# Global asymptotic stability for damped half-linear oscillators

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**Abstract**

A necessary and sufficient condition is established for the equilibrium of the oscillator of half-linear type with a damping term,

$$(\phi_p(x'))' + h(t)\phi_p(x') + \phi_p(x) = 0$$

to be globally asymptotically stable. The obtained criterion is given by the form of a certain growth condition of the damping coefficient $h(t)$ and it can be applied to not only the cases of large damping and small damping but also the case of fluctuating damping. The presented result is new even in the linear cases ($p = 2$). It is also discussed whether a solution of the half-linear differential equation

$$(r(t)\phi_p(x'))' + c(t)\phi_p(x) = 0$$

that converges to a non-zero value exists or not. Some suitable examples are included to illustrate the results in the present paper.

*Key words:* Growth condition; Damped oscillator; Half-linear differential equations; Global asymptotic stability
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## 1. Introduction

The purpose of this paper is to show that a growth condition on $h(t)$ is a necessary and sufficient condition for the equilibrium of the second-order differential equation

$$(\phi_p(x'))' + h(t)\phi_p(x') + \phi_p(x) = 0 \qquad (HL)$$

to be globally asymptotically stable. Here, the prime denotes $d/dt$, the function $\phi_p(z)$ is defined by

$$\phi_p(z) = |z|^{p-2}z, \qquad z \in \mathbb{R}$$

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with $p > 1$, and the damping coefficient $h(t)$ is continuous and nonnegative for $t \geq 0$. Let

$$H(t) = \int_0^t h(s)ds.$$

If $x(t)$ is a solution of ($HL$), then the function $cx(t)$ is another solution of ($HL$), where $c$ is an arbitrary constant except 1. In general, however, the total of two solutions of ($HL$) is not a solution of ($HL$). Hence, the solution space of ($HL$) is homogeneous, but not additive. Because there is only characteristic half of the solution space of linear differential equations, Eq. ($HL$) is often called *half-linear*.

Let $\mathbf{x}(t) = (x(t), x'(t))$ and $\mathbf{x}_0 \in \mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of ($HL$) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. It is clear that Eq. ($HL$) has the equilibrium $\mathbf{x}(t) \equiv \mathbf{0}$.

The equilibrium is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The equilibrium is said to be *attractive* if, for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The equilibrium is said to be *globally attractive* if, for any $t_0 \geq 0$, any $\eta > 0$ and any $\mathbf{x}_0 \in \mathbb{R}^2$, there is a $T(t_0, \eta, \mathbf{x}_0) > 0$ such that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(t_0, \eta, \mathbf{x}_0)$. The equilibrium is *asymptotically stable* if it is stable and attractive. The equilibrium is *globally asymptotically stable* if it is stable and globally attractive. About those definitions, refer to the books [3, 6, 7, 8, 17, 18, 29, 34, 45] for example.

Since $\phi_2(z) = z$, we can consider the damped linear oscillator

$$x'' + h(t)x' + x = 0 \tag{$L$}$$

to be a special case of ($HL$). In the linear differential equations such as Eq. ($L$), it is well known that the equilibrium is attractive (resp., asymptotically stable), then it is globally attractive (resp., globally asymptotically stable). The study of the (global) asymptotic stability for Eq. ($L$) (or its general type) is one of the major themes in the qualitative theory of differential equations. Numerous papers have been devoted to find sufficient conditions and necessary conditions for the asymptotic stability (for example, see [2, 4, 15, 21, 22, 23, 24, 25, 27, 31, 32, 35]).

We can cite Levin and Nohel [27, Theorem 1] as a pioneering work (their result can be applied to more general equations than Eq. ($L$)). They proved that if there exist two positive constants $\underline{h}$ and $\overline{h}$ such that $\underline{h} \leq h(t) \leq \overline{h}$ for $t \geq 0$, then the equilibrium of ($L$) is asymptotically stable. The researches afterwards have advanced toward the direction where at least one of the lower bound $\underline{h}$ or the upper bound $\overline{h}$ is taken off. The case in which $\underline{h} \leq h(t) < \infty$ for $t \geq 0$ and the case in which $0 \leq h(t) \leq \overline{h}$ for $t \geq 0$ are often called *large damping* and *small damping*, respectively.

In the case of large damping, we should first make the special mention of Smith [35, Theorems 1 and 2]. He proved that

$$\int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\, dt = \infty. \tag{1.1}$$

is a necessary and sufficient condition for the equilibrium of $(L)$ to be asymptotically stable. Later, Ballieu and Peiffer [4] obtained several sufficient conditions and necessary conditions for the equilibrium of a certain kind of nonlinear differential equation to be globally asymptotically stable and presented the same criterion as Smith's by using their results (see [4, Corollary 6]). Although the expression of condition (1.1) is very concise, it is not so easy to confirm whether condition (1.1) is satisfied. For this reason, many attempts were carried out to look for other growth conditions that guarantee the asymptotic stability for Eq. $(L)$ or more general nonlinear equations. Artstein and Infante [2] showed that if $H(t)/t^2$ is bounded for $t$ sufficient large, then the equilibrium of $(L)$ is asymptotically stable. When an indefinite integral of $h(t)$ can be obtained, we can confirm their growth condition. In Artstein and Infante's result, the exponent 2 is the best possible in the meaning that it cannot be changed to any $2+\varepsilon$, $\varepsilon > 0$. However, their growth condition is weaker than condition (1.1) because it is sufficient for the asymptotic stability, but not necessary. For example, consider Eq. $(L)$ with large damping $h(t) = (2+t)\log(2+t)$. Then, it is easy to check that $H(t)/t^2$ is unbounded. Hence, Artstein and Infante's result is unavailable. However, it is known that the equilibrium of $(L)$ with $h(t) = (2+t)\log(2+t)$ is asymptotically stable (see [4, Corollary 7]). Hatvani, Krisztin and Totik [23] proved that the growth condition (1.1) on $h(t)$ is equivalent to

$$\sum_{n=1}^{\infty}\bigl(H^{-1}(nc) - H^{-1}((n-1)c)\bigr)^2 = \infty \tag{1.2}$$

for any $c > 0$, where $H^{-1}(s)$ denote the inverse function of $s = H(t)$. The merit of the discrete criterion (1.2) is that it is sometimes easier to check it. For example, we see that if $h(t) = t$, then condition (1.2) is satisfied; if $h(t) = t^2$, then condition (1.2) is not satisfied. However, in general case, it is still difficult to verify condition (1.2). Fortunately, unlike old times, there is a possibility that condition (1.1) can be confirmed by using numerical analysis conducted via personal computer even if it is impossible by the human hand calculation.

Ballieu and Peiffer [4, Theorems 5 and 6] also discussed the case of small damping. From their results, we see that the equilibrium of $(L)$ is asymptotically stable if and only if $H(t)$ tends to $\infty$ as $t \to \infty$, provided that $h(t)$ is positive and nonincreasing for $t \geq 0$. Later, in the case of small damping, Hatvani [21, Corollary 4.4] showed that the weak integral positivity of $h(t)$ implies the asymptotic stability for Eq. $(L)$ (see also [36, 37]). For the definition of the weak integral positivity, see Section 3. Moreover, Hatvani [22, Theorem 1.1] proved that if $\limsup_{t\to\infty} H(t)/t^{2/3} > 0$, then the equilibrium of $(L)$ is asymptotically stable and pointed out that the exponent 2/3 is the best possible in the meaning that it cannot be changed to any $2/3 - \varepsilon$, $\varepsilon > 0$. Although his condition is very sharp, it is not necessary and sufficient for the asymptotic stability. There are a lot of other works in the case of small damping, but no necessary and sufficient condition such as (1.1) has been reported at all.

The case in which $h(t)$ has neither the lower bound $\underline{h}$ nor the upper bound $\overline{h}$ may be most difficult in the study of the asymptotic stability for Eq. $(L)$. Let us call such a case *fluctuating damping*. Pucci and Serrin [31, Theorem A] considered $N$-dimensional nonlinear systems which contain Eq. $(L)$ as a special case and presented sufficient conditions and necessary conditions for the global asymptotic stability applied even to the case of fluctuating damping (see also [32]). As another result that can even be applied to the case of fluctuating damping, we can cite Hatvani and Totik [24, Theorem 3.1]. We will compare our result with the result of them in the last part of this paper.

