Three-dimensional time-varying nonlinear systems containing a Hamilton system

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Abstract

In this paper the following three-dimensional nonlinear system is considered:

$$
\begin{aligned}
x' &= \frac{\partial}{\partial y}H(x,y),\\
y' &= -\frac{\partial}{\partial x}H(x,y) + f(t)z,\\
z' &= -\,g(t)\frac{\partial}{\partial y}H(x,y) - h(t)z,
\end{aligned}
$$

where variable coefficients $f(t)$, $g(t)$ and $h(t)$ are continuous and bounded for $t \geq 0$, but not assumed to be positive. This system contains a subsystem described by a Hamiltonian function. Under the assumption that all orbits of the Hamilton system near to the origin are isolated closed curves surrounding the origin, sufficient conditions are given for the zero solution of the above-mentioned three-dimensional system to tend to the origin as $t \to \infty$. Our main result is compared with the famous Routh-Hurwitz criterion through an example. Some other examples are included to illustrate our main results. Finally, some figures of a positive orbit are also attached to facilitate a deeper understanding.

Key words: Uniform stability; Asymptotic stability; Nonlinear differential systems; Weakly integrally positive; Automatic control theory; Hamilton system

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1. Introduction

Let $B_\rho = \left\{(x, y) \in \mathbb{R}^2\colon 0 < x^2 + y^2 < \rho^2\right\}$ for any $\rho > 0$ and let $H(x, y)$ be a continuous function on B_ρ having continuous first partial derivatives. Suppose there exist constants α_1, α_2, β_1, β_2, γ and μ with $0 < \alpha_1 \le \alpha_2$, $0 < \beta_1 \le \beta_2$, $\gamma > 0$ and $0 < \mu \le 1$ such that

$$\alpha_1\left(x^2 + y^2\right) \le H(x, y) \le \alpha_2\left(x^2 + y^2\right), \tag{C_1}$$

$$\beta_1\left(x^2 + y^2\right) \le x\frac{\partial}{\partial x}H(x, y) + y\frac{\partial}{\partial y}H(x, y) \le \beta_2\left(x^2 + y^2\right), \tag{C_2}$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial}{\partial x}H(x, y) = \lim_{(x,y)\to(0,0)} \frac{\partial}{\partial y}H(x, y) = 0, \tag{C_3}$$

$$|x| < \mu|y| \quad \text{implies} \quad \gamma|y| \le \left|\frac{\partial}{\partial y}H(x, y)\right|. \tag{C_4}$$

Then, in a neighborhood of the origin $(0, 0)$, all solutions of the system

$$\begin{aligned} x' &= \frac{\partial}{\partial y}H(x, y),\\ y' &= -\frac{\partial}{\partial x}H(x, y) \end{aligned} \tag{1.1}$$

are periodic, namely, all orbits near to the origin are isolated closed curves surrounding the origin. Hence, the zero solution of (1.1) is uniformly stable, but not attractive (for the definition, see Section 2).

One of the most simple examples of (1.1) is the pendulum system without friction,

$$\begin{aligned} x' &= y,\\ y' &= -\sin x. \end{aligned} \tag{1.2}$$

In this case, we may consider $H(x, y) = 1 - \cos x + y^2/2$. Hence, for $\rho > 0$ sufficiently small, conditions (C_1)–(C_4) are satisfied with $\alpha_1 = 1/4$, $\alpha_2 = 1/2$, $\beta_1 = 1/2$, $\beta_2 = 1$, $\gamma = 1$ and $\mu = 1$. To take another example of (1.1), we consider the Lotka–Volterra system

$$\begin{aligned} X' &= aX - bXY,\\ Y' &= -cY + dXY \end{aligned}$$

on $\mathbb{R}^2_+$, $\mathbb{R}_+ = (0, \infty)$, where a, b, c and d are positive constants; X and Y are the densities of the prey and predator, respectively. Let $x = -\log(bY/a)$ and $y = -\log(dX/c)$. Then, we can transform the Lotka–Volterra system into the system

$$\begin{aligned} x' &= c(1 - e^{-y}), \\ y' &= a(e^{-x} - 1), \end{aligned} \tag{1.3}$$

which has the form of (1.1) with

$$H(x, y) = a(e^{-x} + x - 1) + c(e^{-y} + y - 1).$$

It is clear that for $\rho > 0$ sufficiently small, conditions (C_1)–(C_4) are satisfied with $\alpha_1 = \min\{a, c\}/4$, $\alpha_2 = \max\{a, c\}$, $\beta_1 = \min\{a, c\}/2$, $\beta_2 = 2\max\{a, c\}$, $\gamma = c/2$ and $\mu = 1$. It is easy to find other nonlinear phenomena described by system (1.1) in pure and applied science.

All solutions $(x(t), y(t))$ of (1.1) do not converge to the origin. Then, can $x(t)$ and $y(t)$ converge to zero by adding the third variable to system (1.1)? To deal with this problem, we consider the three-dimensional time-varying nonlinear system

$$\begin{aligned} x' &= \frac{\partial}{\partial y} H(x, y), \\ y' &= -\frac{\partial}{\partial x} H(x, y) + f(t)z, \\ z' &= -g(t)\frac{\partial}{\partial y} H(x, y) - h(t)z. \end{aligned} \tag{1.4}$$

Our problem bears some relation to automatic control theory. If subsystem (1.1) is linear and the coefficients $f(t)$, $g(t)$ and $h(t)$ are constants, then the well-known Routh-Hurwitz criterion may be useful for our problem. However, if system (1.4) contains a nonlinear subsystem, such as system (1.2) or (1.3), then the Routh-Hurwitz criterion is of no use to system (1.4) directly. Even if subsystem (1.1) is linear or linearization of (1.1) is possible in a neighborhood of the origin, the Routh-Hurwitz criterion cannot be applied to system (1.4) which has time-varying coefficients. As to the Routh-Hurwitz criterion, for example, see [1, 2, 3, 9, 10, 12].

The purpose of this paper is to discuss what kind of condition on time-varying coefficients $f(t)$, $g(t)$ and $h(t)$ will guarantee that the zero solution of (1.4) is uniformly stable and asymptotically stable.

