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Global asymptotic stability for half-linear differential systems with generalized almost periodic coefficients

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Abstract The following system considered in this paper:

$$x' = -e(t)x + f(t)\phi_{p^*}(y), \qquad y' = -(p-1)g(t)\phi_p(x) - (p-1)h(t)y,$$

where $p > 1$, $p^* > 1$ $(1/p + 1/p^* = 1)$ and $\phi_q(z) = |z|^{q-2}z$ for $q = p$ or $q = p^*$. This system is referred to as a half-linear system. The coefficient $f(t)$ is assumed to be bounded, but the coefficients $e(t)$, $g(t)$ and $h(t)$ are not necessarily bounded. Sufficient conditions are obtained for global asymptotic stability of the zero solution. Our results can be applied to not only the case that the signs of $f(t)$ and $g(t)$ change like the periodic function but also the case that $f(t)$ and $g(t)$ irregularly have zeros. Some suitable examples are included to illustrate our results.

Keywords Half-linear differential systems · Global asymptotic stability · Weakly integrally positive · Almost periodic functions

Mathematics Subject Classification (2010) 34D05 · 34D23 · 34D45

1 Introduction

The purpose of this paper is to give sufficient conditions for the zero solution to non-autonomous two dimensional systems of the form

$$\begin{aligned} x' &= -e(t)x + f(t)\phi_{p^*}(y), \\ y' &= -(p-1)g(t)\phi_p(x) - (p-1)h(t)y \end{aligned} \tag{1.1}$$

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to be globally asymptotically stable. Here, the prime denotes d/dt; the coefficients $e(t)$, $f(t)$, $g(t)$, and $h(t)$ are continuous for $t \geq 0$; the two numbers p and p^* are positive and satisfy

$$\frac{1}{p} + \frac{1}{p^*} = 1;$$

the function $\phi_q(z)$ is defined by

$$\phi_q(z) = |z|^{q-2}z, \qquad z \in \mathbb{R}$$

for $q = p$ or $q = p^*$. Note that $p > 1$ and $p^* > 1$. If $(x(t), y(t))$ is a solution of (1.1), then the function $(cx(t), \phi_p(c)y(t))$ is also a solution of (1.1) for any $c \in \mathbb{R}$. However, the sum of two solutions of (1.1) is not always a solution of (1.1). In other words, the solution space of (1.1) is homogeneous, but not additive.

We say that the zero solution of (1.1) is *globally attractive* if every solution $(x(t), y(t))$ of (1.1) tends to the origin $(0,0)$ as time t increases. In addition, if the zero solution of (1.1) is stable, then it is said to be *globally asymptotically stable*. It is not too much to say that the study of the global asymptotic stability of dynamical systems occupies an important position in the qualitative theory of differential equations. There are a lot of applications concerning the global asymptotic stability.

Throughout this paper, we assume that there exist positive numbers α, $\underline{\beta}$ and $\overline{\beta}$ such that

$$E(t) \geq -\alpha \tag{A_1}$$

and

$$\underline{\beta} \leq \exp(pE(t))\frac{f(t)}{g(t)} \leq \overline{\beta} \tag{A_2}$$

for $t \geq 0$, where $E(t) \overset{\text{def}}{=} \int_0^t e(s)ds$. It is not necessarily assumed that $f(t)$ and $g(t)$ are always positive. Note that $f(t)$ and $g(t)$ are allowed to become zero at the same time. For example, if $e(t) = 0$, $f(t) = \sin t$ and $g(t) = \sin t + (\sin 3t)/2$, then assumptions ($A_1$) and ($A_2$) are satisfied with $\alpha = 1$, $\underline{\beta} = 2/5$ and $\overline{\beta} = 2$.