Let us now return to Eq. $(HL)$ that is the research object of this paper. Because Eq. $(HL)$ is a generalization of the damped linear oscillator $(L)$, we will call Eq. $(HL)$ the *damped half-linear oscillator*. In Eq. $(HL)$ as well as Eq. $(L)$, we may classify the damping coefficient $h(t)$ into three types by the presence or absence of the lower bound $\underline{h}$ and the upper bound $\overline{h}$.

Sugie and Onitsuka [40, Theorem 2.1] have considered a system of differential equations of the form

$$\begin{aligned} x' &= -\,e(t)x + f(t)\phi_{p^*}(y), \\ y' &= -\,g(t)\phi_p(x) - h(t)y, \end{aligned} \tag{1.3}$$

where $p^* = p/(p-1)$, and proved that under the assumptions

(i) $E(t) \stackrel{\text{def}}{=} \displaystyle\int_0^t e(s)ds$, $f(t)$, $g(t)$ and $h(t)$ are bounded and $g(t)/f(t)$ is continuously differentiable for $t \geq 0$,

(ii) $f(t)$ and $g(t)$ have the same sign for $t \geq 0$ with $\liminf\limits_{t\to\infty} f(t)g(t) > 0$,

the zero solution of (1.3) is globally asmptotically stable if the function

$$p^* h(t) - p\, e(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'$$

is nonnegative for $t \geq 0$ and weakly integrally positive (as related researches, refer to [38, 39]). As shown in the first paragraph in Section 2, $\phi_{p^*}$ is the inverse function of $\phi_p$. Letting $y = \phi_p(x')$ as a new variable, we see that Eq. $(HL)$ is equivalent to system (1.3) with $e(t) = 0$ and $f(t) = g(t) = 1$. Hence, it turns out that if there exists an $\overline{h}$ such that $0 \leq h(t) \leq \overline{h}$ for $t \geq 0$ and if $h(t)$ is weakly integrally positive, then the equilibrium is globally asymptotically stable. This case is small damping. The above-mentioned result cannot be applied to the cases of large damping and fluctuating damping.

When we consider the cases of large damping and fluctuating damping, we have to take notice of the possibility that the so-called overdamping phenomenon happens. The phenomenon of overdamping is that a solution converging to a non-zero value exists. This phenomenon is caused by too fast growth of the damping coefficient $h(t)$. Recently, Sugie and Hata [38, Section 6] have pointed out that this phenomenon appeared to not only Eq. $(L)$ but also Eq. $(HL)$.

In this paper, we intend to establish a criterion for the equilibrium of $(HL)$ to be globally asymptotically stable which can even be applied to the cases of large damping and fluctuating damping. Our criterion is expressed in the form of a growth condition on $h(t)$ which is a generalization of (1.1). Needless to say, this criterion excludes the phenomenon of overdamping.

## 2. Convergence and divergence of solutions

Consider the half-linear second order differential equation

$$(r(t)\phi_p(x'))' + c(t)\phi_p(x) = 0, \tag{2.1}$$

where $r(t)$ and $c(t)$ are real continuous functions, $r(t) \neq 0$ for $t \geq 0$ and $c(t) \not\equiv 0$. Let $p^*$ be the conjugate number of $p$; namely,

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

then $p^*$ is also greater than $1$. Let

$$w = \phi_p(z) = \begin{cases} z^{p-1} & \text{if } z \geq 0 \\ -(-z)^{p-1} & \text{if } z < 0. \end{cases}$$

Then, $z \geq 0$ if and only if $w \geq 0$, and

$$z = \begin{cases} w^{1/(p-1)} & \text{if } w \geq 0 \\ -(-w)^{1/(p-1)} & \text{if } w < 0. \end{cases}$$

Since $(p-1)(p^*-1) = 1$, it follows that $w^{1/(p-1)} = w^{p^*-1} = |w|^{p^*-2}w$ if $w \geq 0$ and $-(-w)^{1/(p-1)} = -(-w)^{p^*-1} = (-w)^{p^*-2}w = |w|^{p^*-2}w$ if $w < 0$. Hence, $z = \phi_{p^*}(w)$; namely, $\phi_{p^*}$ is the inverse function of $\phi_p$.

It is known that for any $t_0 \geq 0$ and $(c_1, c_2) \in \mathbb{R}^2$, there exists a unique solution of (2.1) satisfying $x(t_0) = c_1$ and $x'(t_0) = c_2$ which is continuable in the future. For details, see Došlý [9, p. 170] or Došlý and Řehák [10, pp. 8–10]. Hence, the global existence and uniqueness of solutions of (2.1) are guaranteed for the initial value problem.

Over the last four decades, a considerable number of studies have been made on the half-linear differential equation (2.1). Especially, many good articles concerning oscillation theory have been presented. Those results can be found in the books [1, 9, 10] and the references cited therein. Even after these books are published, Eq. (2.1) keeps being actively researched (for example, see [5, 11, 12, 13, 14, 30, 33]). In this section, we discuss the asymptotic behavior of solutions of (2.1) from a different angle.

It is clear that

$$\phi_{p^*}(XY) = \phi_{p^*}(X)\phi_{p^*}(Y) \quad \text{for } X \in \mathbb{R} \text{ and } Y \in \mathbb{R}. \tag{2.2}$$

Since $\phi_{p^*}$ is an increasing function, we see that

$$\phi_{p^*}\left(\frac{X+Y}{2}\right) \leq \max\left\{\phi_{p^*}(X), \phi_{p^*}(Y)\right\} \leq \phi_{p^*}(X) + \phi_{p^*}(Y),$$

or

$$\phi_{p^*}(X+Y) \leq \phi_{p^*}(2)\left\{\phi_{p^*}(X) + \phi_{p^*}(Y)\right\}. \tag{2.3}$$

for $X \geq 0$ and $Y \geq 0$. Using the properties (2.2) and (2.3) of $\phi_{p^*}$, we have the following result on the existence of solutions of (2.1) converging to a non-zero value as time increases.

**Theorem 2.1.** *Suppose that*

$$\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t |c(s)|ds}{|r(t)|}\right)dt < \infty. \tag{2.4}$$

*Then every solution* $x(t)$ *of* (2.1) *tends to a finite limit* $x(\infty)$ *as* $t \to \infty$ *and* $x(\infty)$ *does not vanish for at least one solution of* (2.1).

**Proof.** Let $x(t)$ be any solution of (2.1) with the initial time $t_0 \geq 0$. Integrating both sides of (2.1), we obtain

$$r(t)\phi_p(x'(t)) - r(T)\phi_p(x'(T)) + \int_T^t c(s)\phi_p(x(s))ds = 0,$$

where $T$ is a sufficiently large number. Since $r(t) \neq 0$ for $t \geq 0$, we get

$$x'(t) = \phi_{p^*}\left(-\frac{\int_T^t c(s)\phi_p(x(s))ds}{r(t)} + \frac{A}{r(t)}\right), \tag{2.5}$$

where $A = r(T)\phi_p(x'(T))$. Integrate both sides of (2.5) from $T$ to $t$ to obtain

$$x(t) = \int_T^t \phi_{p^*}\left(-\frac{\int_T^s c(\tau)\phi_p(x(\tau))d\tau}{r(s)} + \frac{A}{r(s)}\right)ds + B, \tag{2.6}$$

where $B = x(T)$. Conversely, for every $T > 0$ and any pair of integration constants $A$ and $B$, the function $x(t)$ given by (2.6) is a solution of (2.1) satisfying $A = r(T)\phi_p(x'(T))$ and $B = x(T)$.

