma 2.2. Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function, let A be a nonempty set, and let $B(x)$ be nonempty subsets of $\mathbb{R}^n$ for all $x \in A$. Then

$$\inf_{x\in A} \inf_{y\in B(x)} h(y) = \inf_{y\in \bigcup_{x\in A} B(x)} h(y)$$

Proof. For each $x \in A$, since $B(x) \subseteq \bigcup_{x\in A} B(x)$,

$$\inf_{y\in B(x)} h(y) \geq \inf_{y\in \bigcup_{x\in A} B(x)} h(y).$$

Therefore we have

$$\inf_{x\in A} \inf_{y\in B(x)} h(y) \geq \inf_{y\in \bigcup_{x\in A} B(x)} h(y).$$

Assume that $\inf_{x\in A}\inf_{y\in B(x)} h(y) > \inf_{y\in \bigcup_{x\in A} B(x)} h(y)$. Then there exists β such that

$$\inf_{x\in A}\inf_{y\in B(x)} h(y) > \beta > \inf_{y\in \bigcup_{x\in A} B(x)} h(y).$$

From $\beta > \inf_{y\in \bigcup_{x\in A} B(x)} h(y)$, there exists $y_0 \in \bigcup_{x\in A} B(x)$ such that $h(y_0) < \beta$, and there exist $x_0 \in A$ such that $y_0 \in B(x_0)$. Therefore

$$\inf_{x\in A}\inf_{y\in B(x)} h(y) \leq \inf_{y\in B(x_0)} h(y) \leq h(y_0) < \beta.$$

This contradicts to $\inf_{x\in A}\inf_{y\in B(x)} h(y) > \beta$. So, $\inf_{x\in A}\inf_{y\in B(x)} h(y) = \inf_{y\in \bigcup_{x\in A} B(x)} h(y)$. □

Theorem 2.6. (R. Harada, D. Kuroiwa, [2]) Let $f_i, g_i : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be lower semi continuous proper convex functions for each $i = 0, 1, \ldots, m$, $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \leq 0, \forall i = 1, \ldots, m\}$, let g_i subdifferentiable on S for each $i = 1, \ldots, m$, $\bigcup_{x\in S} \partial g_0(x) \subseteq D_0 \subseteq \mathbb{R}^n$, and $\bigcup_{x\in S}\left(\prod_{i=1}^m \partial g_i(x)\right) \subseteq D \subseteq \mathbb{R}^{nm}$. Then

$$\text{Val(P)}=\inf_{(y_0,(y_i)_{i=1}^m)\in D_0\times D} \text{Val}(\text{P}(y_0,(y_i)_{i=1}^m)).$$

Proof. For any $x \in \mathbb{R}^n$ and $y_0 \in D_0$, we have $g_0(x) + g_0^*(y_0) \geq \langle x, y_0\rangle$, that is, $f_0(x) - g_0(x) \leq f_0(x) - \langle x, y_0\rangle + g_0^*(y_0)$. By using Lemma 2.1,

$$\begin{aligned}\inf_{x\in S}\{f_0(x) - g_0(x)\} &\leq \inf_{x\in S}\{f_0(x) - \langle x, y_0\rangle + g_0^*(x)\}\\ &\leq \inf_{x\in S(y_i)}\{f_0(x) - \langle x, y_0\rangle + g_0^*(x)\}\end{aligned}$$

for any $(y_i)_{i=1}^m \in D$. This shows Val(P)$\leq \inf_{(y_0,(y_i)_{i=1}^m)\in D_0\times D}$ Val(P$(y_0,(y_i)_{i=1}^m)$). Conversely, for each $x \in S$, pick $y_i \in \partial g_i(x)$ for each $i = 1, \ldots, m$, then $f_i(x) -$

$\langle x, y_i\rangle + g_i^*(y_i) = f_i(x) - g_i(x)$. Therefore

$$\begin{aligned}
\inf_{(y_0,(y_i)_{i=1}^m)\in D_0\times D} \mathrm{Val}(\mathrm{P}(y_0,(y_i)_{i=1}^m)) &= \inf_{y_0\in D_0} \inf_{(y_i)_{i=1}^m\in D} \inf_{z\in S(y_i)_{i=1}^m} \{f_0(z) - \langle z, y_0\rangle + g_0^*(y_0)\} \\
&= \inf_{y_0\in D_0} \inf_{z\in \bigcup_{(y_i)_{i=1}^m\in D} S(y_i)_{i=1}^m} \{f_0(z) - \langle z, y_0\rangle + g_0^*(y_0)\} \\
&= \inf_{y_0\in D_0} \inf_{z\in S} \{f_0(z) - \langle z, y_0\rangle + g_0^*(y_0)\} \\
&= \inf_{z\in S} \inf_{y_0\in D_0} \{f_0(z) - \langle z, y_0\rangle + g_0^*(y_0)\} \\
&\le \inf_{z\in S} \inf_{y_0\in \bigcup_{x\in S} \partial g_0(x)} \{f_0(z) - \langle z, y_0\rangle + g_0^*(y_0)\} \\
&\le \inf_{z\in S} \inf_{y_0\in \partial g_0(z)} \{f_0(z) - \langle z, y_0\rangle + g_0^*(y_0)\} \\
&= \inf_{z\in S} \inf_{y_0\in \partial g_0(z)} \{f_0(z) - g_0(z)\} \\
&= \inf_{z\in S} \{f_0(z) - g_0(z)\} \\
&= \mathrm{Val}(\mathrm{P}).
\end{aligned}$$

The second and third equalities hold from Lemma 2.1 and Lemma 2.2, respectively. When $S \neq \emptyset$, this reverse inequality is clear since Val(P)$= +\infty$. This completes the proof. □

From this, we give the following theorem.

Theorem 2.7. (R. Harada, D. Kuroiwa, [2]) Let $f_i, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semi continuous convex functions for each $i = 0, 1, \ldots, m$, $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \le 0, \forall i = 1, \ldots, m\}$ nonempty, $\bigcup_{x\in S} \partial g_0(x) \subseteq D_0 \subseteq \mathbb{R}^n$ and $\bigcup_{x\in S} \left(\prod_{i=1}^m \partial g_i(x)\right) \subseteq D \subseteq \mathbb{R}^{nm}$. If each f_i is continuous at least at one point of $S(y_i)_{i=1}^m = \{x \in \mathbb{R}^n \mid f_i(x) - \langle x, y_i\rangle + g_i^*(y_i) \le 0, \forall i = 1, \ldots, m\}$ and

$$\mathrm{cone\,co} \bigcup_{i=1}^m (\mathrm{epi}\, f_i^* - (y_i, g_i^*(y_i))) + \{0\} \times [0, +\infty) \text{ is closed,} \tag{2.1}$$

for each $(y_i)_{i=1}^m \in D \cap \prod_{i=1}^m \mathrm{dom}\, g_i^*$, then

$$\mathrm{Val(P)} = \inf_{(y_0,(y_i)_{i=1}^m)\in D_0\times D} \max_{\lambda_i\ge 0} \inf_{x\in\mathbb{R}^n} \left\{ f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i=1}^m \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \right\}.$$

Proof. For each $(y_i)_{i=1}^m \in D$,

$$\begin{aligned}
\mathrm{epi}\,(f_i - \langle \cdot, y_i\rangle + g_i^*(y_i))^* &= \mathrm{epi}\,(f_i^*(\cdot + y_i) - g_i^*(y_i)) \\
&= \mathrm{epi} f_i^* - (y_i, g_i^*(y_i))
\end{aligned}$$

for each $i = 1, \dots, m$. Also it is easy to check that $\operatorname{epi} \delta^*_{\mathbb{R}^n} = \{0\} \times [0, +\infty)$. From the assumption of this theorem and Theorem 1.15, we have

$$\begin{aligned} &\inf_{x \in S(y_i)} \{f_0(x) - \langle x, y_0 \rangle + g_0^*(y_0)\} \\ &= \max_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \left\{ f_0(x) - \langle x, y_0 \rangle + g_0^*(y_0) + \sum_{i=1}^m \lambda_i (f_i(x) - \langle x, y_i \rangle + g_i^*(y_i)) \right\}. \end{aligned}$$

Therefore we conclude the final equality by using Theorem 2.6. □

2.2 Applications

We will prove Theorems 2.2, 2.3 and 2.4 by using Theorem 2.7 when all $f_i (i = 1, \dots, m)$ are real-valued functions.
[Theorem 2.7⇒Theorem 2.2]

Assume the assumption of Theorem 2.2, that is,

$$\begin{aligned} &\text{for each } (x_1^*, \dots, x_m^*) \in \prod_{i=1}^m \{g_i^* - f_i^* \leq 0\}, \text{ there exists } \bar{x} \in \mathbb{R}^n \\ &\text{such that } f_i(\bar{x}) - \langle \bar{x}, x_i^* \rangle + g_i^*(x_i^*) < 0, \forall i = 1, \dots, m. \end{aligned}$$

