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Uniform global asymptotic stability for half-linear differential systems with time-varying coefficients

Masakazu Onitsuka
Department of General Education, Miyakonojo National College of Technology, Miyakonojo 885-8567, Japan (onitsuka@cc.miyakonojo-nct.ac.jp)

Jitsuro Sugie
Department of Mathematics and Computer Science, Shimane University, Matsue 690-8504, Japan (jsugie@riko.shimane-u.ac.jp)

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The present paper deals with the following system:

$$\begin{aligned} x' &= -\,e(t)x + f(t)\phi_{p^*}(y),\\ y' &= -(p-1)g(t)\phi_p(x) - (p-1)h(t)y, \end{aligned}$$

where p and p^* are positive numbers satisfying $1/p + 1/p^* = 1$, and $\phi_q(z) = |z|^{q-2}z$ for $q = p$ or $q = p^*$. This system is referred to as a half-linear system. We herein establish conditions on time-varying coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ for the zero solution to be uniformly globally asymptotically stable. If $(e(t), f(t)) \equiv (h(t), g(t))$, then the half-linear system is integrable. We consider two cases: the integrable case $(e(t), f(t)) \equiv (h(t), g(t))$ and the nonintegrable case $(e(t), f(t)) \not\equiv (h(t), g(t))$. Finally, some simple examples are presented to illustrate our results.

1. Introduction

We consider the half-linear system

$$\begin{aligned} x' &= -\,e(t)x + f(t)\phi_{p^*}(y),\\ y' &= -(p-1)g(t)\phi_p(x) - (p-1)h(t)y, \end{aligned} \tag{1.1}$$

where the prime denotes d/dt; the coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are continuous for $t \geq 0$; the numbers p and p^* are positive and satisfy

$$\frac{1}{p} + \frac{1}{p^*} = 1;$$

the function $\phi_q(z)$ is defined by

$$\phi_q(z) = |z|^{q-2}z$$

for $q = p$ or $q = p^*$. Note that p and p^* are naturally greater than 1, and ϕ_{p^*} is the inverse function of ϕ_p. System (1.1) has the zero solution $(x(t), y(t)) \equiv (0, 0)$. If $(x(t), y(t))$ is a solution of (1.1), then $(cx(t), \phi_p(c)y(t))$ is also a solution of (1.1) for any $c \in \mathbb{R}$. However, even if $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two solutions of (1.1), the function $(x_1(t) + x_2(t), y_1(t) + y_2(t))$ is not always a solution of (1.1). If $f(t) \neq 0$ for $t \geq 0$, then by putting

$$w = \exp\biggl(\int_0^t e(\tau)d\tau\biggr)x \quad \text{and} \quad z = \exp\biggl((p-1)\int_0^t h(\tau)d\tau\biggr)y,$$

system (1.1) is transformed into the system

$$\begin{aligned} w' &= \frac{\phi_{p^*}(z)}{r(t)}, \\ z' &= -\, c(t)\phi_p(w), \end{aligned}$$

where

$$r(t) = \phi_p\biggl(\exp\biggl(\int_0^t (h(\tau) - e(\tau))d\tau\biggr)\Big/ f(t)\biggr)$$

and

$$c(t) = (p-1)g(t)\exp\biggl((p-1)\int_0^t (h(\tau) - e(\tau))d\tau\biggr).$$

This system is equivalent to the half-linear differential equation

$$\bigl(r(t)\phi_p(w')\bigr)' + c(t)\phi_p(w) = 0. \tag{HL}$$

It is known that for any $t_0 \geq 0$ and $(c_1, c_2) \in \mathbb{R}^2$, there exists a unique solution of (HL) satisfying $w(t_0) = c_1$ and $w'(t_0) = c_2$ which is continuable in the future. For details, see [5, p. 170] or [6, pp. 8–10]. Hence, the global existence and uniqueness of solutions of (1.1) are guaranteed for the initial value problem.

The purpose of this paper is to present conditions on $e(t)$, $f(t)$, $g(t)$ and $h(t)$ for the zero solution of (1.1) to be uniformly globally asymptotically stable (for the definition, see §2).

In the special case in which $p = 2$, system (1.1) becomes the linear system

$$\mathbf{x}' = A(t)\mathbf{x}, \tag{1.2}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} -\,e(t) & f(t) \\ -\,g(t) & -\,h(t) \end{pmatrix}.$$

Let $\|\mathbf{x}\|$ be the Euclidean norm of a vector $\mathbf{x}$ and let $X(t)$ be a fundamental matrix for system (1.1). We define the norm of $X(t)$ to be

$$\|X(t)\| = \sup_{\|\mathbf{x}\|=1} \|X(t)\mathbf{x}\|.$$

As is well known, the zero solution of (1.2) is merely asymptotically stable; that is, every solution $(x(t), y(t))$ of (1.2) tends to the origin $(0, 0)$ as time t increases if and only if

$$\|X(t)\| \to 0 \quad \text{as} \quad t \to \infty, \tag{1.3}$$

and the zero solution of (1.2) is uniformly asymptotically stable if and only if there exist positive constants R and λ such that

$$\|X(t)X^{-1}(s)\| \le R\exp(-\lambda(t-s)) \quad \text{for } 0 \le s \le t < \infty \tag{1.4}$$

(for example, see [4, p. 54] or [9, p. 84]). Note that if the zero solution of (1.2) is asymptotically stable (uniformly asymptotically stable), then it must become globally asymptotically stable (uniformly globally asymptotically stable). This means that we may remove the term "globally" in the linear system (1.2).

In the general case, where $p \neq 2$, however, the concept of fundamental matrices does not apply, because the sum of two solutions of (1.1) is not always a solution of (1.1), namely, the solution space of (1.1) is not additive. Hence, the criteria (1.3) and (1.4) are useless for verifying that the zero solution of (1.1) is globally asymptotically stable and uniformly globally asymptotically stable, respectively.

Let $S(t)$ be the solution of a basic half-linear differential equation

$$\big(\phi_p(x')\big)' + (p-1)\phi_p(x) = 0$$

satisfying the initial condition $(S(0), S'(0)) = (0,1)$. Then, $S(t)$ satisfies the generalized Pythagorean identity

$$|S(t)|^p + |S'(t)|^p \equiv 1,$$

and $S(t)$ is positive and increasing on $[0, \pi_p/2]$ with $S(\pi_p/2) = 1$ and $S'(\pi_p/2) = 0$, where

$$\pi_p = \int_0^1 \frac{2}{(1-s^p)^{1/p}}\,ds = \frac{2\pi}{p\sin(\pi/p)}.$$

As is customary, we define the generalized sine function $\sin_p \theta$ as follows:

$$\sin_p \theta = \begin{cases} S(\theta) & \text{if } 0 \le \theta \le \pi_p/2 \\ S(\pi_p - \theta) & \text{if } \pi_p/2 < \theta \le \pi_p \end{cases}$$

and

$$\sin_p \theta = \begin{cases} -\sin_p(\theta - \pi_p) & \text{if } \pi_p \le \theta < 2\pi_p \\ \sin_p(\theta - 2n\pi_p) & \text{if } 2n\pi_p \le \theta < 2(n+1)\pi_p \end{cases}$$

for $n = \pm 1, \pm 2, \cdots$. The generalized cosine function $\cos_p \theta$ is defined as $\cos_p \theta = (\sin_p \theta)'$. Then,

$$|\sin_p \theta|^p + |\cos_p \theta|^p = 1 \quad \text{for } \theta \in \mathbb{R}. \tag{1.5}$$

For details concerning π_p, $\sin_p \theta$ and $\cos_p \theta$, see [1, 5–7, 12, 15].