The plan of this paper is as follows. In Section 2, we state our main result on the uniform stability and asymptotic stability of the zero solution of (1.4). To this end, we introduce a characteristic function which used the coefficients $f(t)$, $g(t)$ and $h(t)$. We also make several assumptions on the coefficients. In Section 3, we give the proof of our main result. Since the proof is somewhat long, we show a brief outline of the proof. For illustration of our main theorem, we take some concrete examples in Section 4. We consider the case in which subsystem (1.1) is linear and clarify the relation between the Routh-Hurwitz criterion and our main result. Also, we consider the case where linear approximation cannot be carried out. Moreover, we draw some figures of a positive orbit of the final example.

2. Statement of the main result

Consider a system of differential equations of the form

$$\begin{aligned} x' &= \frac{\partial}{\partial y}H(x,y), \\ y' &= -\frac{\partial}{\partial x}H(x,y) + f(t)z, \\ z' &= -\,g(t)\frac{\partial}{\partial y}H(x,y) - h(t)z, \end{aligned} \tag{E}$$

where the coefficients $f(t)$, $g(t)$ and $h(t)$ are continuous and $g(t)/f(t)$ is differentiable for $t \geq 0$.

Let $\mathbf{x}(t) = (x(t), y(t), z(t))$ and $\mathbf{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, and let $\|\cdot\|$ be the Euclidean norm. We denote the solution of (E) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. It is clear that system (E) has the zero solution $\mathbf{x}(t) \equiv \mathbf{0}$.

The zero solution is said to be *stable*, if for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be *uniformly stable* if it is stable and δ can be chosen independent of t_0. The zero solution is said to be *attractive*, if for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The zero solution of (E) is said to be *asymptotically stable* if it is stable and attractive. The asymptotic stability and the attractivity are completely different concepts in nonlinear systems, such as (E) (refer to the books [1, 2, 3, 4, 5, 11, 17]).

We also assume that $f(t)$ and $g(t)$ are bounded for $t \geq 0$ and that

$$f(t)g(t) > 0 \quad \text{for} \;\; t \geq 0 \quad \text{and} \quad \liminf_{t\to\infty} f(t)g(t) > 0. \tag{C_5}$$

Then there exist positive numbers k and K such that

$$k \leq \frac{f(t)}{g(t)} \leq K \quad \text{for } \ t \geq 0. \tag{2.1}$$

In fact, since $f(t)$ and $g(t)$ are bounded for $t \geq 0$, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$|f(t)| \leq c_1 \quad \text{and} \quad |g(t)| \leq c_2 \tag{2.2}$$

for $t \geq 0$. Because of (C_5), there exists a $c_3 > 0$ with

$$f(t)g(t) \geq c_3 \quad \text{for } \ t \geq 0. \tag{2.3}$$

Hence, it is clear that

$$\frac{f(t)}{g(t)} = \frac{f(t)g(t)}{g^2(t)} \geq \frac{c_3}{c_2^2} \quad \text{for } \ t \geq 0.$$

We can find a number $c_4 > 0$ satisfying

$$|g(t)| \geq c_4 \quad \text{for } \ t \geq 0.$$

If the assertion is false, then there exists a sequence $\{t_n\}$ tending to ∞ such that

$$|g(t_n)| \to 0 \quad \text{as } \ n \to \infty.$$

It then follows from (2.3) that

$$|f(t_n)| \to \infty \quad \text{as } \ n \to \infty.$$

This contradicts (2.2). We therefore conclude that

$$\frac{f(t)}{g(t)} = \frac{|f(t)|}{|g(t)|} \leq \frac{c_1}{c_4} \quad \text{for } \ t \geq 0.$$

Let $k = c_3/c_2^2$ and $K = c_1/c_4$. Then we obtain inequality (2.1).

From (C_1) and (2.1), we can guarantee that all solutions $\mathbf{x}(t)$ of (E) are continuable in the future.

We here introduce a concept that plays an important role in this paper. A nonnegative function ϕ is said to be *weakly integrally positive* if

$$\int_I \phi(s)ds = \infty$$

for every set $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$ such that $\tau_n + \lambda < \sigma_n < \tau_{n+1} \le \sigma_n + \Lambda$ for some $\lambda > 0$ and $\Lambda > 0$. For example, we can cite $1/(1+t)$ or $\sin^2 t/(1+t)$ as a weakly integrally positive function (for example, see [6, 7, 8, 13, 14, 15, 16]). If ϕ is weakly integrally positive, then it naturally satisfies that

$$\lim_{t\to\infty} \int^t \phi(s)ds = \infty.$$

Note that if ϕ is weakly integrally positive and

$$\int_I \phi(s)ds = \sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} \phi(s)ds < \infty$$

for two sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $\tau_n < \sigma_n < \tau_{n+1} \le \sigma_n + \Lambda$, then

$$\liminf_{n\to\infty} (\sigma_n - \tau_n) = 0.$$

Since $g(t)/f(t)$ is differentiable for $t \ge 0$, we may define

$$\psi(t) = 2h(t) + \frac{f(t)}{g(t)} \left(\frac{g(t)}{f(t)} \right)'.$$

We also denote

$$\psi_+(t) = \max\{0, \psi(t)\} \quad \text{and} \quad \psi_-(t) = \max\{0, -\psi(t)\}.$$

In addition to (C_1)–(C_5), we assume that

$$\int_0^{\infty} \psi_-(s)ds < \infty \tag{C_6}$$

and

$$\psi_+(t) \ \text{ is weakly integrally positive.} \tag{C_7}$$

Then we have the following main result.

Theorem 2.1. *Let $f(t)$, $g(t)$ and $h(t)$ be bounded for $t \ge 0$. Suppose that conditions (C_1)–(C_7) are satisfied. Then the zero solution of (E) is uniformly stable and asymptotically stable.*

Let $(x(t), y(t), z(t))$ be any solution of (E) with the initial time $t_0 \geq 0$ and define

$$v(t) = H(x(t), y(t)) + \frac{f(t)}{2g(t)} z^2(t).$$

Then, we have

$$v'(t) = -\frac{f(t)}{2g(t)} \psi(t) z^2(t) \leq \psi_-(t) \frac{f(t)}{2g(t)} z^2(t) \leq \psi_-(t) v(t) \tag{2.4}$$

for $t \geq t_0$. Let

$$\Psi(t) = \int_{t_0}^{t} \psi_-(s) ds.$$

Then, because of (C_6), there exists an $L > 0$ such that $\Psi(t) < L$ for $t \geq t_0$. Integrating (2.4) from t_0 to ∞, we have

$$v(t) \leq v(t_0) e^L,$$

and consequently,

$$v'(t) \leq v(t_0) e^L \psi_-(t).$$

Since the right-hand side of the above inequality is positive for $t \geq t_0$, we see that

$$(v')_+(t) \leq v(t_0) e^L \psi_-(t).$$

Integrate both sides from t_0 to ∞ to obtain

$$\int_{t_0}^{\infty} (v')_+(s) ds \leq v(t_0) e^L L < \infty.$$

On the other hand, since $v(t) \geq 0$ for $t \geq t_0$, we get

$$\int_{t_0}^{\infty} (v')_-(s) ds = \int_{t_0}^{\infty} (v')_+(s) ds - \int_{t_0}^{\infty} v'(s) ds \leq \int_{t_0}^{\infty} (v')_+(s) ds + v(t_0) < \infty.$$