Therefore $\{f_i - \langle \cdot, x_i^* \rangle + g_i^*(x_i^*) \leq 0,\ i = 1, \dots, m\}$ holds the Slater constraint qualification. From Theorem 1.14,

$$\operatorname{cone} \operatorname{co} \bigcup_{i=1}^m \operatorname{epi}(f_i - \langle \cdot, x_i^* \rangle + g_i^*(x_i^*))^* = \operatorname{cone} \operatorname{co} \bigcup_{i=1}^m (\operatorname{epi} f_i^* - (x_i^*, g_i^*(x_i^*))) \text{ is closed.}$$

Let $D_0 = \operatorname{dom} g_0^*$ and $D = \prod_{i=1}^m \{g_i^* - f_i^* \leq 0\}$. It is clear that $\bigcup_{x \in S} \partial g_0(x) \subseteq D_0$. Now we show

$$\bigcup_{x \in S} \left(\prod_{i=1}^m \partial g_i(x) \right) \subseteq D.$$

For each $(y_i)_{i=1}^m \in \bigcup_{x \in S} \left(\prod_{i=1}^m \partial g_i(x) \right)$, there exists $x_0 \in S$ such that $(y_i)_{i=1}^m \in \prod_{i=1}^m \partial g_i(x_0)$. Then, for each $i = 1, \dots, m$, $g_i(x_0) + g_i^*(y_i) = \langle x_0, y_i \rangle$, that is,

$$g_i(x_0) + g_i^*(y_i) = f_i(x_0) + \langle x_0, y_i \rangle - f_i(x_0) \leq f_i(x_0) + f_i^*(y_i),$$

therefore we have $g_i^*(y_i) - f_i^*(y_i) \leq f_i(x_0) - g_i(x_0) \leq 0$. This shows $(y_i)_{i=1}^m \in D$, that is,

$$\bigcup_{x \in S} \left(\prod_{i=1}^m \partial g_i(x) \right) \subseteq D.$$

Therefore the assumption of Theorem 2.7 is satisfied.
[Theorem 2.7⇒Theorem 2.3]

Assume the assumption of Theorem 2.3, that is,

$$\text{for each } (x_1^*, \ldots, x_m^*) \in \Omega = \{(x_1^*, \ldots, x_m^*) \in \mathbb{R}^{nm} \mid \bigcap_{i=1}^m \partial g_i^*(x_i^*) \neq \emptyset\},$$
$$\text{there exists } x_0 \in \mathbb{R}^n \text{ such that } f_i(x_0) - \langle x_0, x_i^* \rangle + g_i^*(x_i^*) < 0,\ \forall i = 1, \ldots, m.$$

Therefore $\{f_i - \langle \cdot, x_i^* \rangle + g_i^*(x_i^*) \leq 0,\ i = 1, \ldots, m\}$ holds the Slater constraint qualification. From Theorem 1.14,

$$\operatorname{cone} \operatorname{co} \bigcup_{i=1}^m \operatorname{epi}(f_i - \langle \cdot, x_i^* \rangle + g_i^*(x_i^*))^* = \operatorname{cone} \operatorname{co} \bigcup_{i=1}^m (\operatorname{epi} f_i^* - (x_i^*, g_i^*(x_i^*))) \text{ is closed.}$$

Let $D_0 = \operatorname{dom} g_0^*$ and $D = \Omega$. It is clear that $\bigcup_{x \in S} \partial g_0(x) \subseteq D_0$. Now we show

$$\bigcup_{x \in S} \left(\prod_{i=1}^m \partial g_i(x) \right) \subseteq D.$$

For each $(y_i)_{i=1}^m \in \bigcup_{x \in S} \left(\prod_{i=1}^m \partial g_i(x) \right)$, there exists $x_0 \in S$ such that $(y_i)_{i=1}^m \in \prod_{i=1}^m \partial g_i(x_0)$. Since $g_i = g_i^{**}$, $x_0 \in \partial g_i(y_i)$ for each $i = 1, \ldots, m$. Therefore $x_0 \in \bigcap_{i=1}^m \partial g_i(y_i)$, and we have $\bigcap_{i=1}^m \partial g_i(y_i) \neq \emptyset$, that is,

$$(y_i)_{i=1}^m \in \Omega = D.$$

Consequently the assumption of Theorem 2.7 is satisfied.

Also, Theorem 2.7 is a strongly generalization of Theorem 2.2 and Theorem 2.3 when all $f_i (i = 1, \ldots, m)$ are real valued functions. For example, let $n = 1$, $m = 1$,

$$f_1(x) = \begin{cases} -x - 1 & x < -1, \\ 0 & x \in [-1, 1], \\ x - 1 & x > 1, \end{cases} \quad \text{and} \quad g_1(x) = 0,\ \forall x \in \mathbb{R}.$$

Then

$$f_1^*(x^*) = |x^*| + \delta_{[-1,1]}(x^*) \quad \text{and} \quad g_1^*(x^*) = \delta_{\{0\}}(x^*).$$

Therefore $\operatorname{cone} \operatorname{epi} f_1^* + \{0\} \times [0, +\infty)$ is closed. In short, the assumption of Theorem 2.7 holds. However, both the assumption of Theorem 2.2 and Theorem 2.3 do not hold because $f_1(x) - \langle x, x^* \rangle + g_1^*(x^*) \geq 0$ for each $x, x^* \in \mathbb{R}$.

But, Theorem 2.7 is not a generalization of Theorem 2.2 and Theorem 2.3 when some $f_i (i = 1, \ldots, m)$ is not a real-valued function. For example, let $n = 1$, $m = 1$,

$$f_0(x) = x, \quad g_0(x) = 0, \quad f_1(x) = \begin{cases} -1 & x \in [-1, 1], \\ +\infty & x \notin [-1, 1] \end{cases} \quad \text{and} \quad g_1(x) = 0, \forall x \in \mathbb{R}.$$

Then,

$$f_1^*(x^*) = |x^*| + 1 \quad \text{and} \quad g_1^*(x^*) = \delta_{\{0\}}, \forall x^* \in \mathbb{R}.$$

Therefore f_0, g_0 f_1 and g_1 satisfy the assumption of Theorem 2.2 and Theorem 2.3. However, $\operatorname{cone}\operatorname{epi} f_1^* + \{0\} \times [0, +\infty)$ is not closed. In short f_1 and g_1 do not satisfy the assumption of Theorem 2.7.

[Theorem 2.7⇒Theorem 2.4]

Assume the assumption to Theorem 2.4. Let $f_0 = \langle a, \cdot \rangle$, $g_0 = 0$, $f_1 = f$, $g_1 = 0$, $f_2 = 0$, and $g_2 = g$. Then $\partial g_0(x) = \{0\}$ for each $x \in S$ and $g_0^* = \delta_{\{0\}}$. Let $D_0 = \{0\}$, $D = \{0\} \times A$, where A is the set satisfying $\bigcup_{x \in S} \partial g(x) \subseteq A$. Then

$$\bigcup_{x \in S} \partial g_0(x) \subseteq D_0 \quad \text{and} \quad \bigcup_{x \in S} (\partial g_1(x) \times \partial g_2(x)) \subseteq D.$$

Also $S(y_1, y_2)$ is nonempty from Lemma 2.1 and

$$\begin{aligned} &\operatorname{cone}\operatorname{co}(\operatorname{epi}(f_1 - \langle \cdot, y_1 \rangle + g_1^*(y_1))^* \cup \operatorname{epi}(f_2 - \langle \cdot, y_2 \rangle + g_2^*(y_2))^*) + \{0\} \times [0, +\infty) \\ &= \operatorname{cone}\operatorname{co}(\operatorname{epi} f^* \cup \{-y_2\} \times [-g^*(y_2), +\infty) \cup \{0\} \times [0, +\infty)) \end{aligned}$$

is closed for each $(y_1, y_2) \in D$. By using Theorem 2.7, we have

$$\begin{aligned} \inf_{f(x) \leq 0, g(x) \geq 0} \langle a, x \rangle &= \inf_{f_i(x) - g_i(x) \leq 0} \{f_0(x) - g_0(x)\} \\ &= \inf_{(y_0, y_1, y_2) \in D_0 \times D} \max_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \\ &\quad \{f_0(x) - \langle x, y_0 \rangle + g_0^*(y_0) + \sum\nolimits_{i=1}^{2} \lambda_i (f_i(x) - \langle x, y_i \rangle + g^*(y_i))\} \\ &= \inf_{y \in A} \max_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \{\langle a, x \rangle + \lambda_1 f(x) + \lambda_2 (- \langle x, y \rangle + g^*(y))\}. \end{aligned}$$

This completes the proof of Theorem 2.4.