If $e(t) \equiv h(t)$ and $f(t) \equiv g(t)$, then system (1.1) is rewritten as the system

$$\begin{aligned} r' &= -h(t)r, \\ \theta' &= g(t) \end{aligned} \tag{1.6}$$

by using the generalized Prüfer transformation

$$x = r\sin_p \theta \quad \text{and} \quad y = \phi_p(r\cos_p \theta).$$

Let $(x(t), y(t))$ be the solution of (1.1) initiating at $t = t_0 \geq 0$. Then, this solution is expressed as

$$x(t) = r(t)\sin_p\theta(t) = r(t_0)\exp\left(-\int_{t_0}^{t} h(\tau)d\tau\right)\sin_p\left(\theta(t_0) + \int_{t_0}^{t} g(\tau)d\tau\right)$$
$$y(t) = \phi_p(r(t)\cos_p\theta(t)) = \phi_p\left(r(t_0)\exp\left(-\int_{t_0}^{t} h(\tau)d\tau\right)\cos_p\left(\theta(t_0) + \int_{t_0}^{t} g(\tau)d\tau\right)\right).$$

We therefore conclude that a necessary and sufficient condition for the zero solution of (1.1) to be globally asymptotically stable is that

$$\int_0^\infty h(t)dt = \infty \tag{1.7}$$

in the special case in which $e(t) \equiv h(t)$ and $f(t) \equiv g(t)$. As proven in §2, the zero solution of (1.1) with $e(t) \equiv h(t)$ and $f(t) \equiv g(t)$ is uniformly globally asymptotically stable if and only if

$$\int_s^t h(\tau)d\tau \geq \lambda(t-s) - \kappa \quad \text{for } 0 \leq s \leq t < \infty \tag{1.8}$$

with $\lambda > 0$ and $\kappa > 0$. For example, if $h(t) = 1/(1+t)$ for $t \geq 0$, then it is easy to confirm that condition (1.7) holds, but condition (1.8) does not hold. Hence, in the case in which $e(t) \equiv h(t) = 1/(1+t)$ and $f(t) \equiv g(t)$, the zero solution of (1.1) is globally asymptotically stable, but not uniformly globally asymptotically stable.

In the general case in which $p \neq 2$ and $(e(t), f(t)) \not\equiv (h(t), g(t))$, we cannot express the solutions of (1.1) accurately. Then, can we decide whether the zero solution is uniformly globally asymptotically stable in the general case? What kind of condition on the coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ will guarantee the uniform global asymptotic stability of the zero solution of (1.1)? As an answer to these questions, in §3, we present sufficient conditions for uniform global asymptotic stability that can be applied even in the case in which $p \neq 2$ and $(e(t), f(t)) \not\equiv (h(t), g(t))$. To present our result, we define a characteristic function obtained from coefficients in system (1.1). In the final section, we take some concrete examples to illustrate the results presented in §3.

2. Definitions

First, let us give some definitions. For this purpose, we denote the solution of (1.1) through $(t_0, \mathbf{x}_0) \in [0, \infty) \times \mathbb{R}^2$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. The zero solution of (1.1) is said to be *uniformly stable* if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution of (1.1) is said to be *uniformly globally attractive* if, for any $\rho > 0$ and any $\eta > 0$, there is a $T(\rho, \eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \rho$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T$. The solutions of (1.1) are said to be *uniformly bounded* if, for any $\rho > 0$, there exists a $B(\rho) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \rho$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < B$ for all $t \geq t_0$. The zero solution of (1.1) is *uniformly globally asymptotically stable* if it is uniformly stable and is uniformly globally attractive, and if the solutions of (1.1) are uniformly bounded. With respect to the various definitions of stability and boundedness, the reader can refer to [2, 8, 14, 17, 22] for example.

The most important points are that δ is independent of the initial time t_0 and that T and B are independent of the initial time t_0 and the initial state $\mathbf{x}_0$ in the definitions above. In the research on uniform global asymptotic stability, it is difficult to choose δ, T and B which do not depend on t_0 or $(t_0, \mathbf{x}_0)$. In order to demonstrate this fact, we have to pay scrupulous attention to its proof.

When δ depends on t_0, or when T or B depends on $(t_0, \mathbf{x}_0)$, even if all solutions approach the origin, each approaching speed or each asymptotic motion is not necessarily the same. This situation is not desirable. For example, if a system has a unique solution with respect to the initial condition $(t_0, \mathbf{x}_0)$ and if the zero solution of the system is uniformly globally asymptotically stable, then there exists a Lyapunov function with suitable properties (i.e., this is the converse Lyapunov theorem). However, as Massera demonstrated in [13, Example 2], non-uniform asymptotic stability of the zero solution of a time varying system does not generally imply the existence of a good Lyapunov function. For details concerning converse Lyapunov theorems and their applications to mathematical control theory, see [2, 8, 14, 17, 18, 22] and the references cited therein.

Converse theorems for uniform (global) asymptotic stability are very useful for dealing with perturbation problems. For example, if the zero solution of (1.2) is uniformly asymptotically stable and if $\mathbf{f}(t, \mathbf{x})$ and $\Gamma(t)$ satisfy the condition $\|\mathbf{f}(t, \mathbf{x})\| \leq \Gamma(t)\|\mathbf{x}\|$ for $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^2$, where

$$\int_0^\infty \Gamma(t) dt < \infty,$$

then the zero solution of the perturbed system

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t, \mathbf{x})$$

is uniformly asymptotically stable. Note that if δ and T depend on t_0, then we cannot derive this conclusion. For details, see [16] (also [3, pp. 169–170] or [9, p. 88]). Therefore, the problems considered in the present study are closely related to perturbation problems.

A scalar function $a : [0, \infty) \to [0, \infty)$ is said to be of the class CIP, if $a(r)$ is continuous and strictly increasing with $a(0) = 0$. For such a function, we write $a(r) \in CIP$. In the definitions of uniform stability and uniform global attractivity of the zero solution of (1.1), we may replace $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ and $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ with

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < a(\varepsilon) \quad \text{and} \quad \|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < a(\eta),$$

respectively, where $a(r) \in CIP$. In the definition of uniform boundedness of the solutions of (1.1), we may replace $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < B$ with

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < a(B),$$

where $a(r) \in CIP$ and $a(r) \to \infty$ as $r \to \infty$.