Consider the special case that $e(t) = 0$, $f(t) = 1$,

$$g(t) = \frac{c(t)}{(p-1)a(t)} \quad \text{and} \quad h(t) = \frac{a'(t) + b(t)}{(p-1)a(t)},$$

where $a(t)$ is positive and differentiable for all $t \geq 0$. Then, taking into account that $y = \phi_p(x')$, we can rewrite system (1.1) as the second-order differential equation

$$\big(a(t)\phi_p(x')\big)' + b(t)\phi_p(x') + c(t)\phi_p(x) = 0, \tag{1.2}$$

which is a basic kinetic equation if $p = 2$. The solution space of (1.2) is also homogeneous, but not additive. For this reason, equation (1.2) is said to be *half-linear*. Since the half-linear differential equation (1.2) is a generalization of the second-order linear differential equation with variable coefficients, many good articles are reported ceaselessly over the past four decades. Those results can be found in the books [1, 14, 15] and the references cited therein. However, strangely, the studies of half-linear differential equations (or systems) are concentrating on oscillation theory. There are not so many researches on the global asymptotic stability of the zero solution of half-linear differential systems. Especially, little is known about the case that the coefficients change the sign like the periodic functions.

Although such a case is significant on the applied aspect, it is hard to examine the asymptotic behavior of solutions.

Very recently, the present authors [53] have discussed the stability problem for system (1.1) in the special case that $f(t)$ and $g(t)$ are the same periodic functions. By putting

$$\psi(t) = ph(t) - pe(t),$$

they reported the following result.

Theorem A *Suppose that*

(i) $f(t) = g(t) \not\equiv 0$ *and* $f(t)$ *is periodic*;

(ii) $E(t)$ *and* $h(t)$ *are bounded for* $t \geq 0$;

(iii) $\psi_+(t)$ *is weakly integrally positive for* $t \geq 0$;

(iv) $\displaystyle\int_0^\infty \psi_-(t)dt < \infty$,

where

$$\psi_+(t) = \max\{0, \psi(t)\} \quad \textit{and} \quad \psi_-(t) = \max\{0, -\psi(t)\}.$$

Then the zero solution of (1.1) *is globally asymptotically stable*.

Note that the weak integral positivity of $\psi_+(t)$ is a stronger assumption than

$$\lim_{t\to\infty} \int^t \psi_+(s)ds = \infty$$

(for the definition of the weak integral positivity, see Sect. 3). It is clear that the above assumptions (i) and (ii) imply assumptions (A_1) and (A_2). Since $f(t)$ is assumed to be periodic, it is allowed to change the sign. Unfortunately, however, assumption (i) is too strong to be applied to any practical case. For example, if $f(t) = g(t) = t\sin t/(1+t)$, then $f(t)$ and $g(t)$ are asymptotically periodic functions, but are not periodic functions. Hence, Theorem A cannot be applied. The following question then arises. In Theorem A, cannot we change only assumption (i) into the assumption that $f(t) = g(t) \not\equiv 0$ and $f(t)$ is asymptotically periodic? Of course, Theorem A is not useful even if $f(t)$ and $g(t)$ are different periodic functions. It may be natural to consider whether or not assumption (i) can be weakened to the assumption that $f(t)$ and $g(t)$ are different periodic functions with the same period. But, if $e(t) = 0$, $f(t) = \sin t$, $g(t) = \cos t$ and $h(t) = 1/(1+t)$, then the zero solution of (1.1) is not globally asymptotically stable though assumptions (ii)–(iv) hold. Here, other questions are caused. What kind of condition on $f(t)$ and $g(t)$ will guarantee the global asymptotic stability of the zero solution of (1.1) under the assumption that $f(t)$ and $g(t)$ are different periodic functions with the same period? To begin with, will some periodicity be necessary for $f(t)$ and $g(t)$? Is not the zero solution of (1.1) globally asymptotically stable when $f(t)$ and $g(t)$ irregularly have zeros?

We answer the above questions in this paper. For this purpose, we define a class of bounded functions, which contains all nontrivial almost periodic functions and asymptotically almost periodic functions (for those definitions, see Sect. 2). To show that this class of functions covers a wide range, we give easy examples in Sect. 2. The main result and its corollary are stated in Sect. 3. A certain function composed by all the coefficients plays a vital role in our results. We call this a characteristic function. To compare our theorems with previous results on the asymptotic stability, we outline the evolution of Lyapunov's direct method. In particular, we explain that our results are not settled by Barbašin and Krasovskiĭ's theorem and Matrosov's theorem. We give the proof of the main result in Sect. 4. This section is the core of the present paper. Our method is to examine the asymptotic behavior of solutions of (1.1) by using a certain function described by coefficients and solutions in addition to a Lyapunov function and the characteristic function. In Sect. 5, we improve the boundedness of $h(t)$. We show that $\liminf_{t\to\infty}|h(t)|$ has only to be finite. Through a simple example, we mention that the zero solution of (1.1) is not necessarily globally asymptotically stable in the case that $\lim_{t\to\infty}|h(t)| = \infty$. Finally, in Sect. 6, we give some examples to illustrate our results.