Putting the two integral estimations together, we have

$$\int_{t_0}^{\infty} |v'(t)| dt = \int_{t_0}^{\infty} \big((v')_+(t) + (v')_-(t)\big) dt < \infty. \tag{2.5}$$

Since $v(t)$ is nonnegative for $t \geq t_0$ and $v'(t)$ is absolutely integrable, it turns out that $v(t)$ has a nonnegative limiting value.

The above argument can be summarised as follows.

Lemma 2.2. *Under condition* (C_6)*, the derivative of* $v(t)$ *is absolutely integrable, and therefore,* $v(t)$ *has an nonnegative limiting value.*

3. Proof of the main result

Before giving the full proof of Theorem 2.1, it is helpful to mention its broad outline. The proof is divided into four parts. To begin with, we will show that

(i) the zero solution of (E) is uniformly stable.

We next show that the zero solution of (E) is asymptotically stable by way of contradiction. For this purpose, we define

$$u(t) = \frac{f(t)}{2g(t)} z^2(t),$$

and then prove

(ii) $\liminf_{t\to\infty} u(t) = 0$;

(iii) $\limsup_{t\to\infty} u(t) = 0$.

The proof of part (ii) is simple, but part (iii) needs a detailed demonstration. Part (iii) is the core of the proof of Theorem 2.1. Using cylindrical coordinates $(x, y, z) \to (r, \theta, z)$ by $x = r\cos\theta$ and $y = r\sin\theta$, we transform system (E) into an equivalent system. We examine any solution $(r(t), \theta(t), z(t))$ of the transformed system in detail. We particularly pay attention to the movement of $(r(t), \theta(t))$ and show that $(r(t), \theta(t))$ stays in an annulus during certain time intervals. From parts (ii) and (iii), we see that $\lim_{t\to\infty} u(t) = 0$. Using this fact and repeating the same argument as in part (iii), we show that

(iv) $z(t)$ does not converge to zero as $t \to \infty$.

However, because $u(t)$ converges to zero, $z(t)$ also converges. This contradicts part (iv).

PROOF OF THEOREM 2.1. (i): Recall that (C_5) implies (2.1), namely,

$$0 < k \le \frac{f(t)}{g(t)} \le K \quad \text{for} \ \ t \ge 0.$$

Define

$$M_1 = \min\left\{\alpha_1, \frac{k}{2}\right\} \quad \text{and} \quad M_2 = \max\left\{\alpha_2, \frac{K}{2}\right\}.$$

To prove uniform stability of the zero solution of (E), for a given $\varepsilon \in (0, \rho)$, we select

$$\delta(\varepsilon) = \sqrt{\frac{M_1}{M_2} e^{-L}}\,\varepsilon.$$

Recall that L is a positive number satisfying $\Psi(t) < L$ for $t \geq t_0$. Needless to say, $\delta < \varepsilon$. Let $t_0 \geq 0$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ be given. We will show that $\|\mathbf{x}_0\| = \sqrt{x_0^2 + y_0^2 + z_0^2} < \delta$ and $t \geq t_0$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$. For convenience of notation, we write $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and $(x(t), y(t), z(t)) = \mathbf{x}(t)$.

Suppose that there exists $t_1 > t_0$ with $\|\mathbf{x}(t_1)\| = \varepsilon$ and

$$\|\mathbf{x}(t)\| < \varepsilon < \rho \quad \text{for } \ t_0 \leq t < t_1,$$

where ρ is the constant given in the first paragraph of Section 1. Note that $(x(t), y(t)) \in B_\rho$ for $t_0 \leq t \leq t_1$. Let

$$v(t) = H(x(t), y(t)) + \frac{f(t)}{2g(t)} z^2(t)$$

for $t \geq t_0$. Then, from (C_1) and (2.1) it turns out that

$$v(t) \geq \alpha_1 \bigl(x^2(t) + y^2(t)\bigr) + \frac{k}{2} z^2(t) \geq M_1 \mathbf{x}^2(t) \tag{3.1}$$

for $t_0 \leq t \leq t_1$. Next, we define

$$w(t) = v(t) \exp(-\Psi(t)) \quad \text{for } \ t \geq t_0.$$

Then, by (2.4) we have

$$w'(t) = (v'(t) - \psi_-(t) v(t)) \exp(-\Psi(t)) \leq 0,$$

so that

$$v(t) \exp(-\Psi(t)) = w(t) \leq w(t_0) = v(t_0)$$

for $t \geq t_0$. Hence, we obtain

$$v(t) < \left\{ H(x_0, y_0) + \frac{f(t_0)}{2g(t_0)} z_0^2 \right\} e^L \quad \text{for } \ t \geq t_0.$$

Since $(x_0, y_0) \in B_\rho$, it follows from (C_1) and (2.1) that

$$v(t) < \left\{ \alpha_2 \bigl(x_0^2 + y_0^2\bigr) + \frac{f(t_0)}{2g(t_0)} z_0^2 \right\} e^L \leq M_2 e^L \delta^2 = M_1 \varepsilon^2$$

for $t \geq t_0$. Hence, together with (3.1), we have

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for } \ t_0 \leq t \leq t_1.$$

This contradicts the assumption that $\|\mathbf{x}(t_1)\| = \varepsilon$. Thus, we see that

$$\|\mathbf{x}(t)\| < \varepsilon < \rho \quad \text{for} \;\; t \geq t_0, \tag{3.2}$$

and therefore, the zero solution of (E) is uniformly stable. This completes the proof of part (i).

Hereafter, we will show that the zero solution of (E) is asymptotically stable. To this end, it is enough to show that it is attractive, namely, $\mathbf{x}(t)$ converges to $\mathbf{0}$ as t increases. By means of Lemma 2.2, the function $v(t)$ has a limiting value $v_0 \geq 0$. If $v_0 = 0$, then by (3.1), the solution $\mathbf{x}(t)$ tends to $\mathbf{0}$ as $t \to \infty$. This completes the proof. Hence, the remainder is the case in which $v_0 > 0$. We will demonstrate that this case does not occur.