Consider again the special case in which $e(t) \equiv h(t)$ and $f(t) \equiv g(t)$. Then, we have the following result.

Theorem 2.1. *Suppose that $e(t) = h(t)$ and $f(t) = g(t)$ for $t \geq 0$. Then the zero solution of* (1.1) *is uniformly globally asymptotically stable if and only if condition* (1.8) *holds.*

To prove theorem 2.1, we prepare the following lemma.

Lemma 2.2. *Let $b(r) \in CIP$ and $c(r) \in CIP$. If*

$$b(|x|) + c(|y|) < b(r), \tag{2.1}$$

then there exists a scalar function $a(r) \in CIP$ such that $\|\mathbf{x}\| = \sqrt{x^2 + y^2} < a(r)$.

Proof. Since $c(r) \in CIP$, there exists the inverse function $c^{-1}(r) \in CIP$. Let $a(r) = r + c^{-1}(b(r))$. Since a composite function of two CIP functions also belong to CIP, $a(r) \in CIP$. By (2.1), we have

$$|x| < r \quad \text{and} \quad |y| < c^{-1}(b(r)).$$

Hence, we obtain

$$\|\mathbf{x}\| \leq |x| + |y| < r + c^{-1}(b(r)) = a(r),$$

as desired. □

Proof of theorem 2.1. From the generalized Prüfer transformation it follows that

$$r^p |\sin_p \theta|^p = |x|^p \quad \text{and} \quad r^p |\cos_p \theta|^p = |\phi_{p^*}(y)|^p = |y|^{(p^*-1)p} = |y|^{p^*}.$$

Hence, by the relation (1.5), we have

$$|x|^p + |y|^{p^*} = r^p. \tag{2.2}$$

Recall that system (1.1) is equivalent to system (1.6) in the case in which $e(t) \equiv h(t)$ and $f(t) \equiv g(t)$. Let $(x(t), y(t)) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and let $r(t; t_0, r_0)$ be the solution of the scalar equation

$$r' = -h(t)r \tag{2.3}$$

corresponding to $(x(t), y(t))$. Then, it is clear that

$$r(t; t_0, r_0) = r_0 \exp\left(-\int_{t_0}^{t} h(\tau) d\tau\right)$$

with $r_0 = \|\mathbf{x}_0\|$. Hence, it is easy to verify that condition (1.8) is necessary and sufficient for the zero solution of (2.3) to be uniformly globally asymptotically stable; that is,

(i) for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $r_0 < \delta$ imply $r(t; t_0, r_0) < \varepsilon$ for all $t \geq t_0$,

(ii) for any $\rho > 0$ and any $\eta > 0$, there is a $T(\rho, \eta) > 0$ such that $t_0 \geq 0$ and $r_0 < \rho$ imply $r(t; t_0, r_0) < \eta$ for all $t \geq t_0 + T$,

(iii) for any $\rho > 0$, there exists a $B(\rho) > 0$ such that $t_0 \geq 0$ and $r_0 < \rho$ imply $r(t; t_0, r_0) < B$ for all $t \geq t_0$.

Suppose that condition (1.8) holds. Then, from (2.2) and (i) we see that

$$|x(t)|^p + |y(t)|^{p^*} = r^p(t;t_0,r_0) < \varepsilon^p \quad \text{for } \ t \geq t_0.$$

Let $b(r) = r^p$ and $c(r) = r^{p^*}$. Then, $b(r) \in CIP$ and $c(r) \in CIP$. Hence, by means of lemma 2.2, we can find a function $a(r) \in CIP$ such that

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < a(\varepsilon) \quad \text{for } \ t \geq t_0,$$

and therefore, the zero solution of (1.1) is uniformly stable. By a similar way, we can show that the zero solution of (1.1) is uniformly globally attractive and the solutions of (1.1) are uniformly bounded. Thus, the zero solution of (1.1) is uniformly globally asymptotically stable.

Conversely, suppose that the zero solution of (1.1) is uniformly globally asymptotically stable. Let $\overline{q} = \max\{1, p/p^*\}$ and $\underline{q} = \min\{1, p/p^*\}$. Since the zero solution of (1.1) is uniformly stable, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$ imply $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < a(\varepsilon)$ for all $t \geq t_0$, where $a(r) = (r/2)^{\overline{q}} \in CIP$. We may assume that ε is sufficiently small. Hence, we have

$$|x(t)| < \left(\frac{\varepsilon}{2}\right)^{\overline{q}} \leq \frac{\varepsilon}{2} \quad \text{and} \quad |y(t)| < \left(\frac{\varepsilon}{2}\right)^{\overline{q}} \leq \left(\frac{\varepsilon}{2}\right)^{p/p^*}$$

for $t \geq t_0$. From these inequalities and (2.2), we see that

$$r^p(t;t_0,r_0) = |x(t)|^p + |y(t)|^{p^*} < \left(\frac{\varepsilon}{2}\right)^p + \left(\frac{\varepsilon}{2}\right)^p < \varepsilon^p$$

for $t \geq t_0$. Thus, (i) holds, and hence the zero solution of (2.3) is uniformly stable. Using the same argument with $a(r) = (r/2)^{\overline{q}} \in CIP$, we can show that the zero solution of (2.3) is uniformly globally attractive, namely (ii). Since the zero solution of (1.1) is uniformly bounded, for any $\rho > 0$, there exists a $B(\rho) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \rho$ imply $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < b(B)$ for all $t \geq t_0$, where $b(r) = (r/2)^{\underline{q}} \in CIP$. We may assume that B is sufficiently large. Hence, we have

$$|x(t)| < \left(\frac{B}{2}\right)^{\underline{q}} \leq \frac{B}{2} \quad \text{and} \quad |y(t)| < \left(\frac{B}{2}\right)^{\underline{q}} \leq \left(\frac{B}{2}\right)^{p/p^*}$$

for $t \geq t_0$. From these inequalities and (2.2), we see that

$$r^p(t;t_0,r_0) = |x(t)|^p + |y(t)|^{p^*} < \left(\frac{B}{2}\right)^p + \left(\frac{B}{2}\right)^p < \varepsilon^p$$

for $t \geq t_0$. Thus, (iii) holds, and hence the zero solution of (2.3) is uniformly bounded. We therefore conclude that the zero solution of (2.3) is uniformly globally asymptotically stable and condition (1.8) holds. □

3. The general case

To state our main result, we need some notations. Let

$$\phi_+(t) = \max\{0, \phi(t)\} \quad \text{and} \quad \phi_-(t) = \max\{0, -\phi(t)\}$$

for a continuous function $\phi(t)$. In other words, the graph of $\phi_+(t)$ is corresponding to the one that removed the negative portion from the graph of $\phi(t)$; the graph of $\phi_-(t)$ is corresponding to the one that removed the negative portion from the graph of $-\phi(t)$. The function $\phi(t)$ and the absolute value $|\phi(t)|$ of the function $\phi(t)$ are expressed as follows:

$$\phi(t) = \phi_+(t) - \phi_-(t) \quad \text{and} \quad |\phi(t)| = \phi_+(t) + \phi_-(t).$$

The function $\phi_+(t)$ is said to be *integrally positive* if

$$\int_I \phi_+(t)dt = \infty$$

for every set $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$ such that $\tau_n + \omega < \sigma_n < \tau_{n+1}$ for some $\omega > 0$. For example, $\sin^2 t$ is an integrally positive function (see [10, 11, 19–21]).