2 Property (P)

Before we go on to the main subject, let us define a family of bounded functions.

Definition 2.1. A nontrivial bounded function $\chi(t)$ is said to have *property* (P) if there exist positive numbers δ, ω and d with $\omega > d$, and a positive sequence $\{t_m\}$ with $t_m < \omega - d$ such that

$$\chi(t) \geq \delta \quad \text{if} \;\; (m-1)\omega + t_m \leq t \leq (m-1)\omega + t_m + d \tag{2.1}$$

or

$$\chi(t) \leq -\delta \quad \text{if} \;\; (m-1)\omega + t_m \leq t \leq (m-1)\omega + t_m + d \tag{2.2}$$

for $m \in \mathbb{N}$ sufficiently large.

It is clear that any nontrivial periodic function has property (P). For example, consider the periodic function $\chi(t) = \sin t$. Then, it has property (P) with $\delta = 1/2$, $\omega = \pi$, $d = 2\pi/3$ and $t_m = \pi/6$ for all $m \in \mathbb{N}$.

There are some definitions of almost periodic functions. The concept was first studied by Harald Bohr. When we limit the domain of an almost periodic function from the whole real line to the nonnegative real line, his definition becomes as follows: a continuous function $\chi(t)$ is said to be *almost periodic* if, for any $\varepsilon > 0$, there exists a positive number $l(\varepsilon)$ such that any interval of length $l(\varepsilon)$ contains at least one number τ for which

$$|\chi(t+\tau) - \chi(t)| < \varepsilon \quad \text{for} \;\; t \geq 0.$$

As to almost periodic functions, for example, see the books [7, 8, 10, 12, 13, 16, 20, 28, 39, 56].

Lemma 2.2 *Any nontrivial almost periodic function has property* (P).

Proof As is well known, any almost periodic function $\chi(t)$ is bounded and uniformly continuous. Since $\chi(t) \not\equiv 0$, there exists a $t^* > 0$ such that $\chi(t^*) > 0$ or $\chi(t^*) < 0$. We may assume that $\chi(t^*) > 0$, because the proof of the case that $\chi(t^*) < 0$ is essentially the same as that of the case that $\chi(t^*) > 0$. Since $\chi(t)$ is continuous for $t \geq 0$, there exist two numbers a and b with $0 < a < t^* < b$ such that

$$\chi(t) \geq \frac{3}{4}\chi(t^*) \quad \text{for} \;\; a \leq t \leq b. \tag{2.3}$$

Since $\chi(t)$ is almost periodic, there exists a $l^* = l(\chi(t^*)/4) > 0$ such that any interval of length l^* contains a τ for which

$$|\chi(t+\tau) - \chi(t)| < \frac{1}{4}\chi(t^*) \quad \text{for } t \geq 0.$$

We may assume without loss of generality that $l^* > b$.

Let $\delta = \chi(t^*)/2$, $\omega = 2l^*$ and $d = b - a$. Then, $\omega > l^* > b > b - a = d$. Consider the interval $[(m-1)\omega, (m-1)\omega + l^*]$ for any $m \in \mathbb{N}$. Note that $[(m-1)\omega, (m-1)\omega + l^*]$ and $[m\omega, m\omega + l^*]$ do not intersect each other. Since the length of $[(m-1)\omega, (m-1)\omega + l^*]$ is l^*, we can find a $\tau_m \in [(m-1)\omega, (m-1)\omega + l^*]$ such that

$$|\chi(t+\tau_m) - \chi(t)| < \frac{1}{4}\chi(t^*) \quad \text{for } t \geq 0.$$