Next, we examine the following DC programming problem with reverse convex inequality system.

$$(\mathrm{R}) \quad \begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & g_i(x) \geq 0, \forall i = 1, 2, \ldots, m, \end{array}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ are convex function for each $i = 1, \ldots, m$. In problem (P), if $f_i = 0$ for each $i = 1, \ldots, m$, problem (P) becomes problem (R). From this, the following theorem hold.

Theorem 2.8. Let $f_0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semi continuous proper convex function, $g_i : \mathbb{R}^n \to \mathbb{R}$ be convex function for each $i = 0, 1, \dots, m$, $\bigcup_{x \in S} \partial g_0(x) \subseteq D_0 \subseteq \mathbb{R}^n$, $\bigcup_{x \in S} \left(\prod_{i=1}^{m} \partial g_i(x) \right) \subseteq D \subseteq \mathbb{R}^{nm}$. If

for each $(y_i)_{i=1}^m \in D \cap \prod_{i=1}^{m} \operatorname{dom} g_i^*$, f_0 is continuous at least at one point of $S(y_i)_{i=1}^m$,

then,

$$\mathrm{Val(R)} = \inf_{(y_0,(y_i)_{i=1}^m) \in D_0 \times D} \max_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \left\{ f_0(x) - \langle x, y_0 \rangle + g_0^*(y_0) + \sum_{i=1}^{m} \lambda_i (- \langle x, y_i \rangle + g_i^*(y_i)) \right\}.$$

Proof. Let $f_i = 0$ for each $i = 1, \dots, m$, then

$$\mathrm{epi} f_i^* = \{0\} \times [0, +\infty).$$

Therefore

$$\begin{aligned} &\operatorname{cone}\operatorname{co} \bigcup_{i=1}^{m} (\operatorname{epi} f_i^* - (y_i, g_i^*(y_i))) + \{0\} \times [0, +\infty) \\ &= \operatorname{cone}\operatorname{co} \bigcup_{i=1}^{m} (\{0\} \times [0, +\infty) - (y_i, g_i^*(y_i))) + \{0\} \times [0, +\infty). \end{aligned}$$

Moreover,

$$\begin{aligned} &\operatorname{cone}\operatorname{co} \bigcup_{i=1}^{m} (\{0\} \times [0, +\infty) - (y_i, g_i^*(y_i))) + \{0\} \times [0, +\infty) \\ &= \operatorname{cone}\operatorname{co} \{(0,1), -(y_1, g_1^*(y_1)), \dots, -(y_m, g_m^*(y_m))\}. \end{aligned}$$

From Theorem 1.1, this set is closed set. Therefore,

$$\mathrm{Val(R)} = \inf_{(y_0,(y_i)_{i=1}^m) \in D_0 \times D} \max_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \left\{ f_0(x) - \langle x, y_0 \rangle + g_0^*(y_0) + \sum_{i=1}^{m} \lambda_i (- \langle x, y_i \rangle + g_i^*(y_i)) \right\}$$

□

Example 2.1. In this example, we calculate the optimal value of the following DC programming problem by using Theorem 2.7:

$$\text{(P)} \quad \begin{array}{ll} \text{minimize} & x - |y| \\ \text{subject to} & x^2 + y^2 - 1 - |x| \leq 0. \end{array}$$

Let $f_0(x,y) = x$, $g_0(x,y) = |y|$, $f_1(x,y) = x^2 + y^2 - 1$, $g_1(x,y) = |x|$, $D_0 = \{0\} \times [-1,1]$, and $D = [-1,1] \times \{0\}$. Then we have $g_0^* = \delta_{\{0\}\times[-1,1]}$ and $g_1^* = \delta_{[-1,1]\times\{0\}}$. For each $y_1 \in D$, we can check that f_1 is continuous at least at one point of $S(y_1)$ because $S(y_1) = \{x \in \mathbb{R}^n \mid f_1(x) - \langle x, y_1\rangle + g_1^*(y_1) \leq 0\}$ is nonempty, and

$$\operatorname{cone}\operatorname{co}\left(\operatorname{epi} f_1^* - (y_1, g_1^*(y_1))\right) + \{0\} \times [0, +\infty)$$

is closed. By using Theorem 3.1,

$$\begin{aligned}
\mathrm{Val(P)} &= \inf_{(t_1,t_2)\in D_0,\,(t_3,t_4)\in D} \max_{\lambda\geq 0} \inf_{(x,y)\in\mathbb{R}^2} \\
&\quad \{f_0(x,y) - \langle (x,y),(t_1,t_2)\rangle + g_0^*(t_1,t_2) + \lambda(f_1(x,y) - \langle (x,y),(t_3,t_4)\rangle + g_1^*(t_3,t_4)\} \\
&= \inf_{t_2,t_3\in[-1,1]} \max_{\lambda\geq 0} \inf_{(x,y)\in\mathbb{R}^2} \{x - t_2 y + \lambda(x^2 + y^2 - 1 - t_3 x)\} \\
&= \inf_{t_2,t_3\in[-1,1]} \max_{\lambda> 0} \inf_{(x,y)\in\mathbb{R}^2} \{x - t_2 y + \lambda(x^2 + y^2 - 1 - t_3 x)\} \\
&= \inf_{t_2,t_3\in[-1,1]} \max_{\lambda> 0} \inf_{(x,y)\in\mathbb{R}^2} \\
&\quad \left\{\lambda\left(\left(x - \frac{\lambda t_3 - 1}{2\lambda}\right)^2 + \left(y - \frac{t_2}{2\lambda}\right)^2 - \left(\frac{\lambda t_3 - 1}{2\lambda}\right)^2 - \left(\frac{t_2}{2\lambda}\right)^2 - 1\right)\right\} \\
&= \inf_{t_2,t_3\in[-1,1]} \max_{\lambda> 0} \left\{\lambda\left(-\left(\frac{\lambda t_3 - 1}{2\lambda}\right)^2 - \left(\frac{t_2}{2\lambda}\right)^2 - 1\right)\right\} \\
&= \inf_{t_2,t_3\in[-1,1]} - \min_{\lambda> 0} \left\{\frac{t_3^2 + 4}{4}\lambda + \frac{t_2^2 + 1}{4\lambda} - \frac{t_3}{2}\right\} \\
&= \inf_{t_2,t_3\in[-1,1]} \left\{-2\sqrt{\frac{t_3^2 + 4}{4} \cdot \frac{t_2^2 + 1}{4}} + \frac{t_3}{2}\right\} \\
&= \inf_{t_3\in[-1,1]} \left\{-2\sqrt{\frac{t_3^2 + 4}{4} \cdot \frac{1^2 + 1}{4}} + \frac{t_3}{2}\right\} \\
&= -\frac{1}{2} - \frac{\sqrt{10}}{2}.
\end{aligned}$$

In the 7th equality, min is attained when $\lambda = \sqrt{\frac{t_2^2+1}{t_3^2+4}}$.

Chapter 3

Another Lagrange-type duality

We observe the following DC programming problem with inequality constraints:

$$\text{(P)} \quad \begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & f_i(x) - g_i(x) \leq 0,\ i = 1, \ldots, m, \end{array}$$

where $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions for each $i = 0, 1, \ldots, m$. Clearly, problem (P) is equivalent to the following problem (P′):

$$\text{(P}'\text{)} \quad \begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & \max\limits_{i=1,\ldots,m} \{f_i(x) - g_i(x)\} \leq 0, \end{array}$$

and problem (P′) is also a DC programming problem. Indeed, form Theorem 1.18,

$$\max_{i=1,\ldots,m} \{f_i(x) - g_i(x)\} = F(x) - G(x),$$

where

$$F = \max_{i=1,\ldots,m} \left\{ f_i + \sum_{j \neq i} g_j \right\},\ G = \sum_{i=1}^{m} g_i,$$

and F and G are convex functions. To our surprise, we can observe that constraint qualifications of two DC inequality systems $\{f_i - g_i \leq 0,\ i = 1, \ldots, m\}$ and $\{F - G \leq 0\}$ have a difference in spite of the two systems being equivalent. This can be seen at the end of Section 3.2. Motivated by the observation, we study other Lagrange-type duality results of the last chapter.

3.1 Maximum function of DC functions

At first, we give another duality result.