Here, we introduce a function that plays an important role in this paper. To this end, we assume that $f(t)g(t) > 0$ and $g(t)/f(t)$ is differentiable for $t \geq 0$. Define

$$\psi(t) = ph(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'.$$

Then, we have the following result.

THEOREM 3.1. *Suppose that $f(t)$, $g(t)$ and $h_+(t)$ are bounded for $t \geq 0$. If*

(i) $f(t)g(t) > 0$ *for* $t \geq 0$ *and* $\liminf_{t\to\infty} f(t)g(t) > 0$,

(ii) $\int_0^\infty e_-(t)dt < \infty$, $\int_0^\infty h_-(t)dt < \infty$ *and* $\int_0^\infty \psi_-(t)dt < \infty$,

(iii) $\psi_+(t)$ *is integrally positive*,

then the zero solution of (1.1) *is uniformly globally asymptotically stable.*

To prove theorem 3.1, we give some brief explanations about the assumptions. Using assumption (i) and the boundedness of $f(t)$, $g(t)$ and $h_+(t)$, we can find positive numbers $\overline{f}$, $\underline{g}$, $\overline{h}$, k and K such that

$$|f(t)| \leq \overline{f}, \quad \underline{g} \leq |g(t)|, \quad h_+(t) \leq \overline{h} \quad \text{and} \quad k \leq \frac{f(t)}{g(t)} \leq K$$

for $t \geq 0$. We may assume without loss of generality that $k \leq 1 \leq K$. From assumption (ii) it follows that there exist positive constants L and M such that

$$L = \int_0^\infty (2e_-(t) + \psi_-(t))dt \quad \text{and} \quad M = \int_0^\infty h_-(t)dt.$$

It is known that assumption (iii) holds if and only if

$$\liminf_{t\to\infty} \int_t^{t+\gamma} \psi_+(\tau)d\tau > 0$$

for every $\gamma > 0$. Hence, there exist an $l > 0$ and a $\hat{t} > 0$ such that

$$\int_t^{t+1} \psi_+(\tau)d\tau \ge l \quad \text{for} \quad t \ge \hat{t}.$$

The values mentioned above are used for the proof of theorem 3.1 without a notice.

Proof of theorem 3.1. We will divide the proof into eight steps. In the first step, we prove that the zero solution of (1.1) is uniformly stable. We next prove that the zero solution of (1.1) is uniformly globally attractive in the seventh step from the second step. Finally, we prove that the solutions of (1.1) are uniformly bounded.

Step 1: Define

$$\overline{p} = \max\{p, p^*\} \quad \text{and} \quad \underline{p} = \min\{p, p^*\}.$$

For an $\varepsilon > 0$ sufficiently small, we choose

$$\delta(\varepsilon) = \left(\frac{k\varepsilon^{\overline{p}}}{2^{\overline{p}/2+1}Ke^L}\right)^{1/\underline{p}}. \tag{3.1}$$

Note that

$$\delta < \varepsilon^{\overline{p}/\underline{p}} \le \varepsilon \ll 1.$$

Let $t_0 \ge 0$ and $\mathbf{x}_0 = (x_0, y_0)$ be given. We will show that $t \ge t_0$ and $\|\mathbf{x}_0\| = \sqrt{x_0^2 + y_0^2} < \delta$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$. For the sake of convenience of notation, we write $(x(t), y(t)) = \mathbf{x}(t; t_0, \mathbf{x}_0)$, where $(x(t), y(t))$ is a solution of (1.1).

Let

$$u(t) = \frac{f(t)}{g(t)}|y(t)|^{p*} \quad \text{and} \quad v(t) = |x(t)|^p + u(t).$$

Then,

$$v(t) \ge |x(t)|^p + k|y(t)|^{p*} \ge k\big(|x(t)|^p + |y(t)|^{p^*}\big)$$

and

$$\begin{aligned} v'(t) &= p\phi_p(x(t))x'(t) + \frac{p^*(g(t)/f(t))\phi_{p^*}(y(t))y'(t) - (g(t)/f(t))'|y(t)|^{p^*}}{(g(t)/f(t))^2} \\ &= -pe(t)|x(t)|^p - \psi(t)u(t) \le (pe_-(t) + \psi_-(t))v(t) \end{aligned}$$

for $t \ge t_0$. Hence, we obtain

$$\begin{aligned} k\big(|x(t)|^p + |y(t)|^{p^*}\big) &\le v(t) \le \exp\left(\int_{t_0}^t (pe_-(\tau) + \psi_-(\tau))d\tau\right)v(t_0) \\ &\le e^L v(t_0) \le Ke^L\big(|x_0|^p + |y_0|^{p^*}\big) \end{aligned} \tag{3.2}$$

for $t \ge t_0$. Since $|x_0| < \delta \ll 1$ and $|y_0| < \delta \ll 1$, we see that

$$k\big(|x(t)|^p + |y(t)|^{p^*}\big) < Ke^L\big(\delta^p + \delta^{p^*}\big) \le 2Ke^L\delta^{\underline{p}} = k\left(\frac{\varepsilon}{\sqrt{2}}\right)^{\overline{p}},$$

and therefore,

$$|x(t)| \le \left(\frac{\varepsilon}{\sqrt{2}}\right)^{\overline{p}/p} \le \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad |y(t)| \le \left(\frac{\varepsilon}{\sqrt{2}}\right)^{\overline{p}/p^*} \le \frac{\varepsilon}{\sqrt{2}}$$

for $t \ge t_0$. Consequently,

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| = \sqrt{x^2(t)+y^2(t)} < \sqrt{\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}} \le \varepsilon \quad \text{for } t \ge t_0.$$

Thus, the zero solution of (1.1) is uniformly stable. This completes the proof of Step 1.