For the sake of simplicity, let

$$u(t) = \frac{f(t)}{2g(t)} z^2(t).$$

Then, we have $v(t) = H(x(t), y(t)) + u(t)$ and $v'(t) = -\psi(t)u(t)$. From (2.1) and (3.2), we see that $u(t)$ is bounded. Hence, $u(t)$ has the inferior limit and the superior limit.

(ii): We will show that the inferior limit of $u(t)$ is zero. Suppose that

$$\liminf_{t\to\infty} u(t) > 0.$$

Then there exist an $\varepsilon_1 > 0$ and a $T_1 \geq t_0$ such that $u(t) > \varepsilon_1$ for $t \geq T_1$. Hence, by (2.5) we have

$$\infty > \int_{t_0}^{\infty} |v'(s)| ds = \int_{t_0}^{\infty} |\psi(s)| u(s) ds \geq \int_{t_0}^{\infty} \psi_+(s) u(s) ds > \varepsilon_1 \int_{T_1}^{\infty} \psi_+(s) ds.$$

This contradicts (C_7). Thus, we see that $\liminf_{t\to\infty} u(t) = 0$. This completes the proof of part (ii).

(iii): We next show that the superior limit of $u(t)$ is zero. The proof is by contradiction. Suppose that $\nu \stackrel{\text{def}}{=} \limsup_{t\to\infty} u(t) > 0$. Since $f(t)$ and $h(t)$ are bounded for $t \geq 0$, there exist numbers $\overline{f} > 0$ and $\overline{h} > 0$ such that

$$|f(t)| \leq \overline{f} \quad \text{and} \quad |h(t)| \leq \overline{h} \tag{3.3}$$

for $t \geq 0$. As we have shown in Section 2, there exists a $\underline{g} > 0$ such that

$$|g(t)| \geq \underline{g} \quad \text{for} \;\; t \geq 0. \tag{3.4}$$

Since $v(t)$ tends to a positive value v_0 as $t \to \infty$, we can choose a $T_2 \geq t_0$ such that

$$0 < \frac{v_0}{2} < v(t) < \frac{3v_0}{2} \quad \text{for} \;\; t \geq T_2. \tag{3.5}$$

Judging from the inequality expression of (C_2), we can exchange the number β_1 with a positive smaller one. In fact, (C_2) implies that

$$\widetilde{\beta}_1\big(x^2+y^2\big) \leq x\frac{\partial}{\partial x}H(x,y) + y\frac{\partial}{\partial y}H(x,y) \leq \beta_2\big(x^2+y^2\big)$$

for any $\widetilde{\beta}_1 < \beta_1$. Hence, we may assume without loss of generality that

$$\beta_1\beta_2 < \frac{\overline{f}\underline{g}\gamma}{\sqrt{2}}.$$

Let $\varepsilon_2 > 0$ be so small that $\varepsilon_2 < \nu/2$,

$$\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0-2\varepsilon_2)}} < \frac{\beta_1}{2\overline{f}}, \tag{3.6}$$

$$\tan\left(\frac{\pi}{2} - \frac{\pi\overline{f}}{2\beta_1}\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0-2\varepsilon_2)}}\right) > \frac{\varepsilon}{\mu}\sqrt{\frac{2\alpha_2}{v_0-2\varepsilon_2}}, \tag{3.7}$$

$$\overline{f}\,\overline{h}\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0-2\varepsilon_2)}} + \beta_1\beta_2 < \frac{\overline{f}\underline{g}\gamma}{\sqrt{2}}, \tag{3.8}$$

where α_2, (β_1,β_2), (γ,μ), k and ε are the numbers given in (C_1), (C_2), (C_4), (2.1) and (3.2), respectively. Because $\liminf_{t\to\infty} u(t) = 0 < \nu = \limsup_{t\to\infty} u(t)$, we can find two divergent sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $T_2 < \tau_n < \sigma_n < \tau_{n+1}$ such that

$$\begin{aligned} u(t) &\geq \varepsilon_2 \quad \text{for} \;\; \tau_n < t < \sigma_n, \\ 0 \leq u(t) &\leq \varepsilon_2 \quad \text{for} \;\; \sigma_n \leq t \leq \tau_{n+1}. \end{aligned} \tag{3.9}$$

and $u(\tau_n) = u(\sigma_n) = \varepsilon_2$. From (2.1), we see that $u(t) \geq kz^2(t)/2$ for $t \geq t_0$. Hence, by (3.9) we have

$$|z(t)| \leq \sqrt{\frac{2}{k}u(t)} \leq \sqrt{\frac{2\varepsilon_2}{k}} \quad \text{for} \;\; \sigma_n \leq t \leq \tau_{n+1}. \tag{3.10}$$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then we can rewrite system (E) as the form

$$\begin{aligned} r' &= \frac{\partial}{\partial y}H(x,y)\cos\theta - \frac{\partial}{\partial x}H(x,y)\sin\theta + f(t)z\sin\theta, \\ \theta' &= \frac{f(t)z}{r}\cos\theta - \frac{1}{r^2}\left\{x\frac{\partial}{\partial x}H(x,y) + y\frac{\partial}{\partial y}H(x,y)\right\}, \\ z' &= -\,g(t)\frac{\partial}{\partial y}H(x,y) - h(t)z. \end{aligned} \qquad (\widetilde{E})$$

Let $(r(t), \theta(t), z(t))$ be the solution of ($\widetilde{E}$) corresponding to $\mathbf{x}(t)$. Using (3.2), (3.5), (3.9) and (C_1), we obtain

$$v_0 - 2\varepsilon_2 < 2(v(t) - u(t)) = 2H(x(t), y(t)) \le 2\alpha_2\big(x^2(t) + y^2(t)\big) < 2\alpha_2\varepsilon^2$$

for $\sigma_n \le t \le \tau_{n+1}$, so that

$$\sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r(t) < \varepsilon \quad \text{for} \ \ \sigma_n \le t \le \tau_{n+1}. \qquad (3.11)$$