Theorem 3.1. (R. Harada, D. Kuroiwa, [1]) Let $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$ be convex functions for each $i = 0, 1, \dots, m$, $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \le 0, \forall i = 1, \dots, m\}$, $\bigcup_{x \in S} \partial g_0(x) \subseteq D_0$ and $D = \bigcup_{x \in S} \sum_{i=1}^{m} \partial g_i(x)$. If

$$\operatorname{cone}\operatorname{co}\left(\bigcup_{i=1}^{m}\Big(\operatorname{epi} f_i^* + \sum_{j \ne i} \operatorname{epi} g_j^*\Big) - \sum_{i=1}^{m}\Big(y_i, g_i^*(y_i)\Big)\right) + \{0\} \times [0, +\infty) \text{ is closed} \quad (3.1)$$

for each $(y_i)_{i=1}^m \in \bigcup_{x \in S} \prod_{i=1}^{m} \partial g_i(x)$, then following Lagrange-type duality holds:

$$\mathrm{Val(P)} = \inf_{(y_0, \hat{y}) \in D_0 \times D} \max_{\substack{\hat{\lambda}, \lambda_i \ge 0 \\ \sum_{i=1}^{m} \lambda_i = \hat{\lambda}}} \inf_{x \in \mathbb{R}^n} \left\{ \begin{array}{l} f_0(x) - \langle x, y_0 \rangle + g_0^*(y_0) \\ + \sum_{i=1}^{m} \lambda_i (f_i(x) - g_i(x)) \\ + \hat{\lambda} \left(\sum_{j=1}^{m} g_j(x) - \langle x, \hat{y} \rangle + (\sum_{j=1}^{m} g_j)^*(\hat{y}) \right) \end{array} \right\}.$$

Also we give a unified result of Theorem 2.7 and Theorem 3.1, as follows:

Theorem 3.2. (R. Harada, D. Kuroiwa, [1]) Let $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$ be convex functions for each $i = 0, 1, \cdots, m$, $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \le 0, \forall i = 1, \dots, m\}$, $I \subseteq \{1, \dots, m\}$, $\bigcup_{x \in S} \partial g_0(x) \subseteq D_0$ and $D = \bigcup_{x \in S} \left(\prod_{i \notin I} \partial g_i(x) \times \sum_{i \in I} \partial g_i(x) \right)$. If

$$\begin{aligned} \operatorname{cone}\operatorname{co}\Bigg(&\bigcup_{i \in I}\Bigg(\Big(\operatorname{epi} f_i^* + \sum_{\substack{j \ne i \\ j \in I}} \operatorname{epi} g_j^*\Big) - \sum_{i \in I}\Big(y_i, g_i^*(y_i)\Big)\Bigg) \\ &\cup \bigcup_{i \notin I}\Big(\operatorname{epi} f_i^* - \Big(y_i, g_i^*(y_i)\Big)\Big)\Bigg) + \{0\} \times [0, +\infty) \end{aligned} \quad (3.2)$$

is closed for each $(y_i)_{i=1}^m \in \bigcup_{x \in S} \prod_{i=1}^{m} \partial g_i(x)$, then

$$\mathrm{Val(P)} = \inf_{(y_0, ((y_i)_{i \notin I}, \hat{y})) \in D_0 \times D} \max_{\substack{\hat{\lambda}, \lambda_i \ge 0 \\ \sum_{i \in I} \lambda_i = \hat{\lambda}}} \inf_{x \in \mathbb{R}^n}$$

$$\left\{ \begin{array}{l} f_0(x) - \langle x, y_0 \rangle + g_0^*(y_0) + \sum_{i \notin I} \lambda_i (f_i(x) - \langle x, y_i \rangle + g_i^*(y_i)) \\ + \sum_{i \in I} \lambda_i (f_i(x) - g_i(x)) + \hat{\lambda} \left(\sum_{j \in I} g_j(x) - \langle x, \hat{y} \rangle + (\sum_{j \in I} g_j)^*(\hat{y}) \right) \end{array} \right\}.$$

Remark 3.1. If $I = \emptyset$, then Theorem 3.2 becomes Theorem 2.7, and if $I = \{1, \cdots, m\}$, then Theorem 3.2 becomes Theorem 3.1. Also, the assumptions of

Theorem 2.7 and Theorem 3.1 have a difference. This can be seen at the end of Section 3. Therefore Theorem 3.2 is a generalization of Theorem 2.7 and Theorem 3.1.

In order to prove Theorem 3.2, we provide the following Theorem 3.3, Lemma 3.1 and Lemma 3.2.

The following theorem will be used in the proof of the main theorem:

Theorem 3.3. (M. Sion, [14]) Let X be a convex set, Y be a compact convex set, $f : X \times Y \to \mathbb{R}$, where $f(x, \cdot)$ is upper semi continuous concave on Y for each $x \in X$ and $f(\cdot, y)$ is lower semi continuous convex on X for each $y \in Y$. Then

$$\inf_{x\in X} \max_{y\in Y} f(x,y) = \max_{y\in Y} \inf_{x\in X} f(x,y).$$

Lemma 3.1. For any $m \in \mathbb{N}$ and for any convex sets $C_i \subseteq \mathbb{R}^n$ $(i = 1, \dots, m)$,

$$\mathrm{co} \bigcup_{i=1}^{m} C_i = \bigcup_{\substack{\lambda_i \geq 0 \\ \sum_{i=1}^{m} \lambda_i = 1}} \sum_{i=1}^{m} \lambda_i C_i. \tag{3.3}$$

Proof. Clearly, (3.3) holds when $m = 1, 2$. Assume that (3.3) holds for some

$m \in \mathbb{N}$. Let $C_i \subseteq \mathbb{R}^n$ be convex sets for all $i = 1, \ldots, m+1$. Then,

$$\begin{aligned}
&\mathrm{co} \bigcup_{i=1}^{m+1} C_i \\
&= \mathrm{co}\left(\bigcup_{i=1}^{m} C_i \cup C_{m+1} \right) \\
&= \mathrm{co}\left(\mathrm{co}\left(\bigcup_{i=1}^{m} C_i \right) \cup C_{m+1} \right) \\
&= \bigcup_{\lambda \in [0,1]} \left(\lambda \mathrm{co} \bigcup_{i=1}^{m} C_i + (1-\lambda) C_{m+1} \right) \quad (\because \text{from the case when } m = 2) \\
&= \bigcup_{\lambda \in [0,1]} \left(\lambda \bigcup_{\substack{\lambda_i \geq 0 \\ \sum_{i=1}^{m} \lambda_i = 1}} \sum_{i=1}^{m} \lambda_i C_i + (1-\lambda) C_{m+1} \right) \quad (\because \text{from the assumption}) \\
&= \bigcup_{\lambda \in [0,1]} \bigcup_{\substack{\lambda_i \geq 0 \\ \sum_{i=1}^{m} \lambda_i = 1}} \left(\sum_{i=1}^{m} \lambda \lambda_i C_i + (1-\lambda) C_{m+1} \right) \\
&= \bigcup_{\substack{\lambda_i \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i = 1}} \sum_{i=1}^{m+1} \lambda_i C_i.
\end{aligned}$$

Therefore (3.3) holds for $m + 1$. From mathematical induction, the proof is completed. □

Lemma 3.2. For any $m \in \mathbb{N}$ and for any convex sets A_i, $B_i \subseteq \mathbb{R}^n$ $(i = 1, \ldots, m)$,

$$\mathrm{co} \bigcup_{\substack{\lambda_i \geq 0 \\ \sum_{i=1}^{m} \lambda_i = 1}} \sum_{i=1}^{m} (\lambda_i A_i + (1-\lambda_i) B_i) = \mathrm{co} \bigcup_{i=1}^{m} (A_i + \sum_{j \neq i} B_j). \tag{3.4}$$

Proof. We may assume that all A_i and B_i are not empty. We show this lemma by using mathematical induction. It is clear that (3.4) holds when $m = 1$. In the case of $m = 2$, (3.4) holds from Lemma 3.1 by putting $C_1 = A_1 + B_2$ and $C_2 = A_2 + B_1$. Assume that (3.4) holds for some $m \in \mathbb{N}$. Let A_i, $B_i \subseteq \mathbb{R}^n$ be convex sets for all $i = 1, \ldots, m+1$. Then,

$$\begin{aligned}
&\operatorname{co} \bigcup_{\substack{\lambda_i \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i = 1}} \sum_{i=1}^{m+1} (\lambda_i A_i + (1-\lambda_i) B_i) \\
&= \operatorname{co} \bigcup_{0 \leq \lambda_1 \leq 1} \Bigg(\bigcup_{\substack{\lambda_2, \ldots, \lambda_{m+1} \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i = 1}} \Big(\sum_{i=1}^{m+1} (\lambda_i A_i + (1-\lambda_i) B_i) \Big) \Bigg) \\
&= \operatorname{co} \Bigg(\bigcup_{0 \leq \lambda_1 < 1} \Bigg(\bigcup_{\substack{\lambda_2, \ldots, \lambda_{m+1} \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i = 1}} \Big(\sum_{i=1}^{m+1} (\lambda_i A_i + (1-\lambda_i) B_i) \Big) \Bigg) \\
&\qquad \cup \Big(A_1 + \sum_{i=2}^{m+1} B_i \Big) \Bigg) \\
&= \operatorname{co} \Bigg(\bigcup_{0 \leq \lambda_1 < 1} \Bigg(\lambda_1 A_1 + (1-\lambda_1) B_1 \\
&\qquad + \bigcup_{\substack{\lambda_2, \ldots, \lambda_{m+1} \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i = 1}} \Big(\sum_{i=2}^{m+1} (\lambda_i A_i + (1-\lambda_i) B_i) \Big) \Bigg) \cup \Big(A_1 + \sum_{i=2}^{m+1} B_i \Big) \Bigg). \\
&= \operatorname{co} \Bigg(\bigcup_{0 \leq \lambda_1 < 1} \Bigg(\lambda_1 A_1 + (1-\lambda_1) B_1 + (1-\lambda_1) \bigcup_{\substack{\lambda_2, \ldots, \lambda_{m+1} \geq 0 \\ \sum_{i=2}^{m+1} \frac{\lambda_i}{1-\lambda_1} = 1}} \\
&\qquad \Big(\sum_{i=2}^{m+1} \big(\frac{\lambda_i}{1-\lambda_1} A_i + \frac{1-\lambda_i}{1-\lambda_1} B_i \big) \Big) \Bigg) \cup \Big(A_1 + \sum_{i=2}^{m+1} B_i \Big) \Bigg).
\end{aligned} \tag{3.5}$$