Step 2: To prove that the zero solution of (1.1) is uniformly globally attractive, for any ρ large enough and any η small enough, we must determine a $T(\rho,\eta) > 0$ such that $t_0 \ge 0$ and $\|\mathbf{x}_0\| < \rho$ imply $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < \eta$ for all $t \ge t_0 + T$. For this purpose, we define several numbers as follows. Let

$$\overline{v} = Ke^L\bigl(\rho^p + \rho^{p^*}\bigr) \quad \text{and} \quad \underline{v} = \frac{k\delta^{\overline{p}}(\eta)}{2^{\overline{p}/2}},$$

where $\delta(\cdot)$ is given in (3.1). Because δ is small enough, we may consider that $\underline{v}/k$ is smaller than 1. Using the numbers $\overline{v}$ and $\underline{v}$, we define

$$\mu = \min\left\{\frac{\underline{v}}{2},\ \frac{k\underline{v}}{2}\left(\frac{\underline{g}}{2\overline{h}}\right)^{p^*}\right\} \quad \text{and} \quad \tau = \hat{t} + \left[\frac{2(1+L)\overline{v}}{l\mu}\right] + 2,$$

where $[c]$ means the greatest integer that is less than or equal to a real number c. Note that the number μ depends only on η and the number τ depends only on ρ and η. Let

$$\nu = \liminf_{t\to\infty} \frac{1}{4\overline{v}} \int_t^{t+\mu k^{1/p}/(4p\overline{f}\overline{v})} \psi_+(\tau)d\tau.$$

The upper limit of integration depends only on ρ and η, and so is the number ν. In addition, ν is positive, because $\psi_+(t)$ is integrally positive. From assumptions (ii) and (iii), we see that there exists a positive number σ depending only on ρ and η such that

$$\int_t^\infty (pe_-(\tau)+\psi_-(\tau))d\tau \le \min\left\{\frac{\mu}{4\overline{v}},\ \frac{\mu\nu}{4}\right\} \tag{3.3}$$

and

$$\int_t^{t+\mu k^{1/p}/(4p\overline{f}\overline{v})} \psi_+(\tau)d\tau \ge 2\nu\overline{v} \tag{3.4}$$

for $t \ge \sigma$. We combine numbers μ, ν, σ and τ, and define

$$T = \sigma + \left(\left[\frac{4}{\mu\nu}\right]+1\right)\left(\frac{3e^M}{(p-1)\overline{h}} + \tau\right)$$

which depends only on ρ and η.

Step 3: Consider a solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ of (1.1) through $(t_0,\mathbf{x}_0)$ with $t_0 \ge 0$ and $\|\mathbf{x}_0\| = \sqrt{x_0^2+y_0^2} < \rho$. To prove that the zero solution of (1.1) is uniformly globally attractive, we have only to show that there exists a $t^* \in [t_0, t_0+T]$ such that

$$|x(t^*)|^p + |y(t^*)|^{p^*} < \frac{\delta^{\overline{p}}(\eta)}{2^{\overline{p}/2}} = \frac{\underline{v}}{k}. \tag{3.5}$$

In fact, if (3.5) holds, then

$$|x(t^*)| < \left(\frac{\underline{v}}{k}\right)^{1/p} \quad \text{and} \quad |y(t^*)| < \left(\frac{\underline{v}}{k}\right)^{1/p^*}.$$

Noticing that

$$\frac{\underline{v}}{k} < 1, \quad \frac{1}{\overline{p}} < \frac{1}{p} < 1 \quad \text{and} \quad \frac{1}{\overline{p}} < \frac{1}{p^*} < 1,$$

we obtain

$$|x(t^*)| < \left(\frac{\underline{v}}{k}\right)^{1/\overline{p}} \quad \text{and} \quad |y(t^*)| < \left(\frac{\underline{v}}{k}\right)^{1/\overline{p}}.$$

Let $\mathbf{x}^* = (x(t^*), y(t^*))$. Then,

$$\|\mathbf{x}^*\| = \sqrt{x^2(t^*) + y^2(t^*)} < \sqrt{2}\left(\frac{\underline{v}}{k}\right)^{1/\overline{p}} = \delta(\eta).$$

Hence, from the conclusion of Step 1, we see that any solution $\mathbf{x}(t; t^*, \mathbf{x}^*)$ of (1.1) through $(t^*, \mathbf{x}^*)$ satisfies that

$$\|\mathbf{x}(t; t^*, \mathbf{x}^*)\| < \eta \quad \text{for } \ t \geq t^*.$$

As mentioned in the top paragraph of §1, system (1.1) has a unique solution of the initial value problem. From this property of solutions of (1.1) and the fact that $\mathbf{x}(t^*; t_0, \mathbf{x}_0) = \mathbf{x}^* = \mathbf{x}(t^*; t^*, \mathbf{x}^*)$, it turns out that $\mathbf{x}(t; t_0, \mathbf{x}_0)$ is corresponding to $\mathbf{x}(t; t^*, \mathbf{x}^*)$ for $t \geq t^*$. Hence, we obtain

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta \quad \text{for } \ t \geq t_0 + T \geq t^*,$$

as required.

By way of contradiction, we will prove that inequality (3.5) holds. Suppose that

$$|x(t)|^p + |y(t)|^{p^*} \geq \frac{\underline{v}}{k} \quad \text{for } \ t_0 \leq t \leq t_0 + T.$$

Then, we have

$$0 < \underline{v} \leq k\big(|x(t)|^p + |y(t)|^{p^*}\big) \leq |x(t)|^p + k|y(t)|^{p*} \leq v(t) \tag{3.6}$$

for $t_0 \leq t \leq t_0 + T$. On the other hand, from (3.2) we see that

$$v(t) \leq Ke^L\big(|x_0|^p + |y_0|^{p^*}\big) < Ke^L\big(\rho^p + \rho^{p^*}\big) = \overline{v} \quad \text{for } \ t \geq t_0. \tag{3.7}$$

Step 4: If $u(t) \geq \mu/2$ for any interval $[\alpha_1, \beta_1] \subset [t_0, t_0 + T]$, then $\beta_1 - \alpha_1 < \tau$, where μ and τ are numbers given in Step 2. In fact, taking into account that

$$\begin{aligned} v'(t) &= -\,pe(t)|x(t)|^p - \psi(t)u(t) \\ &= -\,pe(t)|x(t)|^p + \psi_-(t)u(t) - \psi_+(t)u(t) \end{aligned}$$

for $t \geq t_0$ and using (3.7), we have

$$\begin{aligned} 0 \leq \psi_+(t)u(t) &= -\,v'(t) - pe(t)|x(t)|^p + \psi_-(t)u(t) \\ &\leq -\,v'(t) + (pe_-(t) + \psi_-(t))v(t) \\ &\leq -\,v'(t) + \overline{v}\,(pe_-(t) + \psi_-(t)) \end{aligned} \tag{3.8}$$

for $t \geq t_0$. Integrating the both sides of (3.8) from α_1 to β_1 and using (3.6) and (3.7), we obtain

$$\begin{aligned}\frac{\mu}{2}\int_{\alpha_1}^{\beta_1}\psi_+(\tau)d\tau &\leq \int_{\alpha_1}^{\beta_1}\psi_+(\tau)u(\tau)d\tau \\ &\leq -\int_{\alpha_1}^{\beta_1}v'(\tau)d\tau + \overline{v}\int_{\alpha_1}^{\beta_1}(pe_-(\tau)+\psi_-(\tau))d\tau \\ &\leq v(\alpha_1) - v(\beta_1) + L\overline{v} < (1+L)\overline{v}. \end{aligned} \tag{3.9}$$