Taking into account of (3.10) and (3.11), we see that the solution $(r(t), \theta(t), z(t))$ of ($\widetilde{E}$) stays in the thin disc

$$D = \left\{(r,\theta,z)\colon \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r < \varepsilon,\ -\pi < \theta \le \pi \ \text{ and } \ |z| \le \sqrt{\frac{2\varepsilon_2}{k}}\right\}$$

for $\sigma_n \le t \le \tau_{n+1}$. It follows from ($C_2$), (3.3), (3.10) and (3.11) that

$$\begin{aligned} -\overline{f}\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0 - 2\varepsilon_2)}} - \beta_2 &< -\frac{|f(t)||z(t)|}{r(t)} - \beta_2 \\ &\le \theta'(t) \le \frac{|f(t)||z(t)|}{r(t)} - \beta_1 < \overline{f}\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0 - 2\varepsilon_2)}} - \beta_1 \end{aligned}$$

for $\sigma_n \le t \le \tau_{n+1}$. Let

$$\omega_- = \beta_1 - \overline{f}\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0 - 2\varepsilon_2)}} \quad \text{and} \quad \omega_+ = \beta_2 + \overline{f}\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0 - 2\varepsilon_2)}}.$$

Then, from (3.6), we can estimate that

$$\frac{\beta_1}{2} < \omega_- < \beta_1 \le \beta_2 < \omega_+ < \beta_2 + \frac{\beta_1}{2} \le \frac{3}{2}\beta_2 \qquad (3.12)$$

and

$$-\omega_+ < \theta'(t) < -\omega_- < 0 \quad \text{for} \;\; \sigma_n \le t \le \tau_{n+1}. \tag{3.13}$$

Define a planar region Ω by

$$\Omega = \left\{ (r,\theta) \colon \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r < \varepsilon \;\text{ and }\; \frac{\pi\omega_-}{2\beta_1} \le \theta \le \pi\left(1 - \frac{\omega_-}{2\beta_1}\right) \right\}.$$

The region Ω is non-empty because $\omega_- < \beta_1$.

We will show that

$$\tau_{n+1} - \sigma_n \le \frac{3\pi}{\omega_-} \quad \text{for} \;\; n \in \mathbb{N}. \tag{3.14}$$

If the assertion is not true, then there exists an $n_0 \in \mathbb{N}$ such that $\tau_{n_0+1} - \sigma_{n_0} > 3\pi/\omega_-$. Consider the movement of $(r(t), \theta(t))$. Then, from (3.11) and (3.13), we see that $(r(t), \theta(t))$ stays in the annulus

$$A = \left\{ (r,\theta) \colon \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r < \varepsilon \;\text{ and }\; -\pi < \theta \le \pi \right\} \supset \Omega$$

for $\sigma_{n_0} \le t \le \tau_{n_0+1}$ and it moves clockwise. Integrating (3.13) from σ_{n_0} to τ_{n_0+1}, we obtain

$$\theta(\sigma_{n_0}) - \theta(\tau_{n_0+1}) = -\int_{\sigma_{n_0}}^{\tau_{n_0+1}} \theta'(s)ds > \omega_-(\tau_{n_0+1} - \sigma_{n_0}) > 3\pi.$$

Hence, $(r(t), \theta(t))$ makes at least one and a half rotation in the interval $[\sigma_{n_0}, \tau_{n_0+1}]$. For this reason, we can find two numbers a and b with $\sigma_{n_0} \le a < b \le \tau_{n_0+1}$ such that $\theta(a) - \theta(b) = \pi(1 - \omega_-/\beta_1)$ and

$$(r(t), \theta(t)) \in \Omega \quad \text{for} \;\; a \le t \le b. \tag{3.15}$$

By (3.12) and (3.13), we have

$$\theta(a) - \theta(b) < \omega_+(b-a) < \frac{3}{2}\beta_2(b-a),$$

so that

$$b - a > \frac{2(\theta(a) - \theta(b))}{3\beta_2} = \frac{2\pi(\beta_1 - \omega_-)}{3\beta_1\beta_2} > \frac{2(\beta_1 - \omega_-)}{\beta_1\beta_2}. \tag{3.16}$$

It follows from (3.6) that

$$\frac{\pi\omega_-}{2\beta_1} = \frac{\pi}{2}\left(1 - \frac{\overline{f}}{\beta_1}\sqrt{\frac{4\alpha_2\varepsilon_2}{k(v_0 - 2\varepsilon_2)}}\right)$$
$$> \frac{\pi}{2}\left(1 - \frac{\overline{f}}{\beta_1}\frac{\beta_1}{2\overline{f}}\right) = \frac{\pi}{4}.$$

Hence, together with (3.15), we obtain

$$|y(t)| > \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}}\sin\frac{\pi\omega_-}{2\beta_1} > \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}}\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}}$$

for $a \le t \le b$. Since we can rewrite (3.8) as

$$\overline{h}(\beta_1 - \omega_-) + \beta_1\beta_2 < \frac{\overline{f}\underline{g}\gamma}{\sqrt{2}},$$

we have

$$\frac{1}{\sqrt{2}}\sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} > \frac{\overline{h} + \beta_1\beta_2/(\beta_1 - \omega_-)}{\underline{g}\gamma}\sqrt{\frac{2\varepsilon_2}{k}}.$$

We therefore conclude that

$$|y(t)| > \frac{\overline{h} + \beta_1\beta_2/(\beta_1 - \omega_-)}{\underline{g}\gamma}\sqrt{\frac{2\varepsilon_2}{k}} \quad \text{for } \ a \le t \le b. \tag{3.17}$$

Using (3.15) again, we obtain

$$|x(t)| < \varepsilon\cos\frac{\pi\omega_-}{2\beta_1} \quad \text{for } \ a \le t \le b.$$

Hence, by (3.7) we have

$$|x(t)| < \varepsilon\cos\frac{\pi\omega_-}{2\beta_1} < \varepsilon\sqrt{\frac{2\alpha_2}{v_0 - 2\varepsilon_2}}\frac{|y(t)|}{\tan\left(\pi\omega_-/(2\beta_1)\right)} < \mu|y(t)|$$

for $a \le t \le b$, and therefore, by (C_4) we get

$$\gamma|y(t)| \le \left|\frac{\partial}{\partial y}H(x(t), y(t))\right| \quad \text{for } \ a \le t \le b. \tag{3.18}$$