Define

$$M_t = \max_{T \leq s \leq t} |x(s)|.$$

Of course, $M_t$ is nondecreasing for $t \geq T$. By (2.6), we have

$$\begin{aligned}
|x(t)| &= \left|\int_T^t \phi_{p^*}\left(-\frac{\int_T^s c(\tau)\phi_p(x(\tau))d\tau}{r(s)} + \frac{A}{r(s)}\right)ds + B\right| \\
&\leq \int_T^t \left|\phi_{p^*}\left(-\frac{\int_T^s c(\tau)\phi_p(x(\tau))d\tau}{r(s)} + \frac{A}{r(s)}\right)\right|ds + |B| \\
&\leq \int_T^t \phi_{p^*}\left(\frac{\int_T^s |c(\tau)||\phi_p(x(\tau))|d\tau}{|r(s)|} + \frac{|A|}{|r(s)|}\right)ds + |B| \\
&= \int_T^t \phi_{p^*}\left(\frac{\int_T^s |c(\tau)|\phi_p(|x(\tau)|)d\tau}{|r(s)|} + \frac{|A|}{|r(s)|}\right)ds + |B|.
\end{aligned}$$

Taking into account that

$$\phi_p(|x(\tau)|) \leq \phi_p(M_\tau) \leq \phi_p(M_s)$$

for $s \geq \tau \geq T$, we obtain

$$|x(t)| \leq \int_T^t \phi_{p^*}\biggl(\frac{\phi_p(M_s)\int_T^s |c(\tau)|d\tau}{|r(s)|} + \frac{|A|}{|r(s)|}\biggr)ds + |B|.$$

Hence, using (2.2), (2.3) and the fact that $M_s \leq M_t$ for $t \geq s$, we get

$$\begin{aligned}
|x(t)| &\leq \int_T^t \phi_{p^*}(2)\biggl\{\phi_{p^*}\biggl(\frac{\phi_p(M_s)\int_T^s |c(\tau)|d\tau}{|r(s)|}\biggr) + \phi_{p^*}\biggl(\frac{|A|}{|r(s)|}\biggr)\biggr\}ds + |B| \\
&\leq \phi_{p^*}(2)\int_T^t M_s\phi_{p^*}\biggl(\frac{\int_T^s |c(\tau)|d\tau}{|r(s)|}\biggr)ds + \phi_{p^*}(2)\int_T^t \phi_{p^*}\biggl(\frac{|A|}{|r(s)|}\biggr)ds + |B| \\
&\leq M_t\phi_{p^*}(2)\int_T^t \phi_{p^*}\biggl(\frac{\int_T^s |c(\tau)|d\tau}{|r(s)|}\biggr)ds + \phi_{p^*}(2|A|)\int_T^t \phi_{p^*}\biggl(\frac{1}{|r(s)|}\biggr)ds + |B|.
\end{aligned}$$

Since $c(t) \not\equiv 0$ and $T$ is sufficiently large, we may assume without loss of generality that there exists a $\lambda > 0$ such that

$$\int_0^t |c(s)|ds \geq \lambda \quad \text{for} \;\; t \geq T.$$

From (2.2), (2.4) and this estimation, we see that

$$\begin{aligned}
\infty &> \int_0^\infty \phi_{p^*}\biggl(\frac{\int_0^t |c(s)|ds}{|r(t)|}\biggr)dt \\
&= \int_0^T \phi_{p^*}\biggl(\frac{\int_0^t |c(s)|ds}{|r(t)|}\biggr)dt + \int_T^\infty \phi_{p^*}\biggl(\frac{\int_0^t |c(s)|ds}{|r(t)|}\biggr)dt \\
&\geq \int_T^\infty \phi_{p^*}\biggl(\frac{\lambda}{|r(t)|}\biggr)dt = \phi_{p^*}(\lambda)\int_T^\infty \phi_{p^*}\biggl(\frac{1}{|r(t)|}\biggr)dt,
\end{aligned}$$

and therefore,

$$\int_T^\infty \phi_{p^*}\biggl(\frac{1}{|r(t)|}\biggr)dt < \infty. \tag{2.7}$$

Hence, we conclude that

$$|x(t)| \leq M_t\phi_{p^*}(2)\int_T^t \phi_{p^*}\biggl(\frac{\int_T^s |c(\tau)|d\tau}{|r(s)|}\biggr)ds + C, \tag{2.8}$$

where

$$C = \phi_{p^*}(2|A|)\int_T^\infty \phi_{p^*}\biggl(\frac{1}{|r(t)|}\biggr)dt + |B| \geq 0. \tag{2.9}$$

Let

$$K = K(T) = \phi_{p^*}(2)\int_T^\infty \phi_{p^*}\biggl(\frac{\int_T^t |c(s)|ds}{|r(t)|}\biggr)dt.$$

Then, by (2.8), we have

$$M_t \le KM_t + C \quad \text{for} \ \ t \ge T. \tag{2.10}$$

From (2.4), we see that the integral $K$ is convergent and its value tends to $0$ as $T \to \infty$. Let $T$ be so large that $K$ becomes less than $1$. Then, it follows from (2.10) that

$$|x(t)| \le M_t \le \frac{C}{1-K} \quad \text{for} \ \ t \ge T. \tag{2.11}$$

This means that $x(t)$ is bounded.

Using (2.3) and (2.5), we obtain

$$\begin{aligned}
|x'(t)| &= \phi_{p^*}\left(\left|-\frac{\int_T^t c(s)\phi_p(x(s))ds}{r(t)} + \frac{A}{r(t)}\right|\right) \\
&\le \phi_{p^*}\left(\frac{\int_T^t |c(s)|\phi_p(|x(s)|)ds}{|r(t)|} + \frac{|A|}{|r(t)|}\right) \\
&\le \phi_{p^*}(2)\left\{\phi_{p^*}\left(\frac{\int_T^t |c(s)|\phi_p(|x(s)|)ds}{|r(t)|}\right) + \phi_{p^*}\left(\frac{|A|}{|r(t)|}\right)\right\}
\end{aligned}$$

for $t \ge T$. Combining (2.2), (2.4), (2.7) and (2.11), we get

$$\begin{aligned}
\int_T^\infty |x'(t)|dt &\le \phi_{p^*}(2)\int_T^\infty \phi_{p^*}\left(\frac{\int_T^t |c(s)|\phi_p(|x(s)|)ds}{|r(t)|}\right)dt \\
&\qquad + \phi_{p^*}(2|A|)\int_T^\infty \phi_{p^*}\left(\frac{1}{|r(t)|}\right)dt \\
&\le \frac{C}{1-K}\phi_{p^*}(2)\int_T^\infty \phi_{p^*}\left(\frac{\int_T^t |c(s)|ds}{|r(t)|}\right)dt \\
&\qquad + \phi_{p^*}(2|A|)\int_T^\infty \phi_{p^*}\left(\frac{1}{|r(t)|}\right)dt < \infty.
\end{aligned}$$

In other words, $x(t)$ is of bounded variation for $t \ge T$. Hence, there exists a finite limit $x(\infty)$.

To complete the proof of Theorem 2.1, we have only to show that Eq. (2.1) has a solution $x(t)$ for which $x(\infty) \ne 0$. Since the existence of a finite limit $x(\infty)$ is guaranteed, it follows from (2.6) that

$$x(\infty) = \int_T^\infty \phi_{p^*}\left(-\frac{\int_T^t c(s)\phi_p(x(s))ds}{r(t)} + \frac{A}{r(t)}\right)dt + B. \tag{2.12}$$

Also, since $K(T)$ tends to $0$ as $T \to \infty$, we can choose $T$ so large that $K < 1/2$. Hence, from (2.11), we see that

$$|x(t)| < 2C \quad \text{for} \ \ t \ge T. \tag{2.13}$$

In particular, we consider the case in which $A = 0$ and $B = 1$. Then, from (2.9), (2.12) and (2.13), it turns out that $C = 1$,

$$x(\infty) = -\int_T^\infty \phi_{p^*}\left(\frac{\int_T^t c(s)\phi_p(x(s))ds}{r(t)}\right)dt + 1$$

and

$$|x(t)| < 2 \quad \text{for} \;\; t \geq T,$$

respectively. Hence, we can estimate that

$$\begin{aligned}
1 &\leq |x(\infty)| + \left|\int_T^\infty \phi_{p^*}\left(\frac{\int_T^t c(s)\phi_p(x(s))ds}{r(t)}\right)dt\right| \\
&\leq |x(\infty)| + \int_T^\infty \phi_{p^*}\left(\frac{\int_T^t |c(s)|\phi_p(|x(s)|)ds}{|r(t)|}\right)dt \\
&\leq |x(\infty)| + \int_T^\infty \phi_{p^*}\left(\frac{\phi_p(2)\int_T^t |c(s)|ds}{|r(t)|}\right)dt \\
&= |x(\infty)| + 2\int_T^\infty \phi_{p^*}\left(\frac{\int_T^t |c(s)|ds}{|r(t)|}\right)dt \\
&= |x(\infty)| + \frac{2K}{\phi_{p^*}(2)}
\end{aligned}$$

for $t \geq T$. Since $\phi_{p^*}(2) = 2^{p^*-1} > 1$ and $K < 1/2$, we obtain

$$|x(\infty)| > 1 - 2K > 0.$$

Consequently, Eq. (2.1) has the solution

$$x(t) = -\int_T^t \phi_{p^*}\left(\frac{\int_T^s c(\tau)\phi_p(x(\tau))d\tau}{r(s)}\right)ds + 1 \quad \text{for} \;\; t \geq T,$$

whose finite limit $x(\infty)$ is not zero.