Hence, together with (2.3), we obtain

$$\frac{1}{4}\chi(t^*) > |\chi(t) - \chi(t+\tau_m)| \geq \chi(t) - \chi(t+\tau_m) \geq \frac{3}{4}\chi(t^*) - \chi(t+\tau_m)$$

for $a \leq t \leq b$. By rewriting this, we get

$$\chi(t) \geq \frac{1}{2}\chi(t^*) = \delta \quad \text{for } a + \tau_m \leq t \leq b + \tau_m.$$

Let $t_m = a + \tau_m - (m-1)\omega$ for each $m \in \mathbb{N}$. Then, since $l^* > b$ and $(m-1)\omega \leq \tau_m \leq (m-1)\omega + l^*$, we see that $0 < a \leq t_m \leq a + l^* < a + 2l^* - b = \omega - d$ and

$$\chi(t) \geq \delta \quad \text{for } (m-1)\omega + t_m \leq t \leq (m-1)\omega + t_m + d.$$

We therefore conclude that $\chi(t)$ has property (P). □

Following Fréchet [17], a continuous function $\chi(t)$ is said to be *asymptotically almost periodic* if, it is a sum of an almost periodic function $p(t)$ and a function $q(t)$ which tends to zero as $t \to \infty$; that is,

$$\chi(t) = p(t) + q(t) \quad \text{for } t \geq 0$$

(see also [10, 56]). We can also show that any nontrivial asymptotically almost periodic function has property (P) (we omit the details).

Let us give some examples of the function which has property (P), except for periodic functions, almost periodic functions and asymptotically almost periodic functions. For any $n \in \mathbb{N}$, let $a_n = n^2 + n - 2$. We divide the nonnegative real line $[0, \infty)$ into two sequences of intervals

$$I_n = \left[\frac{\pi a_n}{2}, \frac{\pi(a_n+2)}{2}\right] \quad \text{and} \quad J_n = \left[\frac{\pi(a_n+2)}{2}, \frac{\pi a_{n+1}}{2}\right].$$

Example 2.3 Let $\chi(t)$ be a continuous differentiable function satisfying

$$\chi(t) = \begin{cases} \cos t & \text{if } t \in I_{4k-3} \\ -\cos t & \text{if } t \in I_{4k-2} \\ -\cos t & \text{if } t \in I_{4k-1} \\ \cos t & \text{if } t \in I_{4k} \end{cases}$$

with $k \in \mathbb{N}$ and $\chi(t) = (-1)^n$ for $t \in J_n$. Then, $\chi(t)$ has property (P).

The length of I_n is always π for any $n \in \mathbb{N}$. Since the length of J_n is πn, it tends to ∞ as $n \to \infty$. Hence, $\chi(t)$ given in Example 2.3 is neither an almost periodic function nor an asymptotically almost periodic function.

Note that a_n is even for any $n \in \mathbb{N}$. Then, we can confirm that $\chi(t)$ in Example 2.3 has property (P) with $\delta = 1/2$, $\omega = \pi$, $d = \pi/3$ and $t_m = 0$ for all $m \in \mathbb{N}$.

Here, we define two functions as follows: $r(t) = \sqrt{t}\sin\sqrt{t}$ and

$$s(t) = \begin{cases} t/4 & \text{if } 0 \le t \le 2 \\ 1/2 & \text{if } 2 \le t \le \pi^2 - 2 \\ -(t-\pi^2)/4 & \text{if } \pi^2 - 2 \le t \le \pi^2 \end{cases}$$

with $s(t+\pi^2) = s(t)$ for $t \ge 0$. Needless to say, $s(t)$ is a nonnegative periodic function with period π^2, but $r(t)$ is not even an (asymptotically) almost periodic function. It is clear that $s(t)$ has property (P) with $\delta = 1/2$, $\omega = \pi^2$, $d = \pi^2 - 4$ and $t_m = 2$ for all $m \in \mathbb{N}$. Note that $r(t)$ vanishes at $t = 0$ and $t = \pi^2 n^2$. Taking into account that $r'(\pi^2 n^2) = (-1)^n/2$ and

$$\lim_{t \to 0} r'(t) = 1,$$

we conclude that $s(t) \le |r(t)|$ for $t \ge 0$. Hence, $r(t)$ satisfies the inequality (2.1) or (2.2) with $\delta = 1/2$, $\omega = \pi^2$, $d = \pi^2 - 4$ and $t_m = 2$ for all $m \in \mathbb{N}$. By using the function $r(t)$, it is easy to exhibit another example of property (P).