From the third equation of (E) with (3.3), (3.4), (3.10), (3.17) and (3.18), we obtain

$$\begin{aligned}|z'(t)| &\geq |g(t)|\gamma|y(t)| - |h(t)||z(t)| \\ &> \underline{g}\gamma \frac{\overline{h} + \beta_1\beta_2/(\beta_1 - \omega_-)}{\underline{g}\gamma}\sqrt{\frac{2\varepsilon_2}{k}} - \overline{h}\sqrt{\frac{2\varepsilon_2}{k}} \\ &= \frac{\beta_1\beta_2}{\beta_1 - \omega_-}\sqrt{\frac{2\varepsilon_2}{k}} > 0 \end{aligned} \tag{3.19}$$

for $a \leq t \leq b$. Since $z'(t)$ is continuous for $t \geq t_0$, we see that

$$\left| \int_a^b z'(s)ds \right| = \int_a^b |z'(s)|ds.$$

Hence, by (3.10), (3.16) and (3.19), we have

$$2\sqrt{\frac{2\varepsilon_2}{k}} \geq |z(a)| + |z(b)| \geq \int_a^b |z'(s)|ds > \frac{\beta_1\beta_2}{\beta_1 - \omega_-}\sqrt{\frac{2\varepsilon_2}{k}}(b-a) > 2\sqrt{\frac{2\varepsilon_2}{k}},$$

which is a contradiction. Thus, (3.14) holds.

Let $I = \bigcup_{n=1}^{\infty}[\tau_n, \sigma_n]$. From (2.5) and (3.9), we see that

$$\infty > \int_{t_0}^{\infty} |v'(s)|ds \geq \int_{t_0}^{\infty} \psi_+(s)u(s)ds > \varepsilon_2 \int_I \psi_+(s)ds.$$

Hence, it follows from (C_7) and (3.14) that

$$\liminf_{t\to\infty}(\sigma_n - \tau_n) = 0. \tag{3.20}$$

Since $\liminf_{t\to\infty} u(t) = 0 < \nu = \limsup_{t\to\infty} u(t)$, we can choose two sequences $\{t_n\}$ and $\{s_n\}$ with $T_2 < t_n < s_n < t_{n+1}$ such that $u(t_n) = \nu/2$, $u(s_n) = 3\nu/4$ and

$$\frac{\nu}{2} < u(t) < \frac{3\nu}{4} \quad \text{for} \;\; t_n < t < s_n.$$

Since $\varepsilon_2 < \nu/2$, we may consider that $[t_n, s_n] \subset [\tau_n, \sigma_n]$ for $n \in \mathbb{N}$ (if necessary, we can exchange $\{\tau_n\}$ and $\{\sigma_n\}$ into suitable subsequences of $\{\tau_n\}$ and $\{\sigma_n\}$). Hence, by (3.20) we have

$$\liminf_{n\to\infty}(s_n - t_n) = 0. \tag{3.21}$$

Since $\partial H(x,y)/\partial y$ is continuous, it follows from (C_3) and (3.2) that there exists an $l > 0$ such that

$$\left|\frac{\partial}{\partial y} H(x(t), y(t))\right| \leq l \quad \text{for} \ \ t \geq t_0.$$

Hence, together with (3.3), we have

$$\begin{aligned} u'(t) &= v'(t) - f(t)x'(t)z(t) \leq |v'(t)| + |f(t)||x'(t)||z(t)| \\ &\leq |v'(t)| + \overline{f}\left|\frac{\partial}{\partial y} H(x(t), y(t))\right| |z(t)| \leq |v'(t)| + \overline{f}\, l\, \varepsilon \end{aligned}$$

for $t \geq t_0$. Integrating this inequality from t_n to s_n, we get

$$\frac{\nu}{4} = |u(s_n)| - |u(t_n)| \leq \int_{t_n}^{s_n} |v'(s)| ds + \overline{f}\, l\, \varepsilon (s_n - t_n)$$

for each $n \in \mathbb{N}$. This contradicts (3.21), thereby completing the proof of part (iii).

(iv): From parts (ii) and (iii) above, we see that $\lim_{t\to\infty} u(t) = 0$. Since $u(t) \geq kz^2(t)/2$ for $t \geq t_0$, we have

$$\lim_{t\to\infty} z(t) = 0. \tag{3.22}$$

Because $u(t)$ converges to zero as $t \to \infty$, we can choose a $T_3 \geq t_0$ such that

$$0 \leq u(t) < \varepsilon_2 \quad \text{for} \ \ t \geq T_3.$$

Using this estimation instead of (3.9) and repeating the same process once more, we see that $(r(t), \theta(t))$ remains in the annulus A for $t \geq T_3$ and it rotates clockwise. Hence, we can find two divergent sequences $\{a_n\}$ and $\{b_n\}$ with $T_3 < a_n < b_n$ such that $\theta(a_n) - \theta(b_n) = \pi(1 - \omega_-/\beta_1)$ and $(r(t), \theta(t)) \in \Omega$ for $a_n \leq t \leq b_n$. By the same argument as in the proof of part (iii), we obtain

$$b_n - a_n > \frac{2(\theta(a_n) - \theta(b_n))}{3\beta_2} > \frac{2(\beta_1 - \omega_-)}{\beta_1\beta_2}$$

for each $n \in \mathbb{N}$ and

$$|z'(t)| > \frac{\beta_1\beta_2}{\beta_1 - \omega_-}\sqrt{\frac{2\varepsilon_2}{k}} \quad \text{for} \ \ a_n \leq t \leq b_n.$$

Hence, we have

$$|z(b_n) - z(a_n)| = \int_{a_n}^{b_n} |z'(s)|ds > \frac{\beta_1\beta_2}{\beta_1 - \omega_-}\sqrt{\frac{2\varepsilon_2}{k}}(b_n - a_n) > 2\sqrt{\frac{2\varepsilon_2}{k}}.$$

This contradicts (3.22). Thus, the case of $v_0 > 0$ does not happen. We therefore conclude that the zero solution of (E) is asymptotically stable.

The proof of Theorem 2.1 is thus complete. □

4. Examples

Needless to say, our main theorem can be applied not only to nonlinear systems but also to linear systems. As an example, we may consider the linear system with constant coefficients,

$$\begin{aligned} x' &= ax + by, \\ y' &= -cx - ay + fz, \\ z' &= -agx - bgy - hz. \end{aligned} \tag{4.1}$$

System (4.2) coincides with system (E) with

$$H(x,y) = \frac{c}{2}x^2 + axy + \frac{b}{2}y^2,$$

and $f(t) \equiv f$, $g(t) \equiv g$ and $h(t) \equiv h$.