This completes the proof of Theorem 2.1. $\square$

In Theorem 2.1, $r(t)$ is allowed to be negative for $t \geq 0$ and $c(t)$ is allowed to change its sign. In the special case in which $p = 2$, Eq. (2.1) becomes the linear differential equation

$$(r(t)x')' + c(t)x = 0. \tag{2.14}$$

Wintner [44] have presented the following result (see also Weyl [43]).

**Theorem A.** *Suppose that*

$$\int_0^\infty \frac{\int_0^t |c(s)|ds}{|r(t)|}\,dt < \infty.$$

*Then every solution $x(t)$ of* (2.14) *tends to a finite limit $x(\infty)$ as $t \to \infty$ and $x(\infty)$ does not vanish for at least one solution of* (2.14).

Theorem 2.1 is a natural generalization of Theorem A from the linear differential equation (2.14) to the half-linear differential equation (2.1).

As Theorem 2.1 shown, condition (2.4) is sufficient for Eq. (2.1) to have a solution $x(t)$ converging to a non-zero value as $t \to \infty$. The following result shows that condition (2.4) is also necessary for this problem in the case in which $r(t)$ and $c(t)$ have the same sign for $t \geq 0$.

**Theorem 2.2.** *Suppose that $r(t)c(t) > 0$ for $t \geq 0$. If*

$$\int_0^\infty |c(t)|dt = \infty \tag{2.15}$$

*and*

$$\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t |c(s)|ds}{|r(t)|}\right)dt = \infty, \tag{2.16}$$

*then Eq.* (2.1) *fails to have a solution $x(t)$ with $x(\infty) \neq 0$* (*a finite limit $x(\infty)$ might be infinity or not exists*).

**Proof.** By way of contradiction, we suppose that there exists a solution of (2.1) whose finite limit $x(\infty)$ is positive. The proof of the case in which $x(\infty) < 0$ is carried out in the same way as the proof of the case in which $x(\infty) > 0$.

We can find a $T > 0$ such that

$$x(t) > \frac{1}{2}x(\infty) > 0 \quad \text{for} \;\; t \geq T. \tag{2.17}$$

Integrating both sides of (2.1) twice, we obtain

$$x(t) = \int_T^t \phi_{p^*}\left(-\frac{\int_T^s c(\tau)\phi_p(x(\tau))d\tau}{r(s)} + \frac{A}{r(s)}\right)ds + B, \tag{2.18}$$

where $A = r(T)\phi_p(x'(T))$ and $B = x(T)$. Taking into account that $r(t)$ and $c(t)$ have the same sign for $t \geq 0$, we get

$$\begin{aligned} -\frac{\int_T^t c(s)\phi_p(x(s))ds}{r(t)} + \frac{A}{r(t)} &= -\frac{\int_T^t |c(s)|\phi_p(x(s))ds}{|r(t)|} + \frac{A}{r(t)} \\ &\leq -\frac{\int_T^t |c(s)|\phi_p(x(s))ds}{|r(t)|} + \frac{|A|}{|r(t)|} \end{aligned}$$

for $t \geq T$. Since

$$\phi_p(x(t)) > \phi_p(x(\infty)/2) \quad \text{for} \;\; t \geq T$$

by (2.17), it follows that

$$-\frac{\int_T^t c(s)\phi_p(x(s))ds}{r(t)} + \frac{A}{r(t)} < -\frac{\phi_p(x(\infty)/2)\int_T^t |c(s)|ds}{|r(t)|} + \frac{|A|}{|r(t)|} \tag{2.19}$$

for $t \geq T$. From (2.15), we can choose $T_1$ so large that $T_1 > T$ and

$$\left(1 - \frac{1}{\phi_p(2)}\right) \int_T^{T_1} |c(t)| dt > \max \left\{ \frac{|A|}{\phi_p(x(\infty)/2)},\ \frac{1}{\phi_p(2)} \int_0^T |c(t)| dt \right\}$$

Since $\phi_p(2) > 1$,

$$\left(1 - \frac{1}{\phi_p(2)}\right) \int_T^{t} |c(s)| ds > \max \left\{ \frac{|A|}{\phi_p(x(\infty)/2)},\ \frac{1}{\phi_p(2)} \int_0^T |c(t)| dt \right\}$$

for $t \geq T_1$. From this inequality with (2.2), we see that

$$\begin{aligned} -\phi_p(x(\infty)/2) \int_T^t |c(s)| ds + |A| &< -\phi_p(x(\infty)/4) \int_T^t |c(s)| ds \\ &< -\frac{\phi_p(x(\infty)/4)}{\phi_p(2)} \int_0^t |c(s)| ds \\ &= -\phi_p(x(\infty)/8) \int_0^t |c(s)| ds \end{aligned}$$

for $t \geq T_1$. Hence, together with (2.18) and (2.19), we can estimate that

$$\begin{aligned} x(t) &= \int_{T_1}^t \phi_{p^*}\left( -\frac{\int_T^s c(\tau)\phi_p(x(\tau)) d\tau}{r(s)} + \frac{A}{r(s)} \right) ds \\ &\quad + \int_T^{T_1} \phi_{p^*}\left( -\frac{\int_T^s c(\tau)\phi_p(x(\tau)) d\tau}{r(s)} + \frac{A}{r(s)} \right) ds + B \\ &< \int_{T_1}^t \phi_{p^*}\left( -\frac{\phi_p(x(\infty)/2) \int_T^s |c(\tau)| d\tau}{|r(s)|} + \frac{|A|}{|r(s)|} \right) ds + C + B \\ &< \int_{T_1}^t \phi_{p^*}\left( -\frac{\phi_p(x(\infty)/8) \int_0^s |c(\tau)| d\tau}{|r(s)|} \right) ds + C + B \end{aligned}$$

for $t \geq T_1$, where

$$C = \int_T^{T_1} \phi_{p^*}\left( -\frac{\int_T^s c(\tau)\phi_p(x(\tau)) d\tau}{r(s)} + \frac{A}{r(s)} \right) ds.$$

We therefore conclude that

$$\begin{aligned} x(t) &< -\frac{1}{8} x(\infty) \int_{T_1}^t \phi_{p^*}\left( \frac{\int_0^s |c(\tau)| d\tau}{|r(s)|} \right) ds + C + B \\ &= -\frac{1}{8} x(\infty) \int_0^t \phi_{p^*}\left( \frac{\int_0^s |c(\tau)| d\tau}{|r(s)|} \right) ds + D + C + B \end{aligned}$$

for $t \geq T_1$, where

$$D = \frac{1}{8} x(\infty) \int_0^{T_1} \phi_{p^*}\left( \frac{\int_0^t |c(s)| ds}{|r(t)|} \right) dt.$$

Consequently, we obtain

$$0 < x(\infty) \le -\frac{1}{8}x(\infty)\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t |c(s)|ds}{|r(t)|}\right)dt + D + C + B.$$

This is a contradiction because of (2.16).

The proof of Theorem 2.2 is now complete. □

The following result is a direct conclusion of Theorems 2.1 and 2.2.

**Theorem 2.3.** *Under the assumption that* $r(t)c(t) > 0$ *for* $t \ge 0$ *and condition* (2.15), *condition* (2.4) *is necessary and sufficient for Eq.* (2.1) *to have a solution converging to a non-zero limit.*

To illustrate Theorems 2.1 and 2.2, we give simple examples.

**Example 2.1.** Consider Eq. (2.1) with

$$r(t) = \left(1 + t^{p^*}\right)^{2(p-1)} \quad \text{and} \quad c(t) = \left(\frac{p^*(1 + t^{p^*})}{2 + t^{p^*}}\right)^{p-1}. \tag{2.20}$$

Then it has a solution $x(t)$ for which $x(\infty) \ne 0$.