Let

$$m = \left[\frac{2(1+L)\overline{v}}{l\mu}\right] + 1 \in \mathbb{N}.$$

Then, $m \geq 2(1+L)\overline{v}/(l\mu)$. Hence, we obtain

$$\begin{aligned}\int_t^{t+m}\psi_+(\tau)d\tau &= \int_t^{t+1}\psi_+(\tau)d\tau + \int_{t+1}^{t+2}\psi_+(\tau)d\tau + \cdots + \int_{t+m-1}^{t+m}\psi_+(\tau)d\tau \\ &\geq lm \geq \frac{2(1+L)\overline{v}}{\mu}\end{aligned}$$

for $t \geq \hat{t}$. If $\alpha_1 \geq \hat{t}$, then by (3.9), we have

$$\int_{\alpha_1}^{\beta_1}\psi_+(\tau)d\tau \leq \frac{2(1+L)\overline{v}}{\mu} \leq \int_{\alpha_1}^{\alpha_1+m}\psi_+(\tau)d\tau,$$

and therefore, $\beta_1 - \alpha_1 \leq m < \tau$. Otherwise, using (3.9) again, we get

$$\int_{\alpha_1}^{\beta_1}\psi_+(\tau)d\tau \leq \frac{2(1+L)}{\mu} \leq \int_{\hat{t}}^{\hat{t}+m}\psi_+(\tau)d\tau \leq \int_{\alpha_1}^{\alpha_1+\hat{t}+m}\psi_+(\tau)d\tau.$$

Hence, $\beta_1 - \alpha_1 \leq \hat{t} + m < \tau$. Thus, we conclude that $u(t) \geq \mu/2$ for $\alpha_1 \leq t \leq \beta_1$ implies $\beta_1 - \alpha_1 < \tau$.

Step 5: If $u(t) \leq \mu$ for any interval $[\alpha_2, \beta_2] \subset [t_0, t_0+T]$, then $\beta_2 - \alpha_2 \leq 2e^M/(p-1)\overline{h}$. In fact, from

$$u(t) = \frac{f(t)}{g(t)}|y(t)|^{p*}, \quad v(t) = |x(t)|^p + u(t) \quad \text{and} \quad \mu = \min\left\{\frac{\underline{v}}{2},\ \frac{k\underline{v}}{2}\left(\frac{\underline{g}}{2\overline{h}}\right)^{p^*}\right\},$$

it follows that

$$|x(t)| = (v(t)-u(t))^{1/p} \geq (\underline{v}-\mu)^{1/p} \geq \left(\frac{\underline{v}}{2}\right)^{1/p} \tag{3.10}$$

and

$$|y(t)| = \left(\frac{g(t)u(t)}{f(t)}\right)^{1/p^*} \leq \left(\frac{\mu}{k}\right)^{1/p^*} \leq \frac{\underline{g}}{2\overline{h}}\left(\frac{\underline{v}}{2}\right)^{1/p^*} \tag{3.11}$$

for $\alpha_2 \leq t \leq \beta_2$. Noticing that

$$y'(t) - (p-1)h_-(t)y(t) = -(p-1)g(t)\phi_p(x(t)) - (p-1)h_+(t)y(t)$$

for $t \geq t_0$ and using (3.10) and (3.11), we obtain

$$\begin{aligned}\left|\left(\exp\left(-(p-1)\int_{t_0}^{t} h_-(\tau)d\tau\right) y(t)\right)'\right| &= (p-1)\exp\left(-(p-1)\int_{t_0}^{t} h_-(\tau)d\tau\right) \\ &\quad\times \left|g(t)\phi_p(x(t)) + h_+(t)y(t)\right| \\ &\geq (p-1)\exp\left(-(p-1)\int_{t_0}^{t} h_-(\tau)d\tau\right) \\ &\quad\times \left(|g(t)||x(t)|^{p-1} - h_+(t)|y(t)|\right) \\ &\geq (p-1)e^{-M}\left(\underline{g}\left(\frac{\underline{v}}{2}\right)^{1/p^*} - \overline{h}|y(t)|\right) \\ &\geq \frac{(p-1)\underline{g}e^{-M}}{2}\left(\frac{\underline{v}}{2}\right)^{1/p^*} > 0\end{aligned}$$

for $\alpha_2 \leq t \leq \beta_2$. Hence, combining this estimation with (3.11), we get

$$\begin{aligned}\frac{\underline{g}}{\overline{h}}\left(\frac{\underline{v}}{2}\right)^{1/p^*} &\geq |y(\beta_2)| + |y(\alpha_2)| \\ &\geq \left|\exp\left(-(p-1)\int_{t_0}^{\beta_2} h_-(\tau)d\tau\right) y(\beta_2)\right. \\ &\qquad \left. - \exp\left(-(p-1)\int_{t_0}^{\alpha_2} h_-(\tau)d\tau\right) y(\alpha_2)\right| \\ &= \left|\int_{\alpha_2}^{\beta_2}\left(\exp\left(-(p-1)\int_{t_0}^{t} h_-(\tau)d\tau\right) y(t)\right)' dt\right| \\ &= \int_{\alpha_2}^{\beta_2}\left|\left(\exp\left(-(p-1)\int_{t_0}^{t} h_-(\tau)d\tau\right) y(t)\right)'\right| dt \\ &\geq \frac{(p-1)\underline{g}e^{-M}}{2}\left(\frac{\underline{v}}{2}\right)^{1/p^*}(\beta_2 - \alpha_2),\end{aligned}$$

namely, $\beta_2 - \alpha_2 \leq 2e^M/(p-1)\overline{h}$. Thus, we conclude that $u(t) \leq \mu$ for $\alpha_2 \leq t \leq \beta_2$ implies $\beta_2 - \alpha_2 \leq 2e^M/(p-1)\overline{h}$.

Step 6: Let us divide the interval $[t_0+\sigma, t_0+T]$ into several pieces. To this end, we define

$$J_i = \left[t_0 + \sigma + (i-1)\left(\frac{3e^M}{(p-1)\overline{h}} + \tau\right),\ t_0 + \sigma + i\left(\frac{3e^M}{(p-1)\overline{h}} + \tau\right)\right]$$

for any $i \in \mathbb{N}$. Then, it is clear that the length of J_i is $3e^M/(p-1)\overline{h} + \tau$ for each $i \in \mathbb{N}$. Hence, we can denote the interval $[t_0+\sigma, t_0+T]$ by

$$[t_0+\sigma, t_0+T] = J_1 \cup J_2 \cup \cdots \cup J_{[4/(\mu\nu)]+1}.$$

Let us pay attention to the motion of $u(t)$ in the subinterval J_1. We will show that $u(t)$ moves from $\mu/2$ to μ in the subinterval J_1. Suppose that $u(t) \geq \mu/2$ for $t \in [t_0+\sigma, t_0+\sigma+\tau] \subset [t_0, t_0+T]$. Then, we may consider α_1 and β_1 in Step 4 to be $t_0+\sigma$ and $t_0+\sigma+\tau$, respectively. From the conclusion of Step 4 it follows that