For all $i = 2, \ldots, m+1$, since B_i are convex sets, $1 - \lambda_i = (1 - \lambda_1 - \lambda_i) + \lambda_1$, and $1 - \lambda_1 - \lambda_i \geq 0$, we have

$$\frac{1-\lambda_i}{1-\lambda_1} B_i = \frac{1-\lambda_1-\lambda_i}{1-\lambda_1} B_i + \frac{\lambda_1}{1-\lambda_1} B_i = \left(1 - \frac{\lambda_i}{1-\lambda_1}\right) B_i + \frac{\lambda_1}{1-\lambda_1} B_i$$

and then

$$\bigcup_{\substack{\lambda_2,\ldots,\lambda_{m+1}\geq 0\\ \sum_{i=2}^{m+1}\frac{\lambda_i}{1-\lambda_1}=1}}\Big(\sum_{i=2}^{m+1}(\frac{\lambda_i}{1-\lambda_1}A_i+\frac{1-\lambda_i}{1-\lambda_1}B_i)\Big)$$

$$=\bigcup_{\substack{\lambda_2,\ldots,\lambda_{m+1}\geq 0\\ \sum_{i=2}^{m+1}\frac{\lambda_i}{1-\lambda_1}=1}}\Big(\sum_{i=2}^{m+1}(\frac{\lambda_i}{1-\lambda_1}A_i+(1-\frac{\lambda_i}{1-\lambda_1})B_i+\frac{\lambda_1}{1-\lambda_1}B_i\Big)$$

$$=\frac{\lambda_1}{1-\lambda_1}\sum_{i=2}^{m+1}B_i+\bigcup_{\substack{\lambda_2,\ldots,\lambda_{m+1}\geq 0\\ \sum_{i=2}^{m+1}\frac{\lambda_i}{1-\lambda_1}=1}}\Big(\sum_{i=2}^{m+1}(\frac{\lambda_i}{1-\lambda_1}A_i+(1-\frac{\lambda_i}{1-\lambda_1})B_i)\Big)$$

$$=\frac{\lambda_1}{1-\lambda_1}\sum_{i=2}^{m+1}B_i+\bigcup_{\substack{\lambda'_2,\ldots,\lambda'_{m+1}\geq 0\\ \sum_{i=2}^{m+1}\lambda'_i=1}}\Big(\sum_{i=2}^{m+1}(\lambda'_iA_i+(1-\lambda'_i)B_i)\Big).$$

Hence,

$$\begin{aligned}&(3.5)\\ &=\mathrm{co}\Bigg(\bigcup_{0\leq\lambda_1<1}\Bigg(\lambda_1A_1+(1-\lambda_1)B_1+\lambda_1\sum_{i=2}^{m+1}B_i\\ &+(1-\lambda_1)\bigcup_{\substack{\lambda'_2,\ldots,\lambda'_{m+1}\geq 0\\ \sum_{i=2}^{m+1}\lambda'_i=1}}\Big(\sum_{i=2}^{m+1}(\lambda'_iA_i+(1-\lambda'_i)B_i)\Big)\Bigg)\cup\Big(A_1+\sum_{i=2}^{m+1}B_i\Big)\Bigg)\end{aligned} \tag{3.6}$$

$$\begin{aligned}&=\mathrm{co}\Bigg(\bigcup_{0\leq\lambda_1<1}\Bigg(\lambda_1A_1+(1-\lambda_1)B_1+\lambda_1\sum_{i=2}^{m+1}B_i\\ &+(1-\lambda_1)\mathrm{co}\bigcup_{\substack{\lambda'_2,\ldots,\lambda'_{m+1}\geq 0\\ \sum_{i=2}^{m+1}\lambda'_i=1}}\Big(\sum_{i=2}^{m+1}(\lambda'_iA_i+(1-\lambda'_i)B_i)\Big)\Bigg)\cup\Big(A_1+\sum_{i=2}^{m+1}B_i\Big)\Bigg).\end{aligned}$$

From the assumption,

$$\begin{aligned}
(3.6) &= \mathrm{co}\Bigg(\bigcup_{0\le\lambda_1<1}\Big(\lambda_1 A_1 + (1-\lambda_1)B_1 + \lambda_1\sum_{i=2}^{m+1} B_i \\
&\qquad +(1-\lambda_1)\mathrm{co}\bigcup_{i=2}^{m+1}\Big(A_i + \sum_{\substack{j\neq i\\ 2\le j\le m+1}} B_j\Big)\Big)\cup\Big(A_1+\sum_{i=2}^{m+1} B_i\Big)\Bigg) \\
&= \mathrm{co}\Bigg(\bigcup_{0\le\lambda_1<1}\Big(\lambda_1 A_1 + (1-\lambda_1)B_1 + \lambda_1\sum_{i=2}^{m+1} B_i \\
&\qquad +(1-\lambda_1)\bigcup_{i=2}^{m+1}\Big(A_i + \sum_{\substack{j\neq i\\ 2\le j\le m+1}} B_j\Big)\Big)\cup\Big(A_1+\sum_{i=2}^{m+1} B_i\Big)\Bigg) \qquad (3.7)\\
&= \mathrm{co}\Bigg(\bigcup_{0\le\lambda_1\le1}\Big(\lambda_1\Big(A_1 + \sum_{i=2}^{m+1} B_i\Big) \\
&\qquad + (1-\lambda_1)\Big(B_1 + \bigcup_{i=2}^{m+1}(A_i + \sum_{\substack{j\neq i\\ 2\le j\le m+1}} B_j)\Big)\Big)\Bigg) \\
&= \mathrm{co}\Bigg(\bigcup_{0\le\lambda_1\le1}\Big(\lambda_1\Big(A_1 + \sum_{i=2}^{m+1} B_i\Big) + (1-\lambda_1)\Big(\bigcup_{i=2}^{m+1}(A_i + \sum_{j\neq i} B_j)\Big)\Big)\Bigg).
\end{aligned}$$

By using Lemma 3.1,

$$\begin{aligned}
(3.7) &= \mathrm{co}\Bigg(\Big(A_1 + \sum_{i=2}^{m+1} B_i\Big)\cup\Big(\bigcup_{i=2}^{m+1}(A_i + \sum_{j\neq i} B_j)\Big)\Bigg) \\
&= \mathrm{co}\bigcup_{i=1}^{m+1}(A_i + \sum_{j\neq i} B_j).
\end{aligned}$$

Consequently, (3.4) holds for $m+1$. □

Proof of Theorem 3.2. Let $F = \max_{i\in I}\{f_i + \sum_{\substack{j\neq i\\ j\in I}} g_j\}$ and $G = \sum_{i\in I} g_i$. We can see the problem (P) is converted to the following equivalent problem (P″):

$$(\mathrm{P}'') \quad \begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & f_i(x) - g_i(x) \le 0, \forall i \notin I, \\ & F(x) - G(x) \le 0. \end{array}$$

From (1.2),

$$D = \bigcup_{x\in S}\Bigg(\prod_{i\notin I}\partial g_i(x)\times\sum_{i\in I}\partial g_i(x)\Bigg) = \bigcup_{x\in S}\Bigg(\prod_{i\notin I}\partial g_i(x)\times\partial\sum_{i\in I} g_i(x)\Bigg).$$

For each $((y_i)_{i\notin I}, \hat{y}) \in D \cap (\prod_{i\notin I} \operatorname{dom} g_i^* \times \operatorname{dom} G^*)$, there exists $\hat{x} \in S$ such that $y_i \in \partial g_i(\hat{x})$ for each $i \notin I$ and $\hat{y} \in \partial \sum_{i\in I} g_i(\hat{x})$, that is,

$$g_i(\hat{x}) + g_i^*(y_i) = \langle \hat{x}, y_i \rangle \ (i \notin I), \quad \Big(\sum_{i\in I} g_i\Big)(\hat{x}) + \Big(\sum_{i\in I} g_i\Big)^*(\hat{y}) = \langle \hat{x}, \hat{y} \rangle .$$