It is clear from (2.20) that

$$\int_0^t |c(s)|ds \le \int_0^t (p^*)^{p-1}ds = (p^*)^{p-1}t \quad \text{for } t \ge 0,$$

and therefore,

$$\begin{aligned}\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t |c(s)|ds}{|r(t)|}\right)dt &\le \int_0^\infty \phi_{p^*}\left(\left(\frac{p^* t^{p^*-1}}{(1 + t^{p^*})^2}\right)^{p-1}\right)dt \\ &= \int_0^\infty \frac{p^* t^{p^*-1}}{(1 + t^{p^*})^2}dt = 1.\end{aligned}$$

Hence, condition (2.4) is satisfied. Thus, by means of Theorem 2.1, we conclude that every solution $x(t)$ of (2.1) with (2.20) tends to a finite limit $x(\infty)$ as $t \to \infty$ and $x(\infty)$ does not vanish for at least one solution.

The function

$$x(t) = \frac{2 + t^{p*}}{p^*(1 + t^{p^*})}$$

is a solution of (2.1) with (2.20) converging to $1/p^*$ as $t \to \infty$. In fact, since

$$x'(t) = -\frac{t^{p^*-1}}{(1 + t^{p^*})^2},$$

it is obvious that

$$\phi_p(x(t)) = \left(\frac{2+t^{p*}}{p^*(1+t^{p^*})}\right)^{p-1} \quad \text{and} \quad \phi_p(x'(t)) = -\left(\frac{t^{p^*-1}}{(1+t^{p^*})^2}\right)^{p-1}.$$

Hence, it is easy to verify that

$$\begin{aligned}(r(t)\phi_p(x'(t)))' + c(t)\phi_p(x(t)) &= -\left(\left(1+t^{p^*}\right)^{2(p-1)}\left(\frac{t^{p^*-1}}{(1+t^{p^*})^2}\right)^{p-1}\right)' \\ &\qquad + \left(\frac{p^*(1+t^{p^*})}{2+t^{p^*}}\right)^{p-1}\left(\frac{2+t^{p*}}{p^*(1+t^{p^*})}\right)^{p-1} \\ &= -t' + 1 = 0.\end{aligned}$$

**Example 2.2.** Consider Eq. (2.1) with

$$r(t) = (1+t)^{p-1} \quad \text{and} \quad c(t) = \frac{p-1}{1+t}. \tag{2.21}$$

Then every solution fails to have a non-zero limit.

It follows from (2.21) that

$$r(t)c(t) = (p-1)(1+t)^{p-2} > 0 \quad \text{for } t \geq 0,$$

$$\int_0^t |c(s)|ds = (p-1)\log(1+t) \to \infty \quad \text{as } t \to \infty$$

and

$$\begin{aligned}\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t |c(s)|ds}{|r(t)|}\right)dt &= \int_0^\infty \phi_{p^*}\left(\frac{(p-1)\log(1+t)}{(1+t)^{p-1}}\right)dt \\ &= \phi_{p^*}(p-1)\int_0^\infty \left(\frac{\log(1+t)}{(1+t)^{p-1}}\right)^{p^*-1} dt \\ &= (p-1)^{p^*-1}\int_0^\infty \frac{\left(\log(1+t)\right)^{p^*-1}}{(1+t)} dt = \infty.\end{aligned}$$

Hence, all conditions of Theorem 2.2 are satisfied. Thus, Eq. (2.1) with (2.21) fails to have a solution converging to a non-zero limit.

To be more precise, we can express any solution of (2.1) with (2.21) satisfying the initial condition $(x(t_0), x'(t_0)) = (\alpha, \beta)$ as

$$\begin{aligned}x(t) = &\sqrt[p]{|\alpha|^p + |\phi_p((1+t_0)\beta))|^{p^*}} \\ &\times \sin_p\left(\operatorname{arctan}_p \frac{\alpha}{(1+t_0)\beta} + \log(1+t)\right),\end{aligned} \tag{2.22}$$

where $\sin_p t$ is the solution of a basic half-linear differential equation

$$\big(\phi_p(x')\big)' + (p-1)\phi_p(x) = 0$$

satisfying the initial condition $(x(0), x'(0)) = (0,1)$; $\cos_p t = (\sin_p t)'$,

$$\tan_p t = \frac{\sin_p t}{\cos_p t}$$

and $\arctan_p$ is the inverse function of $\tan_p$ in the domain $(-\pi_p/2, \pi_p/2)$;

$$\pi_p = \int_0^1 \frac{2}{(1-t^p)^{1/p}}\,dt = \frac{2\pi}{p\sin(\pi/p)}.$$

The functions $\sin_p$ and $\cos_p$ are periodic with period $2\pi_p$ and the functions $\sin_p$, $\cos_p$ and $\tan_p$ are usually called the generalized sine, cosine and tangent functions, respectively. For details about the generalized trigonometric functions, see Došlý [9, p. 168–169] or Došlý and P. Řehák [10, pp. 4–6]. It is clear that the solution given by (2.22) does not have a non-zero limit.

Let us consider Example 2.2 from a different point of view. For this purpose, we put

$$y = \phi_p((1+t)x')$$

as a new variable. Then, we can rewrite Eq. (2.1) with (2.21) as the system

$$\begin{aligned} x' &= \frac{1}{1+t}\phi_{p^*}(y), \\ y' &= -\frac{p-1}{1+t}\phi_p(x). \end{aligned} \tag{2.23}$$

Let $(x(t), y(t))$ be any solution of (2.23). Then, the solution satisfies that $|x(t)|^p + |y(t)|^{p^*} = c$ for some $c \ge 0$. In fact,

$$\begin{aligned} \frac{d}{dt}\big(|x(t)|^p + |y(t)|^{p^*}\big) &= p\phi_p(x(t))x'(t) + p^*\phi_{p^*}(y(t))y'(t) \\ &= \frac{p}{1+t}\phi_p(x(t))\phi_{p^*}(y(t)) - \frac{p^*(p-1)}{1+t}\phi_{p^*}(y(t))\phi_p(x(t)) \\ &= 0. \end{aligned}$$

Hence, each nontrivial positive orbit of (2.23) is a closed curve surrounding the origin $(0,0)$ and it moves clockwise around the origin as time increases. This means that every solution of (2.1) with (2.21) fails to have a non-zero limit.

## 3. Necessary and sufficient conditions for global attractivity

Before we advance to the main subject, it is very helpful to describe several relations of the parameters $p$ and $p^*$ and the function $\phi_q(z)$ with $q = p$ or $q = p^*$. Since

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

it follows that

$$(p-1)(p^*-1) = 1 \quad \text{and} \quad p = p^*(p-1).$$

The following formulae concerning differentiation hold:

$$\frac{d}{dz}\phi_q(z) = (q-1)|z|^{q-2} \quad \text{and} \quad \frac{d}{dz}|z|^q = q\phi_q(z).$$

Let $x(t)$ be any solution of ($HL$) with the initial time $t_0 \geq 0$ and define

$$v(t) = \frac{|x(t)|^p}{p} + \frac{|x'(t)|^p}{p^*}.$$

Since $|\phi_p(x'(t))|^{p^*} = |x'(t)|^{p^*(p-1)} = |x'(t)|^p$, we can rewrite $v(t)$ as

$$v(t) = \frac{|x(t)|^p}{p} + \frac{|\phi_p(x'(t))|^{p^*}}{p^*}.$$

Recall that $\phi_{p^*}$ is the inverse function of $\phi_p$. Then, we get

$$\begin{aligned} v'(t) &= \phi_p(x(t))x'(t) + \phi_{p^*}\big(\phi_p(x'(t))\big)\big(\phi_p(x'(t))\big)' \\ &= \phi_p(x(t))x'(t) - x'(t)\big(h(t)\phi_p(x'(t)) + \phi_p(x(t))\big) \\ &= -\,h(t)\phi_p(x'(t))x'(t) = -\,h(t)|x'(t)|^p \end{aligned}$$

for $t \geq t_0$. Since $h(t) \geq 0$ for $t \geq 0$, we see that

$$v(t) \leq v(t_0) \quad \text{for} \;\; t \geq t_0.$$

Hence, we obtain the following result.

**Proposition 3.1.** *The equilibrium of* ($HL$) *is stable.*

In the linear differential equations such as Eq. ($L$), it is well known that if the equilibrium is attractive, then it is stable (for example, see Coppel [8, p. 54]). However, in nonlinear differential equations, the attractivity does not always imply the stability. The stability and the attractivity are completely different concepts in general. Although Eq. ($HL$) is nonlinear, Proposition 3.1 shows that it has this inclusion relation if $h(t)$ is nonnegative for $t \geq 0$.