We first consider the case $b > 0$. If $bc > a^2$, then $H(x,y) \geq \alpha_1(x^2 + y^2)$ for $\alpha_1 > 0$ sufficiently small. It is clear that $H(x,y) \leq \alpha_2(x^2 + y^2)$ for $\alpha_2 > 0$ sufficiently large. Note that

$$x\frac{\partial}{\partial x}H(x,y) + y\frac{\partial}{\partial y}H(x,y) = 2H(x,y).$$

Hence, conditions (C_1) and (C_2) are satisfied, provided that $bc > a^2$. Since

$$\frac{\partial}{\partial x}H(x,y) = cx + ay \to 0 \quad \text{and} \quad \frac{\partial}{\partial y}H(x,y) = ax + by \to 0$$

as $(x,y) \to (0,0)$, condition (C_3) holds. It is clear that (C_4) is satisfied with $0 < \mu < \min\{1, |b|/|a|\}$. Because $f(t)$ and $g(t)$ are constants, if $fg > 0$, then condition (C_5) is also satisfied. Moreover, it turns out that $\psi(t) = 2h$ for $t \geq 0$.

Hence, if $h > 0$, then $\psi_+(t) \equiv 2h$ and $\psi_-(t) \equiv 0$, and therefore, conditions (C_6) and (C_7) hold.

We next consider the case $b < 0$. By changing y into $-y$, system (4.2) becomes the system

$$\begin{aligned} x' &= ax - by, \\ y' &= cx - ay - fz, \\ z' &= -agx + bgy - hz, \end{aligned} \tag{4.2}$$

which has the form of (E) with

$$H(x, y) = -\frac{c}{2}x^2 + axy - \frac{b}{2}y^2,$$

and $f(t) \equiv -f$, $g(t) \equiv g$ and $h(t) \equiv h$. It is easy to verify that conditions (C_1)–(C_7) are satisfied if $bc > a^2$, $fg < 0$ and $h > 0$.

Thus, by means of Theorem 2.1, we have the following example.

Example 4.1. If $bc > a^2$, $bfg > 0$ and $h > 0$, then the zero solution of (4.2) is asymptotically stable.

Remark 4.2. Since system (4.2) is autonomous, if the zero solution is stable, then it is uniformly stable. Hence, the term "uniformly stable" is omitted in Example 4.1.

System (4.2) is equivalent to the third-order equation

$$x''' + hx'' + (bc - a^2 + bfg)x' + (bc - a^2)hx = 0.$$

By using the Routh-Hurwitz criterion, it can be shown that the zero solution of (4.2) is asymptotically stable if and only if the determinants $\mathit{\Delta}_1 = h$,

$$\Delta_2 = \det\begin{pmatrix} h & (bc - a^2)h \\ 1 & bc - a^2 + bfg \end{pmatrix} = bfgh$$

and

$$\Delta_3 = \det\begin{pmatrix} h & (bc - a^2)h & 0 \\ 1 & bc - a^2 + bfg & 0 \\ 0 & h & (bc - a^2)h \end{pmatrix} = bfgh^2(bc - a^2)$$

are positive, namely, $h > 0$, $bfg > 0$ and $bc > a^2$. This condition is the same as in Example 4.1. Hence, it is safe to say that our main theorem is considerably sharp.

To illustrate Theorem 2.1 from a different point of view, we will take another example. Define

$$H(x,y) = x^2 + y^2 + \frac{2xy(x^2 - y^2)}{x^2 + y^2}, \qquad H(0,0) = 0. \tag{4.3}$$

Let $x = r\cos\theta$ and $y = \sin\theta$. Then we have

$$\begin{aligned} H(r\cos\theta, r\sin\theta) &= r^2 + 2\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)r^2 \\ &= (1 + 2\cos\theta\sin\theta\cos 2\theta)r^2 \\ &= (1 + \sin 2\theta\cos 2\theta)r^2 \\ &= \left(1 + \frac{1}{2}\sin 4\theta\right)r^2. \end{aligned}$$

Hence, $H(x,y)$ is continuous at the origin and condition (C_1) is satisfied with $\alpha_1 = 1/2$ and $\alpha_2 = 3/2$. Since

$$x\frac{\partial}{\partial x}H(x,y) + y\frac{\partial}{\partial y}H(x,y) = 2H(x,y),$$

condition (C_2) is satisfied with $\beta_1 = 1$ and $\beta_2 = 3$. By a simple calculation, we obtain

$$\frac{\partial}{\partial x}H(x,y) = 2x + \frac{2(x^4y + 4x^2y^3 - y^5)}{(x^2+y^2)^2}$$

and

$$\frac{\partial}{\partial y}H(x,y) = 2y + \frac{2(x^5 - 4x^3y^2 - xy^4)}{(x^2+y^2)^2}$$

for $(x,y) \neq (0,0)$. Hence, we can verify that

$$\lim_{(x,y)\to(0,0)} \frac{\partial}{\partial x}H(x,y) = \lim_{(x,y)\to(0,0)} \frac{\partial}{\partial y}H(x,y) = 0;$$

that is, condition (C_3) holds. We can estimate that

$$\begin{aligned} \left|\frac{\partial}{\partial y}H(x,y)\right| &\geq 2|y| - \frac{2(|x|^5 + 4|x|^3|y|^2 + |x||y|^4)}{(x^2+y^2)^2} \\ &\geq 2|y| - \frac{2(|x|^5 + 4|x|^3|y|^2 + |x||y|^4)}{|y|^4} \end{aligned}$$

for $|y| \neq 0$. Let $\gamma > 0$ and $0 < \mu \leq 1$ be numbers satisfying $\gamma = 2(1 - \mu - 4\mu^3 - \mu^5)$. Then, it turns out that $0 < |x| \leq \mu|y|$ implies $|\partial H(x, y)/\partial y| \geq \gamma|y|$. We therefore conclude that condition (C_4) also holds.

Let $f(t)$, $g(t)$ and $h(t)$ be constants f, g and h, respectively. Then, $fg > 0$ and $h > 0$ imply that conditions (C_5)–(C_7) hold. Thus, by virtue of Theorem 2.1, we have the following example.

Example 4.3. Consider system (E) with (4.3), and $f(t)$, $g(t)$ and $h(t)$ being constants f, g and h, respectively. If $fg > 0$ and $h > 0$, then the zero solution of (E) is asymptotically stable.

Remark 4.4. The Routh-Hurwitz criterion is inapplicable to system (E) given in Example 4.3, because we cannot find the corresponding linear approximation.