From (1.4), there exists $y_i (i \in I)$ such that $(\sum_{i\in I} g_i)^*(\hat{y}) = \sum_{i\in I} g_i^*(y_i)$ and $\sum_{i\in I} y_i = \hat{y}$. Then

$$\sum_{i\in I} (g_i(\hat{x}) + g_i^*(y_i)) = \sum_{i\in I} \langle \hat{x}, y_i \rangle ,$$

and since $g_i(\hat{x}) + g_i^*(y_i) \geq \langle \hat{x}, y_i \rangle$ for each $i \in I$, we have

$$g_i(\hat{x}) + g_i^*(y_i) = \langle \hat{x}, y_i \rangle , \text{ that is } y_i \in \partial g_i(\hat{x})$$

for each $i \in I$. Therefore

$$(y_i)_{i=1}^m \in \prod_{i=1}^m \partial g_i(\hat{x}) \subseteq \bigcup_{x\in S} \prod_{i=1}^m \partial g_i(x). \tag{3.8}$$

From $\hat{y} \in \partial \sum_{i\in I} g_i(\hat{x})$ and $\hat{x} \in S$,

$$\begin{aligned} F(x) - \langle \hat{x}, \hat{y} \rangle + G^*(\hat{y}) &= \max_{i\in I} \{ f_i(\hat{x}) + \sum_{\substack{j\neq i \\ j\in I}} g_j(\hat{x}) \} - \langle \hat{x}, \hat{y} \rangle + (\sum_{i\in I} g_i)^*(\hat{y}) \\ &= \max_{i\in I} \{ f_i(\hat{x}) + \sum_{\substack{j\neq i \\ j\in I}} g_j(\hat{x}) \} - \sum_{i\in I} g_i(\hat{x}) \\ &= \max_{i\in I} \{ f_i(\hat{x}) - g_i(\hat{x}) \} \leq 0. \end{aligned}$$

From $y_i \in \partial g_i(\hat{x})$ for each $i \notin I$ and $\hat{x} \in S$, $f_i(\hat{x}) - \langle \hat{x}, y_i \rangle + g_i^*(y_i) = f_i(\hat{x}) - g_i(\hat{x}) \leq 0$. Therefore $\hat{x}$ is an element of $\{x \in \mathbb{R}^n \mid f_i(x) - \langle x, y_i \rangle + g_i^*(y_i) \leq 0, \forall i \notin I, F(x) - \langle x, \hat{y} \rangle + G^*(\hat{y}) \leq 0\}$ and this set is non-empty. For each $i \in I$, let

$F_i = f_i + \sum_{\substack{j\neq i\\ j\in I}} g_j$. Now we have

$$\begin{aligned}
\operatorname{epi} F^* &= \operatorname{co}\bigcup_{i\in I}\operatorname{epi} F_i^* \qquad (\because \text{from (1.1)})\\
&= \bigcup_{\substack{\lambda_i\geq 0\\ \sum_{i\in I}\lambda_i=1}} \sum_{i\in I}\lambda_i \operatorname{epi} F_i^* \qquad (\because \text{by using Lemma 3.1})\\
&= \bigcup_{\substack{\lambda_i\geq 0\\ \sum_{i\in I}\lambda_i=1}} \sum_{i\in I}\lambda_i (\operatorname{epi} f_i^* + \sum_{\substack{j\neq i\\ j\in I}}\operatorname{epi} g_j^*) \qquad (\because \text{from (1.5)})\\
&= \bigcup_{\substack{\lambda_i\geq 0\\ \sum_{i=1}^{m}\lambda_i=1}} \sum_{i\in I}(\lambda_i \operatorname{epi} f_i^* + (1-\lambda_i)\operatorname{epi} g_i^*)\\
&= \operatorname{co}\bigcup_{\substack{\lambda_i\geq 0\\ \sum_{i=1}^{m}\lambda_i=1}} \sum_{i\in I}(\lambda_i \operatorname{epi} f_i^* + (1-\lambda_i)\operatorname{epi} g_i^*)\\
&= \operatorname{co}\bigcup_{i\in I}(\operatorname{epi} f_i^* + \sum_{\substack{j\neq i\\ j\in I}}\operatorname{epi} g_i^*). \qquad (\because \text{from Lemma 2})
\end{aligned}$$

Therefore

$$\operatorname{epi} F^* - (\hat{y}, G^*(\hat{y})) = \operatorname{co}\left(\bigcup_{i\in I}(\operatorname{epi} f_i^* + \sum_{\substack{j\in I\\ j\neq i}}\operatorname{epi} g_j^*) - \sum_{i\in I}(y_i, g_i^*(y_i))\right),$$

and hence

$$\begin{aligned}
&\operatorname{cone}\operatorname{co}\left(\bigcup_{i\notin I}(\operatorname{epi} f_i^* - (y_i, g_i^*(y_i))) \cup (\operatorname{epi} F^* - (\hat{y}, G^*(\hat{y})))\right) + \{0\}\times[0,+\infty)\\
&= \operatorname{cone}\operatorname{co}\left(\bigcup_{i\notin I}(\operatorname{epi} f_i^* - (y_i, g_i^*(y_i))) \cup \left(\bigcup_{i\in I}(\operatorname{epi} f_i^* + \sum_{\substack{j\in I\\ j\neq i}}\operatorname{epi} g_j^*) - \sum_{i\in I}(y_i, g_i^*(y_i))\right)\right)\\
&\qquad + \{0\}\times[0,+\infty),
\end{aligned}$$

because $\operatorname{co}(A \cup \operatorname{co} B) = \operatorname{co}(A\cup B)$ for any A, $B \subseteq \mathbb{R}^n$. From (3.2), this set is closed. By using Theorem 2.7,

$$\begin{aligned}
\mathrm{Val(P)} = &\inf_{(y_0,((y_i)_{i\notin I},\hat{y}))\in D_0\times D}\ \max_{\hat{\lambda},\,\lambda_i\geq 0}\ \inf_{x\in\mathbb{R}^n}\\
&\left\{\begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I}\lambda_i(f_i(x) - \langle x, y_i\rangle + g_i^*(y_i))\\ +\hat{\lambda}(F(x) - \langle x, \hat{y}\rangle + G^*(\hat{y}))\end{array}\right\}
\end{aligned}$$

holds. For any $(y_0, ((y_i)_{i\notin I}, \hat{y})) \in D_0 \times D$,

$$\max_{\substack{\lambda_i\geq 0 (i\notin I)\\ \hat{\lambda}\geq 0}} \inf_{x\in\mathbb{R}^n} \left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ +\hat{\lambda}(F(x) - \langle x, \hat{y}\rangle + G^*(\hat{y})) \end{array} \right\}$$

$$= \max_{\substack{\lambda_i\geq 0 (i\notin I)\\ \hat{\lambda}\geq 0}} \inf_{x\in\mathbb{R}^n} \left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ +\hat{\lambda}\Big(\max_{i\in I}\Big\{ f_i(x) + \sum_{j\neq i,\, j\in I} g_j(x)\Big\} - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})\Big) \end{array} \right\}$$

$$= \max_{\substack{\lambda_i\geq 0 (i\notin I)\\ \hat{\lambda}\geq 0}} \inf_{x\in\mathbb{R}^n}$$
$$\left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ +\hat{\lambda}\Big(\max_{\substack{\lambda_i\geq 0 (i\in I)\\ \sum_{i\in I}\lambda_i = 1}} \sum_{i\in I} \lambda_i (f_i(x) + \sum_{j\neq i,\, j\in I} g_j(x)) - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})\Big) \end{array} \right\}$$

$$= \max_{\substack{\lambda_i\geq 0 (i\notin I)\\ \hat{\lambda}\geq 0}} \inf_{x\in\mathbb{R}^n} \max_{\substack{\lambda_i\geq 0 (i\in I)\\ \sum_{i\in I}\lambda_i = 1}}$$
$$\left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ +\hat{\lambda} \sum_{i\in I} \lambda_i (f_i(x) + \sum_{j\neq i,\, j\in I} g_j(x)) - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})) \end{array} \right\}$$

$$= \max_{\substack{\lambda_i\geq 0 (i\notin I)\\ \hat{\lambda}\geq 0}} \max_{\substack{\lambda_i\geq 0 (i\in I)\\ \sum_{i\in I}\lambda_i = 1}} \inf_{x\in\mathbb{R}^n}$$
$$\left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ +\hat{\lambda} \sum_{i\in I} \lambda_i (f_i(x) + \sum_{j\neq i,\, j\in I} g_j(x)) - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})) \end{array} \right\}$$

$$= \max_{\substack{\hat{\lambda},\, \lambda_i\geq 0\\ \sum_{i\in I}\lambda_i = 1}} \inf_{x\in\mathbb{R}^n}$$
$$\left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ +\hat{\lambda}(\sum_{i\in I} \lambda_i (f_i(x) - g_i(x) + \sum_{j\in I} g_j(x)) - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})) \end{array} \right\}$$

$$= \max_{\substack{\hat{\lambda},\, \lambda_i\geq 0\\ \sum_{i\in I}\lambda_i = 1}} \inf_{x\in\mathbb{R}^n}$$
$$\left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ +\hat{\lambda} \sum_{i\in I} \lambda_i (f_i(x) - g_i(x)) + \hat{\lambda}(\sum_{j\in I} g_j(x) - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})) \end{array} \right\}$$

$$= \max_{\substack{\hat{\lambda},\, \lambda_i\geq 0\\ \sum_{i\in I}\lambda_i = \hat{\lambda}}} \inf_{x\in\mathbb{R}^n}$$
$$\left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I} \lambda_i (f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ + \sum_{i\in I} \lambda_i (f_i(x) - g_i(x)) + \hat{\lambda}(\sum_{j\in I} g_j(x) - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})) \end{array} \right\}.$$