We next present necessary conditions for the equilibrium of ($HL$) to be attractive.

**Theorem 3.2.** *If the equilibrium of* $(HL)$ *is attractive, then*

$$H(t) \to \infty \quad \textit{as} \;\; t \to \infty$$

*and*

$$\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)dt = \infty. \tag{3.1}$$

**Proof.** As mentioned above, since $h(t) \geq 0$ for $t \geq 0$, it follows that

$$v'(t) = -\,h(t)|x'(t)|^p \geq -\,p^*h(t)\left\{\frac{|x(t)|^p}{p} + \frac{|x'(t)|^p}{p^*}\right\} = -\,p^*h(t)v(t);$$

namely, $v'(t) + p^*h(t)v(t) \geq 0$ for $t \geq t_0$. This differential inequality yields that

$$v(t) \geq v(t_0)\exp\bigl\{p^*(H(t_0) - H(t))\bigr\} \quad \text{for} \;\; t \geq t_0.$$

Since the equilibrium of $(HL)$ is attractive, the function $v(t)$ tends to $0$ as $t \to \infty$. Hence, we see that $H(t)$ diverges to $\infty$ as $t \to \infty$.

Multiplying both sides of Eq. $(HL)$ by $e^{H(t)}$, we obtain

$$\begin{aligned}\bigl(e^{H(t)}\phi_p(x')\bigr)' + e^{H(t)}\phi_p(x) &= e^{H(t)}(\phi_p(x'))' + h(t)e^{H(t)}\phi_p(x') + e^{H(t)}\phi_p(x)\\ &= 0.\end{aligned}$$

Hence, Eq. $(HL)$ becomes Eq. (2.1) with $r(t) = c(t) = e^{H(t)}$. For this reason, if (3.1) does not hold, then by virtue of Theorem 2.1, Eq. $(HL)$ has a solution $x(t)$ which tends to a non-zero limit $x(\infty)$ as $t$ increases. This means that the equilibrium of $(HL)$ is not attractive. Hence, (3.1) is also satisfied. □

So far as the special case in which $p = 2$ is concerned, Smith [35] has already proved the following result.

**Theorem B.** *If the equilibrium of* $(L)$ *is attractive, then*

$$H(t) \to \infty \quad \textit{as} \;\; t \to \infty \tag{3.2}$$

*and*

$$\int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt = \infty. \tag{3.3}$$

As shown in Theorem B, conditions (3.2) and (3.3) are necessary for the equilibrium of $(L)$ to be attractive. On the other hand, Hatvani and Totik [24, Example 3.2] have reported that conditions (3.2) and (3.3) alone are not sufficient for the equilibrium of $(L)$ to be attractive. Afterwards, many efforts has been made to bridge this gap. For example, refer to [15, 21, 22, 23, 25, 31, 32]. An attempt was to strengthen condition (3.2) as follows (see [16, 20, 28, 41, 42, 46]).

**Definition 3.1.** A nonnegative function $\psi(t)$ is said to be *integrally positive* if

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n}\psi(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n + \omega < \sigma_n \le \tau_{n+1}$ for some $\omega > 0$.

It is known that $\psi(t)$ is integrally positive if and only if

$$\liminf_{t\to\infty}\int_t^{t+\gamma}\psi(s)ds > 0$$

for every $\gamma > 0$. For example, the function $\sin^2 t$ is integrally positive. The integral positivity is rather stringent restriction than

$$\lim_{t\to\infty}\int^t\psi(s)ds = \infty.$$

If $\psi(t)$ is nonincreasing for $t \ge 0$ and tends to zero as $t \to \infty$, then it is not integrally positive any longer. The following class of functions was introduced to weaken the concept of the integral positivity.

**Definition 3.2.** An integrally positive function $\psi(t)$ is said to be *weakly integrally positive* if $\tau_{n+1} \le \sigma_n + \Omega$ for some $\Omega > 0$.

We can find the concept of the weak integral positivity in the papers [19, 21, 26, 36, 37, 38, 40]. A typical example of weakly integrally positive functions is $1/(1+t)$ or $\sin^2 t/(1+t)$. These are not integrally positive (for the proof, see Sugie *et al*. [39, Proposition 2.1]).

Hereafter, assuming that $h(t)$ is integrally positive or weakly integrally positive, we advance our discussion. To begin with, we show that the integral positivity of $h(t)$ and condition (3.1) are sufficient for the equilibrium of ($HL$) to be globally attractive. Needless to say, Theorem 3.3 below can be applied to Eq. ($L$). Therefore, it is a natural generalization of Hatvani [21, Corollary 4.3].

**Theorem 3.3.** *Suppose that $h(t)$ is integrally positive and it satisfies* (3.1). *Then the equilibrium of* ($HL$) *is globally attractive*.

**Proof.** As mentioned in the proof of Theorem 3.2, Eq. ($HL$) is equivalent to Eq. (2.1) with $r(t) = c(t) = e^{H(t)}$. It is clear that $r(t)c(t) > 0$ for $t \ge 0$. Since $h(t) \ge 0$ for $t \ge 0$, we see that $H(t) \ge H(0) = 0$ for $t \ge 0$. Hence,

$$\int_0^\infty |c(t)|dt = \int_0^\infty e^{H(t)}dt \ge \int_0^\infty dt = \infty;$$

namely, condition (2.15) is satisfied. Since

$$\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t |c(s)|ds}{|r(t)|}\right)dt = \int_0^\infty \phi_{p^*}\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)dt = \infty,$$

condition (2.16) is also satisfied. Consequently, by means of Theorem 2.2, every solution of $(HL)$ fails to have a non-zero limit.

The proof is by contradiction. Suppose that the equilibrium of $(HL)$ is not globally attractive. Then, Eq. $(HL)$ has a solution $x(t)$ for which

$$v(t) = \frac{|x(t)|^p}{p} + \frac{|x'(t)|^p}{p^*} \not\to 0 \quad \text{as} \ \ t \to \infty.$$

Let $t_0$ be the initial time of the solution $x(t)$. Since $v'(t) = -\,h(t)|x'(t)|^p \le 0$ for $t \ge t_0$, we see that $v(t)$ is nonincreasing for $t \ge t_0$ and it has a positive limit $v_0$. Hence, if $|x'(t)|^p$ tends to $0$ as $t \to \infty$, then $|x(t)|^p$ have to approach the positive value $pv_0$ as $t \to \infty$. This contradicts the fact that every solution of $(HL)$ fails to have a non-zero limit. We therefore conclude that $\limsup_{t\to\infty} |x'(t)|^p > 0$. Let $\mu$ be so small that

$$\limsup_{t\to\infty} |x'(t)|^p > \mu p v_0. \tag{3.4}$$

It is possible to find such a positive number $\mu$. On the other hand, if there exist a $\nu > 0$ and a $T_1 \ge t_0$ such that $|x'(t)|^p > \nu$ for $t \ge T_1$, then we have

$$v'(t) = -\,h(t)|x'(t)|^p \le -\,\nu h(t)$$

for $t \ge T_1$. Integrating this inequality from $T_1$ to $t$, we obtain

$$-\,v(T_1) \le v(t) - v(T_1) = \int_{T_1}^t v'(s)ds \le -\,\nu \int_{T_1}^t h(s)ds,$$

which tends to $-\infty$ as $t \to \infty$ because $H(t)$ diverges to $\infty$ as $t \to \infty$. This is a contradiction. Thus, we see that

$$\liminf_{t\to\infty} |x'(t)|^p = 0 \tag{3.5}$$

Since $v(t)$ is nonincreasing for $t \ge t_0$, there exists a $T_2 \ge t_0$ such that

$$0 < v_0 \le v(t) \le 2v_0 \quad \text{for} \ \ t \ge T_2. \tag{3.6}$$

Because of (3.4) and (3.5), we can choose three divergent sequences $\{\tau_n\}$, $\{t_n\}$ and $\{\sigma_n\}$ with $T_2 < \tau_n < t_n < \sigma_n \le \tau_{n+1}$ such that $|x'(\tau_n)|^p = |x'(\sigma_n)|^p = \mu v_0$, $|x'(t_n)|^p = p\mu v_0$ and

$$|x'(t)|^p > \mu v_0 \quad \text{for} \ \ \tau_n < t < \sigma_n \tag{3.7}$$

and

$$\mu v_0 < |x'(t)|^p < p\mu v_0 \quad \text{for} \ \ \tau_n < t < t_n. \tag{3.8}$$