$$\tau = t_0 + \sigma + \tau - (t_0 + \sigma) = \beta_1 - \alpha_1 < \tau,$$

which is a contradiction. Hence, we see that there exists a $t_1 \in [t_0+\sigma, t_0+\sigma+\tau] \subset J_1$ such that $u(t_1) < \mu/2$. Next, suppose that $u(t) \le \mu$ for $t \in [t_0+\sigma+\tau, \gamma] \subset [t_0, t_0+T]$, where $\gamma = t_0 + \sigma + 3e^M/(p-1)\overline{h} + \tau$. Then, we may consider α_2 and β_2 in Step 5 to be $t_0 + \sigma + \tau$ and γ, respectively. From the conclusion of Step 5 it follows that

$$\frac{3e^M}{(p-1)\overline{h}} = \gamma - (t_0 + \sigma + \tau) = \beta_2 - \alpha_2 \le \frac{2e^M}{(p-1)\overline{h}}.$$

This is a contradiction. Hence, we see that that there exists a $t_2 \in [t_0+\sigma+\tau, \gamma] \subset J_1$ such that $u(t_2) > \mu$. Since $u(t)$ is continuous for $t \ge t_0$, there exists an interval $[\alpha, \beta] \subset [t_1, t_2]$ such that $u(\alpha) = \mu/2$, $u(\beta) = \mu$ and

$$\frac{\mu}{2} \le u(t) \le \mu \quad \text{for } \alpha \le t \le \beta. \tag{3.12}$$

Hence, together with (3.3) and (3.7), we have

$$\begin{aligned}
\frac{\mu}{2} &= u(\beta) - u(\alpha) = \int_\alpha^\beta u'(\tau)d\tau \\
&= \int_\alpha^\beta \left(-\psi(\tau)u(\tau) - pf(\tau)\phi_p(x(\tau))\phi_{p^*}(y(\tau))\right) d\tau \\
&\le \int_\alpha^\beta \left(\psi_-(\tau)v(\tau) + p|f(\tau)||x(\tau)|^{p-1}|y(\tau)|^{p^*-1}\right) d\tau \\
&\le \frac{\mu}{4} + p\overline{f}\int_\alpha^\beta |x(\tau)|^{p-1}|y(\tau)|^{p^*-1}d\tau,
\end{aligned}$$

and therefore,

$$\frac{\mu}{4p\overline{f}} \le \int_\alpha^\beta |x(\tau)|^{p-1}|y(\tau)|^{p^*-1}d\tau.$$

Using (3.7) again, we obtain

$$|x(t)| = (v(t) - u(t))^{1/p} < \overline{v}^{\,1/p} \quad \text{and} \quad |y(t)| = \left(\frac{g(t)u(t)}{f(t)}\right)^{1/p^*} \le \left(\frac{\overline{v}}{k}\right)^{1/p^*}$$

for $t \ge t_0$. From these inequalities and the relation that $1/p + 1/p^* = 1$, we conclude that

$$\frac{\mu k^{1/p}}{4p\overline{f}\overline{v}} < \beta - \alpha. \tag{3.13}$$

Step 7: We may understand $v(t)$ as an energy-type function. Let us examine a change of the energy in the subinterval J_1. We will estimate the difference between the values of $v(t_0+\sigma)$, $v(\alpha)$, $v(\beta)$ and $v(\gamma)$ in particular. By using (3.3), (3.4), (3.8), (3.12) and (3.13), we obtain

$$\begin{aligned}
\mu\nu\overline{v} &\le \frac{\mu}{2}\int_\alpha^{\alpha+\mu k^{1/p}/(4p\overline{f})} \psi_+(\tau)d\tau \le \frac{\mu}{2}\int_\alpha^\beta \psi_+(\tau)d\tau \\
&\le \int_\alpha^\beta \psi_+(\tau)u(\tau)d\tau \le \int_\alpha^\beta \left\{-v'(\tau) + \overline{v}\left(pe_-(\tau) + \psi_-(\tau)\right)\right\}d\tau \\
&= v(\alpha) - v(\beta) + \overline{v}\int_\alpha^\beta (pe_-(\tau) + \psi_-(\tau))d\tau \le v(\alpha) - v(\beta) + \frac{\mu\nu\overline{v}}{4},
\end{aligned}$$

and therefore,

$$v(\beta) - v(\alpha) \le -\frac{3\mu\nu\overline{v}}{4}.$$

It also follows from (3.3) and (3.8) that

$$v(\alpha) - v(t_0+\sigma) = \int_{t_0+\sigma}^{\alpha} v'(\tau)d\tau \le \overline{v}\int_{t_0+\sigma}^{\alpha}(pe_-(\tau)+\psi_-(\tau))d\tau \le \frac{\mu\nu\overline{v}}{4}$$

and

$$v(\gamma) - v(\beta) = \int_{\beta}^{\gamma} v'(\tau)d\tau \le \overline{v}\int_{\beta}^{\gamma}(pe_-(\tau)+\psi_-(\tau))d\tau \le \frac{\mu\nu\overline{v}}{4}.$$

We therefore conclude that

$$\begin{aligned}\int_{J_1} v'(\tau)d\tau &= v(\gamma) - v(\beta) + v(\beta) - v(\alpha) + v(\alpha) - v(t_0+\sigma)\\ &\le \frac{\mu\nu\overline{v}}{4} - \frac{3\mu\nu\overline{v}}{4} + \frac{\mu\nu\overline{v}}{4} = -\frac{\mu\nu\overline{v}}{4}.\end{aligned}$$

By means of the same process as in the proof of Steps 6 and 7, we see that

$$\int_{J_i} v'(\tau)d\tau \le -\frac{\mu\nu\overline{v}}{4} \quad \text{for } 1 \le i \le [4/(\mu\nu)]+1,$$

and therefore,

$$v(t_0+T) - v(t_0+\sigma) = \sum_{i=1}^{[4/(\mu\nu)]+1}\int_{J_i} v'(s)ds \le -\frac{\mu\nu}{4}\left(\left[\frac{4}{\mu\nu}\right]+1\right) < -\overline{v}.$$

Hence, from (3.7) it follows that

$$v(t_0+T) < v(t_0+\sigma) - \overline{v} < 0.$$

This contradicts the fact that $v(t) \ge 0$ for $t \ge t_0$. This contradiction is caused because it had been assumed that inequality (3.5) did not hold. Thus, (3.5) is true. Consequently, as shown in Step 3, the zero solution of (1.1) is uniformly globally attractive.