The fourth equality of the previous equalities follows from Theorem 3.3. Hence we have

$$\mathrm{Val(P)} = \inf_{(y_0,((y_i)_{i\notin I},\hat{y}))\in D_0\times D} \max_{\substack{\hat{\lambda},\lambda_i\geq 0\\ \sum_{i\in I}\lambda_i=\hat{\lambda}}} \inf_{x\in\mathbb{R}^n} \left\{ \begin{array}{l} f_0(x) - \langle x, y_0\rangle + g_0^*(y_0) + \sum_{i\notin I}\lambda_i(f_i(x) - \langle x, y_i\rangle + g_i^*(y_i)) \\ + \sum_{i\in I}\lambda_i(f_i(x) - g_i(x)) + \hat{\lambda}(\sum_{j\in I} g_j(x) - \langle x, \hat{y}\rangle + (\sum_{j\in I} g_j)^*(\hat{y})) \end{array} \right\}.$$

This completes the proof. □

3.2 Comparison

Now we can apply Theorem 3.1 to DC programming problems.

Example 3.1. Consider the following DC programming problem:

$$\text{(P)} \quad \begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & x = (x_1, x_2) \in \mathbb{R}^2,\, f_i(x) - g_i(x) \leq 0,\ i = 1, 2, \end{array}$$

where $f_0(x_1, x_2) = x_1^2 - x_2$, $g_0(x_1, x_2) = 0$, $f_1(x_1, x_2) = x_2$, $g_1(x_1, x_2) = |x_1|$, $f_2(x_1, x_2) = -x_2$, and $g_2(x_1, x_2) = |x_1|$. This mathematical programming problem is neither convex nor differentiable, therefore the previous theorems concerned with convex or differentiable programming problems can not be applied directly. Let $D_0 = \bigcup_{x\in S} \partial g_0(x) = \{(0, 0)\}$ and $D = \bigcup_{x\in S} (\partial g_1(x) + \partial g_2(x)) = [-2, 2] \times \{0\}$. We can check that the assumption of Theorem 3.1 holds. Therefore,

$$\begin{aligned} \mathrm{Val(P)} &= \inf_{\hat{y}_1\in[-2,2]} \max_{\lambda_1,\lambda_2\geq 0} \inf_{x_1,x_2\in\mathbb{R}} \\ &\quad \left\{x_1^2 - x_2 + \lambda_1(|x_1| + x_2) + \lambda_2(|x_1| - x_2) - (\lambda_1 + \lambda_2)\hat{y}_1 x_1\right\} \\ &= \inf_{\hat{y}_1\in[-2,2]} \max_{\lambda_1,\lambda_2\geq 0} \inf_{x_1,x_2\in\mathbb{R}} \\ &\quad \left\{x_1^2 + (\lambda_1 + \lambda_2)(|x_1| - \hat{y}_1 x_1) + (-1 + \lambda_1 - \lambda_2)x_2\right\} \\ &= \inf_{\hat{y}_1\in[-2,2]} \max_{\lambda_2\geq 0} \inf_{x_1\in\mathbb{R}} \left\{x_1^2 + (2\lambda_2 + 1)(|x_1| - \hat{y}_1 x_1)\right\} \\ &= \inf_{\hat{y}_1\in[-2,2]} \max_{\lambda_2\geq 0} \min\{\inf_{x_1\geq 0} \left\{x_1^2 + (2\lambda_2 + 1)(1 - \hat{y}_1)x_1\right\}, \\ &\qquad\qquad \inf_{x_1\leq 0} \left\{x_1^2 - (2\lambda_2 + 1)(1 + \hat{y}_1)x_1\right\}\}, \end{aligned}$$

and we can see that

$$\inf_{x_1\geq 0} \left\{x_1^2 + (2\lambda_2 + 1)(1 - \hat{y}_1)x_1\right\} = \begin{cases} -\frac{1}{4}(2\lambda_2 + 1)^2(1 - \hat{y}_1)^2 & \text{if } \hat{y}_1 \in [1, 2] \\ 0 & \text{if } \hat{y}_1 \in [-2, 1) \end{cases}$$

$$\inf_{x_1\leq 0} \left\{x_1^2 - (2\lambda_2 + 1)(1 + \hat{y}_1)x_1\right\} = \begin{cases} -\frac{1}{4}(2\lambda_2 + 1)^2(1 + \hat{y}_1)^2 & \text{if } \hat{y}_1 \in [-2, -1] \\ 0 & \text{if } \hat{y}_1 \in (-1, 2], \end{cases}$$

then we have

$$\begin{aligned}\mathrm{Val}(\mathrm{P}) &= \inf_{|\hat{y}_1|\in[1,2]} \max_{\lambda_2\geq 0} \left\{ -\frac{1}{4}(2\lambda_2+1)^2(1-|\hat{y}_1|)^2 \right\} \\ &= \inf_{|\hat{y}_1|\in[1,2]} \left\{ -\frac{1}{4}(1-|\hat{y}_1|)^2 \right\} \\ &= -\frac{1}{4}.\end{aligned}$$

This example shows that Theorem 3.1 contributes to solving DC programming problems.

Next, we provide an observation that Theorem 3.1 has no relevance to Theorem 2.7. At first, we give a DC inequality system which holds the assumption of Theorem 3.1 but does not hold the assumption of Theorem 2.7 in the following example:

Example 3.2. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ as

$$f_1(x) = \begin{cases} \frac{1}{4}x^2 - x + 1 & \text{if } x \geq 2, \\ 0 & \text{if } -2 < x < 2, \\ \frac{1}{4}x^2 + x + 1 & \text{otherwise,} \end{cases} \qquad f_2(x) = \frac{1}{25}x^2 - \frac{1}{4},$$

$$g_1(x) = \frac{1}{5}x^2 \quad \text{and} \quad g_2(x) = \left[\frac{x+1}{2}\right] x - \left[\frac{x+1}{2}\right]^2,$$

where $[\cdot]$ is the greatest integer function. Since $g_2(x) = kx - k^2$ if $x \in [2k-1, 2k+1)$ where $k \in \mathbb{Z}$, g_2 is also a convex function. Also we can see that

$$f_1^*(y) = \begin{cases} y^2 + 2y & \text{if } y \geq 0, \\ y^2 - 2y & \text{otherwise,} \end{cases} \qquad f_2^*(y) = 5y^2 + \frac{1}{4},$$

$$g_1^*(y) = \frac{5}{4}y^2 \quad \text{and} \quad g_2^*(y) = (2[y]+1)y - [y]^2 - [y].$$

Put $F = \max\{f_1 + g_2, f_2 + g_1\}$ and $G = g_1 + g_2$. For each $\hat{y} \in D = \bigcup_{x\in S}(\partial g_1(x) + \partial g_2(x))$, there exists $\hat{x} \in S$, $y_1 \in \partial g_1(\hat{x})$, $y_2 \in \partial g_2(\hat{x})$ such that $\hat{y} = y_1 + y_2$ and $G^*(\hat{y}) = g_1^*(y_1) + g_2^*(y_2)$ from (1.4). Since $\operatorname{epi} F^* = \operatorname{co}((\operatorname{epi} f_1^* + \operatorname{epi} g_2^*) \cup (\operatorname{epi} f_2^* + \operatorname{epi} g_1^*))$,

$$\begin{aligned}&\operatorname{cone}\operatorname{co}(\operatorname{epi} F^* - (\hat{y}, G^*(\hat{y}))) + \{0\} \times [0, +\infty) \\ &= \operatorname{cone}\operatorname{co}(\{(n, n^2) \mid n \in \mathbb{Z}\} - (y_1 + y_2, g_1^*(y_1) + g_2^*(y_2))) + \{0\} \times [0, +\infty).\end{aligned}$$