In fact, from (3.5) it follows that $|x'(t^*)|^p \le \mu v_0$ for some $t^* > T_2$. Let

$$t_1 = \min\bigl\{t > t^*\colon |x'(t)|^p = p\mu v_0\bigr\},$$

$$\tau_1 = \max\bigl\{t < t_1\colon |x'(t)|^p = \mu v_0\bigr\}$$

and

$$\sigma_1 = \min\bigl\{t > t_1\colon |x'(t)|^p = \mu v_0\bigr\}.$$

The existence of such numbers is guaranteed by (3.4), (3.5) and the continuity of $|x'(t)|^p$. Using $\sigma_1$ instead of $t^*$, we define $t_2$, $\tau_2$ and $\sigma_2$ similarly to $t_1$, $\tau_1$ and $\sigma_1$, and so on. Then, $T_2 < \tau_n < t_n < \sigma_n \le \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$. Also, (3.7) and (3.8) are satisfied.

Let us estimate the distance between $\tau_n$ and $t_n$ for $n \in \mathbb{N}$. Since

$$\begin{aligned}\bigl(|\phi_p(x'(t))|^{p^*}\bigr)' &= \frac{dy}{dt}\,\frac{d}{dy}|y|^{p^*}\Big|_{y=\phi_p(x'(t))} = y' p^* \phi_{p^*}(y)\Big|_{y=\phi_p(x'(t))}\\ &= p^*\bigl(\phi_p(x'(t))\bigr)'\phi_{p^*}\bigl(\phi_p(x'(t))\bigr) = p^*\bigl(\phi_p(x'(t))\bigr)'x'(t),\end{aligned}$$

we obtain

$$\begin{aligned}(p-1)\mu v_0 = |x'(t_n)|^p - |x'(\tau_n)|^p &= \int_{\tau_n}^{t_n} (|x'(t)|^p)'dt\\ &= \int_{\tau_n}^{t_n} \bigl(|\phi_p(x'(t))|^{p^*}\bigr)'dt = p^*\int_{\tau_n}^{t_n} \bigl(\phi_p(x'(t))\bigr)'x'(t)dt.\end{aligned}$$

Taking into account that $h(t) \ge 0$ for $t \ge 0$, we get

$$\begin{aligned}(p-1)\mu v_0 &= p^*\int_{\tau_n}^{t_n} \bigl(-h(t)\phi_p(x'(t)) - \phi_p(x(t))\bigr)x'(t)dt\\ &= p^*\int_{\tau_n}^{t_n} \bigl(-h(t)|x'(t)|^p - \phi_p(x(t))x'(t)\bigr)dt\\ &\le p^*\int_{\tau_n}^{t_n} |\phi_p(x(t))||x'(t)|dt.\end{aligned}$$

From (3.6) and (3.8) it follows that

$$|x(t)|^p \le 2pv_0 \quad \text{for} \;\; t \ge T_2$$

and

$$|x'(t)| < (p\mu v_0)^{\frac{1}{p}} \quad \text{for} \;\; \tau_n < t < t_n,$$

respectively. Hence, we can estimate that

$$(p-1)\mu v_0 < p^*\int_{\tau_n}^{t_n} 2^{\frac{p-1}{p}} p \mu^{\frac{1}{p}} v_0\, dt = 2^{\frac{p-1}{p}} pp^* \mu^{\frac{1}{p}} v_0 (t_n - \tau_n);$$

namely,

$$t_n - \tau_n > \left(\frac{\mu}{2}\right)^{\frac{p-1}{p}} \frac{p-1}{pp^*} = \left(\frac{\mu}{2}\right)^{\frac{1}{p^*}} \left(\frac{1}{p^*}\right)^2$$

for each $n \in \mathbb{N}$. It is clear that the number $(\mu/2)^{1/p^*}(1/p^*)^2$ is independent of $n$.

Since $[\tau_n, t_n] \subset [\tau_n, \sigma_n]$, it follows that $\tau_n + \omega < \sigma_n$ for each $n \in \mathbb{N}$, where $\omega = (\mu/2)^{1/p^*}(1/p^*)^2$. Hence,

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty \tag{3.9}$$

because $h(t)$ is integrally positive. However, from (3.7), we see that

$$\begin{aligned} v(\sigma_n) - v(\tau_1) &= \int_{\tau_1}^{\sigma_n} v'(t)dt = -\int_{\tau_1}^{\sigma_n} h(t)|x'(t)|^p dt \\ &\le -\sum_{i=1}^{n} \int_{\tau_i}^{\sigma_i} h(t)|x'(t)|^p dt \le -\mu v_0 \sum_{i=1}^{n} \int_{\tau_i}^{\sigma_i} h(t)dt. \end{aligned}$$

This contradicts (3.9). We have thus proved Theorem 3.3. □

As already mentioned, if $h(t)$ is nonincreasing for $t \ge 0$ and tends to zero as $t \to \infty$, then it is not integrally positive. For this reason, we cannot apply Theorem 3.3 to the damped oscillator

$$(\phi_p(x'))' + \frac{p-1}{1+t}\phi_p(x') + \phi_p(x) = 0.$$

although condition (3.1) is satisfied. In fact, since $h(t) = (p-1)/(1+t)$, we can easily confirm that

$$e^{H(t)} = (1+t)^{p-1} \quad \text{and} \quad \int_0^t e^{H(s)}ds = \frac{1}{p}\{(1+t)^p - 1\}.$$

Since $p > 1$, we see that $2^p - 1 > p$. Hence, we obtain

$$\int_0^{\infty} \phi_{p^*}\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)dt > \int_1^{\infty} \phi_{p^*}\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)dt > \int_1^{\infty} \frac{1}{1+t} = \infty;$$

namely, condition (3.1).

To overcome this weak point, we assume that $h(t)$ is weakly integrally positive instead of integrally positive. Then, we need an additional condition concerning $h(t)$.

**Theorem 3.4.** *Suppose that $h(t)$ is uniformly continuous for $t \ge 0$ and weakly integrally positive and suppose that* (3.1) *holds. Then the equilibrium of* $(HL)$ *is globally attractive.*

**Proof.** Since $h(t)$ is uniformly continuous for $t \ge 0$, we can find a $\delta > 0$ so that $|h(\alpha) - h(\beta)| < 1$ whenever $\alpha \ge 0$ and $\beta \ge 0$ with $|\alpha - \beta| < \delta$. Note that $\delta$ is independent of $t$.

The proof is by contradiction. Suppose that the equilibrium of $(HL)$ is not globally attractive. Then, Eq. $(HL)$ has a solution $x(t)$ satisfying

$$v(t) = \frac{|x(t)|^p}{p} + \frac{|x'(t)|^p}{p^*} \searrow v_0 > 0 \quad \text{as} \ \ t \to \infty.$$

Let $t_0$ be the initial time of the solution $x(t)$. Since $v(t)$ is nonincreasing for $t \geq t_0$, there exists a $T \geq t_0$ such that

$$v_0 \leq v(t) \leq 2v_0 \quad \text{for} \ \ t \geq T. \tag{3.10}$$

By means of the same argument as in the proof of Theorem 3.3, we can show that $\limsup_{t\to\infty} |x'(t)|^p$ is positive and $\liminf_{t\to\infty} |x'(t)|^p$ is zero. We may assume without loss of generality that

$$\limsup_{t\to\infty} |x'(t)|^p > p\mu v_0,$$

where

$$0 < \mu < \frac{1}{(3+2/\delta)^{p^*} + p/p^*}. \tag{3.11}$$

Since $\liminf_{t\to\infty} |x'(t)|^p = 0 < \mu p v_0 < \limsup_{t\to\infty} |x'(t)|^p$, we can select three divergent sequences $\{\tau_n\}$, $\{t_n\}$ and $\{\sigma_n\}$ with $T < \tau_n < t_n < \sigma_n \leq \tau_{n+1}$ such that $|x'(\tau_n)|^p = |x'(\sigma_n)|^p = \mu v_0$, $|x'(t_n)|^p = p\mu v_0$ and

$$|x'(t)|^p > \mu v_0 \quad \text{for} \ \ \tau_n < t < \sigma_n, \tag{3.12}$$

$$\mu v_0 < |x'(t)|^p < p\mu v_0 \quad \text{for} \ \ \tau_n < t < t_n. \tag{3.13}$$

and

$$|x'(t)|^p \leq p\mu v_0 \quad \text{for} \ \ \sigma_n \leq t \leq \tau_{n+1}. \tag{3.14}$$