Example 4.5. Consider system (E) with (4.3). Suppose that $f(t) = g(t) = h(t) = 1/(2+\sin t)$. Then the zero solution of (E) is uniformly stable and asymptotically stable.

Since $H(x, y)$ is the same as that of Example 4.3, we see that conditions (C_1)–(C_4) hold. The difference between Examples 4.3 and 4.5 is whether the coefficients converge or not. In Example 4.5, $f(t)$, $g(t)$ and $h(t)$ oscillate periodically between $1/3$ and 1, and therefore, they do not converge to any constants. It is easy to verify that $f(t)g(t) \geq 1/9$, $\psi_+(t) = 2h(t) \geq 2/3$ and $\psi_-(t) = 0$ for $t \geq 0$. Hence, conditions (C_5)–(C_7) are satisfied.

In the figure on the next page, we draw a positive orbit of Example 4.5 and its projections onto the x-y plane, the y-z plane and the x-z plane. The initial time $t_0 = 0$ and the starting point $(x_0, y_0, z_0) = (1, 1, 1)$. The positive orbit approaches the origin $(0, 0, 0)$ by regular but somewhat complicated motion.

Although $h(t)$ is positive for $t \geq 0$ in Example 4.5, the positivity of $h(t)$ is not necessarily required for our main theorem. For example, let

$$h(t) = \frac{\sin^2 t}{1 + t} - \frac{\cos^2 t}{(1 + t)^2}. \tag{4.4}$$

Then conditions (C_6) and (C_7) are satisfied, but $h(t)$ is negative at $t = n\pi$ for $n \in \mathbb{N}$. Hence, we can rewrite Example 4.5 as follows.

Example 4.6. Consider system (E) with (4.3) and (4.4). Suppose that $f(t) = g(t) = 1/(2+\sin t)$. Then the zero solution of (E) is uniformly stable and asymptotically stable.

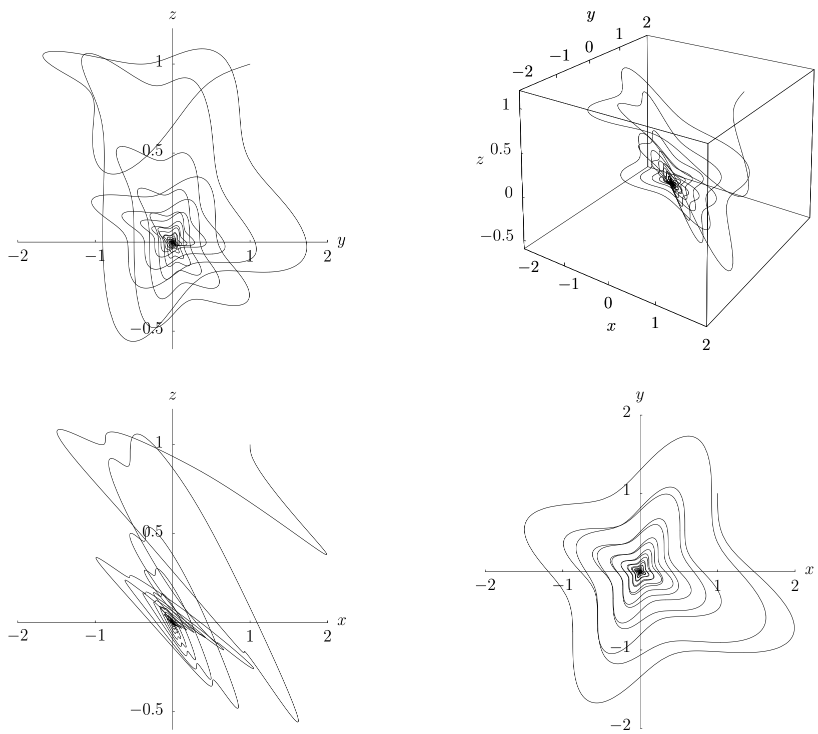

Figure 1: A positive orbit of Example 4.5.

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References

[1] F. Brauer, J. Nohel, Qualitative Theory of Ordinary Differential Equations, W. A. Benjamin, New York and Amsterdam, 1969.

[2] L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, 2nd edition, Springer-Verlag, Berlin, 1963.

[3] W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath, Boston, 1965.

[4] A. Halanay, Differential Equations: Stability, Oscillations, Time Lags, Academic Press, New York and London, 1966.

[5] J. K. Hale, Ordinary Differential Equations, Wiley-Interscience, New York, London and Sydney, 1969; (revised) Krieger, Malabar, 1980.

[6] L. Hatvani, A generalization of the Barbashin-Krasovskij theorems to the partial stability in nonautonomous systems, Qualitative Theory of Differential Equations I, II (ed. M. Farkas), Colloq. Math. Soc. János Bolyai **30**, North-Holland, Amsterdam and New York, 1981, pp. 381–409.

[7] L. Hatvani, On the uniform attractivity of solutions of ordinary differential equations by two Lyapunov functions, Proc. Japan Acad. 67 (1991) 162–167.

[8] L. Hatvani, On the asymptotic stability for a two-dimensional linear nonautonomous differential system, Nonlinear Anal. 25 (1995) 991–1002.

[9] A. Hurwitz, Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt, Math. Ann. 46 (1895) 273–284.

[10] D. R. Merkin, Introduction to the Theory of Stability, Texts in Applied Mathematics 24, Springer-Verlag, New York, Berlin and Heidelberg, 1997.

[11] N. Rouche, P. Habets, M. Laloy, Stability Theory by Liapunov's Direct Method, Applied Mathematical Sciences 22, Springer-Verlag, New York, Heidelberg and Berlin, 1977.

[12] E. J. Routh, Treatise on the Stability of a Given State of Motion, Macmillan, London, 1877.

[13] J. Sugie, Influence of anti-diagonals on the asymptotic stability for linear differential systems, Monatsh. Math. 157 (2009) 163–176.

[14] J. Sugie, Convergence of solutions of time-varying linear systems with integrable forcing term, Bull. Austral. Math. Soc. 78 (2008) 445–462.

[15] J. Sugie and Y. Ogami, Asymptotic stability for three-dimensional linear differential systems with time-varying coefficients, Quart. Appl. Math. 67 (2009) 687–705.

[16] J. Sugie, S. Hata, M. Onitsuka, Global asymptotic stability for half-linear differential systems with periodic coefficients, J. Math. Anal. Appl. 371 (2010) 95–112.

[17] T. Yoshizawa, Stability Theory by Liapunov's Second Method, Math. Soc. Japan, Tokyo, 1966.