The latter set is always closed. In general,

$$\operatorname{cone}\operatorname{co}(\{(n, n^2) \mid n \in \mathbb{Z}\} - (a, b)) = \begin{cases} \operatorname{epi} h & \text{if } a \notin \mathbb{Z},\ \alpha \leq \beta \text{ or } a \in \mathbb{Z},\ a^2 - b \geq 0, \\ \mathbb{R}^2 & \text{otherwise,} \end{cases}$$

where $a,\, b \in \mathbb{R}$, $\alpha = \min\left\{\frac{n^2-b}{n-a} \mid n \in \mathbb{Z},\, n > a\right\}$, $\beta = \max\left\{\frac{n^2-b}{n-a} \mid n \in \mathbb{Z},\, n > a\right\}$, and $h(x) = \begin{cases} \alpha x & \text{if } x \geq 0, \\ \beta x & \text{otherwise.} \end{cases}$ From this, $\operatorname{cone}\operatorname{co}(\{(n,n^2) \mid n \in \mathbb{Z}\} - (a,b))$ is always closed. Therefore $\{F - G \leq 0\}$ holds condition (2.1). Also $S(\hat{y}) \neq \emptyset$ because $F(\hat{x}) - \langle \hat{x}, \hat{y}\rangle + G^*(\hat{y}) \leq 0$. Therefore $\{F - G \leq 0\}$ holds the assumption of Theorem 3.1. However,

$$\begin{aligned}&\operatorname{cone}\operatorname{co}((\operatorname{epi} f_1^* - (0, g_1^*(0))) \cup (\operatorname{epi} f_2^* - (0, g_2^*(0)))) + \{0\} \times [0, +\infty) \\ &= \{(x,\alpha) \mid 2|x| < \alpha\} \cup \{(0,0)\}\end{aligned}$$

is not closed, that is, $\{f_1 - g_1 \leq 0,\, f_2 - g_2 \leq 0\}$ does not hold (2.1).

Next, we give a DC inequality system which holds the assumption of Theorem 2.7 but does not hold the assumption of Theorem 3.1 in the following example:

Example 3.3. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned}&f_1(x) = \left[\frac{x+1}{2}\right] x - \left[\frac{x+1}{2}\right]^2, \quad f_2(x) = \left[\frac{2x+1}{2}\right] x - \frac{1}{2}\left[\frac{2x+1}{2}\right]^2, \\ &g_1(x) = \frac{1}{4}x^2, \quad \text{and} \quad g_2(x) = \frac{1}{2}x^2.\end{aligned}$$

We can see that

$$\begin{aligned}&f_1^*(y) = (2[y]+1)y - [y]^2 - [y], \quad f_2^*(y) = ([y] + \frac{1}{2})y - \frac{1}{2}[y]^2 - \frac{1}{2}[y], \\ &g_1^*(y) = y^2 \quad \text{and} \quad g_2^*(x) = \frac{1}{2}y^2,\end{aligned}$$

and then

$$\begin{aligned}&\operatorname{cone}\operatorname{co}((\operatorname{epi} f_1^* - (y_1, g_1^*(y_1))) \cup (\operatorname{epi} f_2^* - (y_2, g_2^*(y_2)))) + \{0\} \times [0, +\infty) \\ &= \operatorname{cone}\operatorname{co}\big((\{(n,n^2) \mid n \in \mathbb{Z}\} - (y_1, g_1^*(y_1))) \\ &\qquad \cup \left(\left\{\left(n, \frac{1}{2}n^2\right) \mid n \in \mathbb{Z}\right\} - (y_2, g_2^*(y_2))\right)\bigg) + \{0\} \times [0, +\infty),\end{aligned}$$

for each $(y_1, y_2) \in \bigcup_{x \in S}(\partial g_1(x) \times \partial g_2(x))$. The latter set is always closed in the similar way to Example 3.2. Also, for each $(y_1, y_2) \in \bigcup_{x\in S}(\partial g_1(x) \times \partial g_2(x))$, there exists $z \in \mathbb{R}$ such that $y_1 = \frac{1}{2}z$, $y_2 = z$, then

$$\begin{aligned}S(y_1,y_2) &= \{x \in \mathbb{R} \mid f_i(x) - xy_i + g_i^*(y_i) \leq 0,\, i = 1,2\} \\ &= \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{l} \left[\frac{x+1}{2}\right] x - \left[\frac{x+1}{2}\right]^2 - \frac{1}{2}xz + \frac{1}{4}z^2 \leq 0, \\ \left[\frac{2x+1}{2}\right] x - \frac{1}{2}\left[\frac{2x+1}{2}\right]^2 - xz + \frac{1}{2}z^2 \leq 0 \end{array} \right\} \\ &\supseteq \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{l} \frac{1}{4}x^2 - \frac{1}{2}xz + \frac{1}{4}z^2 \leq 0, \\ \frac{1}{2}x^2 - xz + \frac{1}{2}z^2 \leq 0 \end{array} \right\} \\ &\ni z,\end{aligned}$$

Then $S(y_1, y_2)$ is non-empty. Therefore $\{f_1 - g_1 \leq 0,\ f_2 - g_2 \leq 0\}$ holds the assumption of Theorem 2.7. However,

$$\begin{aligned}
&\operatorname{cone}\operatorname{co}((\operatorname{epi} f_1^* + \operatorname{epi} g_2^*) \cup (\operatorname{epi} f_2^* + \operatorname{epi} g_1^*) - (0 + 0, g_1^*(0) + g_2^*(0))) \\
&\qquad + \{0\} \times [0, +\infty) \\
&= \mathbb{R} \times (0, +\infty) \cup \{(0, 0)\}
\end{aligned}$$

is not a closed set, that is, (3.1) does not hold.

Conclusions

In this paper, we studied Lagrange-type duality for DC programming problems. In Chapter 2, we introduced previous Lagrange-type duality theorems in previous research and we gave a Lagrange-type duality theorem, as Theorem 2.7. Also we showed that Theorem 2.7 is a generalization of known previous Lagrange-type duality results for DC programming problems in the real-valued case. In Chapter 3, we used the fact that the maximum of DC functions is also a DC function. Based on this idea, we presented Theorem 3.1, which is a Lagrange-type duality theorem for the maximum DC inequality constraint of the original DC inequality constraints. We observed that Theorem 3.1 has no relevance to Theorem 2.7, more precisely, Theorem 3.1 does not imply Theorem 2.7 and Theorem 2.7 does not imply Theorem 3.1. Also we proved Theorem 3.2, which is a unified Lagrange-type duality result of Theorem 2.7 and Theorem 3.1. Consequently, the class of DC programming problems to which Lagrange-type duality theorems can be applied was broader than the class in previous research.

Bibliography

[1] R. Harada, D. kuroiwa: Lagrange-type duality in DC programming problems with equivalent DC inequalities. Journal of Inequalities and Applications. (2016)

[2] R. Harada, D. Kuroiwa: Lagrange-type duality in DC programming. J. Math. Anal. Appl. 418, 415-424 (2014)

[3] Y. Fujiwara, D. Kuroiwa,: Lagrange duality in canonical DC programming, J. Math. Anal. Appl. 408, 476-483 (2013)

[4] Goberna, M. A., Jeyakumar, V., López, M. A.: Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities. Nonlinear Anal. 68 1184-1194 (2008)

[5] Burachik, R. S., Jeyakumar, V.: A dual condition for the convex subdifferential sum formula with applications. Journal of Convex Analysis 12, No. 2, 279-290 (2005)

[6] Jeyakumar, V., Dinh, N., Lee, G. M.: A new closed cone constraint qualification for convex optimization. Research Report AMR 04/8, Department of Applied Mathematics, University of New South Wales (2004)

[7] Horst, R., Thoai, N. V.: DC programming: overview. World Scientific, Singapore (1999)

[8] Martínez-Legaz, J. E., Volle, M.: Duality in DC programming: the case of several DC constraints. J. Math. Anal. Appl. 237 657-671 (1999)

[9] Rockafellar, R. T.: Convex analysis. Princeton Paperbacks. (1996)

[10] Horst, R.,Pardalos, P. M., and Thoai, N. V.: Introduction to global optimization. Kluwer academic publishers, Dordrecht, Holland (1995)

[11] Hiriart-Urruty, J., Baptiste, J., Lemarechal, C.: Convex analysis and minimization algorithms I. Advanced theory and bundle methods. Grundlehren der mathematischen wissenschaften. Springer-Verlag, Berlin. (1993)

[12] Hiriart-Urruty, J., Baptiste, J., Lemarechal, C.: Convex analysis and minimization algorithms II. Advanced theory and bundle methods. Grundlehren der mathematischen wissenschaften. Springer-Verlag, Berlin. (1993)

[13] On functions representable as a difference of convex functions, Hartman, Pacific J. Math. 9 707-713 (1959)

[14] Sion, M.: On general minimax theorems. Pacific J. Math. 8 171-176 (1958)