Step 8: Consider again a solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ of (1.1) through $(t_0,\mathbf{x}_0)$ with $t_0 \ge 0$ and $\|\mathbf{x}_0\| = \sqrt{x_0^2+y_0^2} < \rho$ for any ρ large enough. Let

$$B(\rho) = \sqrt{\left(\frac{\overline{v}}{k}\right)^{2/p} + \left(\frac{\overline{v}}{k}\right)^{2/p^*}},$$

where $\overline{v}$ is the number given in Step 2. We will show that $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < B$ for $t \ge t_0$. Recall that $(x(t),y(t)) = \mathbf{x}(t;t_0,\mathbf{x}_0)$. By the same way as in Step 1, we have the estimation (3.2). Since $|x_0| < \rho$ and $|y_0| < \rho$, we see that

$$|x(t)|^p + |y(t)|^{p*} < \frac{Ke^L(\rho^p+\rho^{p^*})}{k} = \frac{\overline{v}}{k}$$

and therefore,

$$|x(t)| \le \left(\frac{\overline{v}}{k}\right)^{1/p} \quad \text{and} \quad |y(t)| \le \left(\frac{\overline{v}}{k}\right)^{1/p^*}$$

for $t \ge t_0$. Consequently,

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| = \sqrt{x^2(t) + y^2(t)} < B \quad \text{for} \ \ t \ge t_0.$$

Thus, the solutions of (1.1) are uniformly bounded.

The proof of theorem 3.1 is thus complete. □

By transforming $(x, y) \to (y, x)$, system (1.1) becomes the system

$$\begin{aligned} x' &= -\,\widetilde{e}(t)x + \widetilde{f}(t)\phi_{\widetilde{p}^*}(y), \\ y' &= -(p-1)\widetilde{g}(t)\phi_{\widetilde{p}}(x) - (p-1)\widetilde{h}(t)y, \end{aligned}$$

where $\widetilde{p} = p^*$, $\widetilde{p}^* = p$; $\widetilde{e}(t) = (p-1)h(t)$, $\widetilde{f}(t) = -(p-1)g(t)$, $\widetilde{g}(t) = -f(t)/(p-1)$ and $\widetilde{h}(t) = e(t)/(p-1)$ for $t \ge 0$. Let

$$\widetilde{\psi}(t) = \frac{p}{(p-1)^2}\,e(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'.$$

Then, we have the following result.

THEOREM 3.2. *Suppose that $e_+(t)$, $f(t)$, and $g(t)$ are bounded for $t \ge 0$. If*

(i) $f(t)g(t) > 0$ *for* $t \ge 0$ *and* $\liminf_{t\to\infty} f(t)g(t) > 0$,

(ii) $\int_0^\infty e_-(t)dt < \infty$, $\int_0^\infty h_-(t)dt < \infty$ *and* $\int_0^\infty \widetilde{\psi}_-(t)dt < \infty$,

(iii) $\widetilde{\psi}_+(t)$ *is integrally positive*,

then the zero solution of (1.1) *is uniformly globally asymptotically stable.*

4. Examples

To illustrate theorems 3.1 and 3.2, we give simple examples in which $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are periodic functions with period 2π. We consider the positive number p in system (1.1) as a parameter. Before we present the examples, it is helpful to mention a property of a periodic function $w(t)$ defined by

$$w(t) = ce^{\sin t} - \sin t$$

for any positive c. By a straightforward calculation, we can confirm that $w(t)$ is nonnegative for $t \ge 0$ if and only if $c \ge 1/e$. Since $w(t)$ is periodic, we see that $c \ge 1/e$ is a necessary and sufficient condition for $w(t)$ to be integrally positive.

EXAMPLE 4.1. Consider system (1.1) with

$$e(t) = -\frac{1}{(1+t)^2}, \quad f(t) = e^{\sin t}, \quad g(t) = e^{\cos t} \quad \text{and} \quad h(t) = \frac{1}{\sqrt{2}}e^{\sin(t+\pi/4)-1}. \tag{4.1}$$

Then, $\psi_+(t)$ is integrally positive if and only if

$$p \geq 2. \tag{4.2}$$

If (4.2) holds, then the zero solution is uniformly globally asymptotically stable.

From (4.1) it follows that

$$\begin{aligned}\psi(t) &= \frac{p}{\sqrt{2}e}e^{\sin(t+\pi/4)} + e^{(\sin t - \cos t)}\left(e^{(\cos t - \sin t)}\right)' \\ &= \frac{p}{\sqrt{2}e}e^{\sin(t+\pi/4)} - \sin t - \cos t \\ &= \sqrt{2}\left(\frac{p}{2e}e^{\sin(t+\pi/4)} - \sin(t+\pi/4)\right)\end{aligned}$$

for $t \geq 0$. Hence, (4.2) is a necessary and sufficient condition under which $\psi_+(t)$ is integrally positive.

It is clear that $f(t)$ and $g(t)$ are bounded and $g(t)/f(t)$ is differentiable for $t \geq 0$. Since $e_-(t) = h_-(t) = 0$ and

$$h_+(t) = \frac{1}{\sqrt{2}}e^{\sin(t+\pi/4)-1},$$

$e_-(t)$ and $h_-(t)$ are integrable and $h_+(t)$ is bounded for $t \geq 0$. We have

$$f(t)g(t) = e^{\sin t + \cos t} \geq e^{-\sqrt{2}} > 0 \quad \text{for} \quad t \geq 0.$$

If (4.2) holds, then $\psi(t)$ is nonnegative for $t \geq 0$, and therefore, $\psi_-(t) \equiv 0$. Thus, by virtue of Theorem 3.1, we conclude that the zero solution is uniformly globally asymptotically stable provided that $p \geq 2$.

Example 4.2. Consider system (1.1) with

$$e(t) = \frac{1}{\sqrt{2}}e^{\sin(t+\pi/4)-1}, \quad f(t) = e^{\sin t}, \quad g(t) = e^{\cos t} \quad \text{and} \quad h(t) = -\frac{1}{(1+t)^2}. \tag{4.3}$$

Then, $\widetilde{\psi}_+(t)$ is integrally positive if and only if

$$1 < p \leq 2. \tag{4.4}$$

If (4.4) holds, then the zero solution is uniformly globally asymptotically stable.

From (4.1) we see that

$$\begin{aligned}\widetilde{\psi}_+(t) &= \frac{p}{\sqrt{2}(p-1)^2e}e^{\sin(t+\pi/4)} + e^{(\sin t - \cos t)}\left(e^{(\cos t - \sin t)}\right)' \\ &= \frac{p}{\sqrt{2}(p-1)^2e}e^{\sin(t+\pi/4)} - \sin t - \cos t \\ &= \sqrt{2}\left(\frac{p}{2(p-1)^2e}e^{\sin(t+\pi/4)} - \sin(t+\pi/4)\right)\end{aligned}$$

for $t \geq 0$. It is clear that $p/(p-1)^2 \geq 2$ if and only if $1/2 \leq p \leq 2$. Recall that $p > 1$. Hence, (4.4) is a necessary and sufficient condition under which $\widetilde{\psi}_+(t)$ is integrally positive.

It is easy to confirm that all of the assumptions in Theorem 3.2 are satisfied if $1 < p \leq 2$. We omit the details.

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