Using (3.10) and (3.13) instead of (3.6) and (3.8), respectively, and following the same process as in the proof of Theorem 3.3, we can estimation that $\tau_n + \omega < \sigma_n$ for each $n \in \mathbb{N}$, where

$$\omega = \left(\frac{\mu}{2}\right)^{\frac{1}{p^*}} \left(\frac{1}{p^*}\right)^2.$$

From the uniform continuity of $h(t)$, we see that

$$|h(t) - h(\sigma_n)| < 1 \quad \text{for} \ \ \sigma_n - \delta < t < \sigma_n + \delta. \tag{3.15}$$

Let us pay attention to the value of $h(t)$ at $t = \sigma_n$ for each $n \in \mathbb{N}$. Define

$$S = \{n \in \mathbb{N} \colon h(\sigma_n) \geq 2\}.$$

**Claim 1.** The number of elements in the set $S$ is finite.

Suppose that $S$ is infinite. Let $d = \min\{\delta, \omega\}$. Then, from (3.12) and the fact that $\tau_n + \omega < \sigma_n$ for each $n \in \mathbb{N}$, we obtain

$$|x'(t)|^p \geq \mu v_0 \quad \text{for} \ \ \sigma_n - d \leq t \leq \sigma_n. \tag{3.16}$$

From (3.15) it follows that $n \in S$ implies that

$$h(t) \geq 1 \quad \text{for } \sigma_n - d \leq t \leq \sigma_n.$$

Hence, together with (3.16), we get

$$\int_{\sigma_n - d}^{\sigma_n} h(t)|x'(t)|^p dt \geq d\mu v_0 \quad \text{if } n \in S,$$

and therefore,

$$\begin{aligned} v(t) - v(t_0) &= \int_{t_0}^{t} v'(s)ds = -\int_{t_0}^{t} h(s)|x'(s)|^p ds \\ &\leq -\sum_{n\in S} \int_{\sigma_n}^{\sigma_n} h(t)|x'(t)|^p dt \leq -\infty. \end{aligned}$$

This is a contradiction. Thus, Claim 1 is proved.

From Claim 1, we see that there exists an $N \in \mathbb{N}$ such that

$$h(\sigma_n) < 2 \quad \text{for } n \geq N. \tag{3.17}$$

Next, let us pay attention to the distance between intervals $[\tau_n, \sigma_n]$ and $[\tau_{n+1}, \sigma_{n+1}]$ for each $n \in \mathbb{N}$. Taking into account that

$$\frac{|x(t)|^p}{p} = v(t) - \frac{|x'(t)|^p}{p^*}$$

and using (3.10) and (3.14), we obtain

$$|x(t)|^{p-1} \geq \left( p v_0 \left( 1 - \frac{p\mu}{p^*} \right) \right)^{\frac{p-1}{p}} = \left( p v_0 \left( 1 - \frac{p\mu}{p^*} \right) \right)^{\frac{1}{p^*}}$$

and

$$|\phi_p(x'(t))| = |x'(t)|^{p-1} \leq (p\mu v_0)^{\frac{1}{p^*}} \tag{3.18}$$

for $\sigma_n \leq t \leq \tau_{n+1}$. Hence, we have

$$\begin{aligned} \left| \big(\phi_p(x'(t))\big)' \right| &\geq |\phi_p(x(t))| - h(t)|\phi_p(x'(t))| = |x(t)|^{p-1} - h(t)|\phi_p(x'(t))| \\ &\geq \left( p v_0 \left( 1 - \frac{p\mu}{p^*} \right) \right)^{\frac{1}{p^*}} - h(t)(p\mu v_0)^{\frac{1}{p^*}} \end{aligned} \tag{3.19}$$

for $\sigma_n \leq t \leq \tau_{n+1}$.

**Claim 2.** The sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfy $\tau_{n+1} - \sigma_n \leq \delta$ for $n \geq N$.

Suppose that there exists an $n_0 \geq N$ such that $\tau_{n_0+1} - \sigma_{n_0} > \delta$. Then, from (3.15) and (3.17) it follows that

$$h(t) \leq 1 + h(\sigma_{n_0}) < 3 \quad \text{for} \ \ \sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta.$$

Hence, from (3.11) and (3.19), we can estimate that

$$\left|\big(\phi_p(x'(t))\big)'\right| > \left(pv_0\left(1 - \frac{p\mu}{p^*}\right)\right)^{\frac{1}{p^*}} - 3(p\mu v_0)^{\frac{1}{p^*}} > \frac{2}{\delta}(p\mu v_0)^{\frac{1}{p^*}} > 0$$

for $\sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta < \tau_{n_0+1}$. Integrating this inequality from $\sigma_{n_0}$ to $\sigma_{n_0} + \delta$, we obtain

$$\begin{aligned}
|\phi_p(x'(\sigma_{n_0} + \delta))| + |\phi_p(x'(\sigma_{n_0}))| &= \left|\int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta} \big(\phi_p(x'(t))\big)' dt\right| \\
&= \int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta} \left|\big(\phi_p(x'(t))\big)'\right| dt > 2(p\mu v_0)^{\frac{1}{p^*}},
\end{aligned}$$

which contradicts (3.18). Thus, Claim 2 is proved.

From Claim 2, we see that there exists an $\Omega \geq \delta$ such that $\tau_{n+1} \leq \sigma_n + \Omega$ for each $n \in \mathbb{N}$. Recall that $\tau_n + \omega < \sigma_n$ for each $n \in \mathbb{N}$. Hence,

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty \tag{3.20}$$

because $h(t)$ is weakly integrally positive. However, from (3.12), we see that

$$\begin{aligned}
v(\sigma_n) - v(\tau_1) &= \int_{\tau_1}^{\sigma_n} v'(t)dt = -\int_{\tau_1}^{\sigma_n} h(t)|x'(t)|^p dt \\
&\leq -\sum_{i=1}^{n} \int_{\tau_i}^{\sigma_i} h(t)|x'(t)|^p dt \leq -\mu v_0 \sum_{i=1}^{n} \int_{\tau_i}^{\sigma_i} h(t)dt.
\end{aligned}$$

This contradicts (3.20). Thus, $v(t)$ fails to have any positive limit $v_0$.

The proof of Theorem 3.4 is thus complete. □

Theorem 3.4 is new even in the linear case in which $p = 2$. We can now combine Theorems 3.2–3.4 with Proposition 3.1 to obtain the following result.

**Theorem 3.5.** *Suppose that one of the following assumptions*

(i) $h(t)$ *is integrally positive,*

(ii) $h(t)$ *is uniformly continuous for* $t \geq 0$ *and weakly integrally positive*

*holds. Then the equilibrium of* $(HL)$ *is globally asymptotically stable if and only if condition* (3.1) *holds.*

Hatvani and Totik [24, Theorem 3.1] have established a criterion for judging whether the equilibrium of damped linear oscillators is asymptotically stable or not. Applying their criterion to Eq. $(L)$, we have the following result.

**Theorem C.** *Suppose that there exists a* $\gamma_0$ *with* $0 < \gamma_0 < \pi$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+\gamma_0} h(s)ds > 0.$$

*Then the equilibrium of* $(L)$ *is asymptotically stable if and only if condition* (3.3) *holds.*

In addition, they pointed out that the requirement that $0 < \gamma_0 < \pi$ in Theorem C was the best possible (see [24, Example 3.2]). Recall that $h(t)$ is integrally positive if and only if

$$\liminf_{t\to\infty} \int_t^{t+\gamma} h(s)ds > 0$$

for every $\gamma > 0$. Hence, the assumption regarding $h(t)$ of Theorem C is weaker than the integral positivity. Even in this sense, Theorem C is very nice. Unfortunately, however, Theorem C cannot be applied to the case in which $h(t)$ disappears as $t \to \infty$. On the other hand, Theorem 3.5 has a strong point that it is possible to apply to even in such a case.


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