Example 2.4 The function

$$\chi(t) = \max\{\min\{r(t), 1\}, -1\}$$

has property (P).

3 Our main result and development of Lyapunov's direct method

Here, we give the definition of the weak integral positivity that is assumed in Theorem A. The weak integral positivity is an important concept even in the present paper.

Definition 3.1. A nonnegative function φ is said to be *weakly integrally positive* if

$$\int_I \varphi(t)dt = \infty$$

for every set $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$ such that $\tau_n + \delta < \sigma_n < \tau_{n+1} \le \sigma_n + \Delta$ for some $\delta > 0$ and $\Delta > 0$.

We can also find the concept in the paper [22, 24, 25, 31, 51, 52]. A typical example of weakly integrally positive function is $1/(1+t)$ or $\sin^2 t/(1+t)$ (for the proof, see Proposition 2.1 in [53]). Any nonnegative periodic function is also weakly integrally positive. If φ is weakly integrally positive, then it naturally follows that

$$\lim_{t \to \infty} \int^t \varphi(s)ds = \infty.$$

To state our main result, we assume that $f(t)/g(t)$ is differentiable for $t \geq 0$ and define the characteristic function

$$\Psi(t) = ph(t) - pe(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'.$$

This assumption does not mean that both $f(t)$ and $g(t)$ are differentiable for $t \geq 0$. For the sake of brevity, we write

$$\Psi_+(t) = \max\{0, \Psi(t)\} \quad \text{and} \quad \Psi_-(t) = \max\{0, -\Psi(t)\}.$$

In the case that $f(t) = g(t)$, then $\Psi_+(t)$ and $\Psi_-(t)$ are the same as $\psi_+(t)$ and $\psi_-(t)$ in Theorem A, respectively.

Theorem 3.2 *Let assumptions* (A_1) *and* (A_2) *hold and suppose that* $f(t)/g(t)$ *is differentiable for* $t \geq 0$. *Suppose also that*

$$g(t)/\exp((p-1)E(t)) \textit{ has property } (P); \tag{3.1}$$

$$h(t) \textit{ is bounded for } t \geq 0; \tag{3.2}$$

$$\Psi_+(t) \textit{ is weakly integrally positive}; \tag{3.3}$$

$$\int_0^\infty \Psi_-(t)dt < \infty. \tag{3.4}$$

Then the zero solution of (1.1) *is globally asymptotically stable*.

As known well, Lyapunov's direct method is a tool which is effective to examine the asymptotic behavior of solutions of differential systems. Although this method is convenient to deal with stability problems roughly, it is not so easy to seek available Lyapunov functions. For example, to show that the zero solution is globally asymptotically stable, we need to find a Lyapunov function which is positive definite and radially unbounded and whose total derivative along any solution is negative definite (for example, see [2, 6, 41–43]). Unfortunately, however, it is very difficult to construct such a suitable Lyapunov function for a concrete system. Even if we choose the total energy as a Lyapunov function, the derivative of the Lyapunov function is not always negative definite. Hence, such Lyapunov-type theorems has a big weak point.

To overcome this weak point, a great deal of efforts has been made. The main attempt was to weaken the negative definiteness of the total derivative. For example, Barbašin and Krasovskiĭ [6] have presented asymptotic stability criteria for autonomous systems under the assumption that the total derivative of Lyapunov function is nonpositive and that the set where the derivative is zero contains no complete trajectories except the origin. Krasovskiĭ [32] proved that these criteria can be applied even to periodic systems. However, it is necessary to find a suitable periodic Lyapunov function. LaSalle [35] called the essence of Barbašin and Krasovskiĭ's argument 'invariance principle' and extended it to a particular case of nonautonomous systems (see also [34, 36]). Note that Barbašin and Krasovskiĭ's theorem cannot be extended to a general case of nonautonomous systems (refer to [44]). Much ink has been spent on extensions of the Barbašin-Krasovskiĭ-LaSalle method (for example, see [4, 9, 18, 19, 23, 29, 30, 38, 55]).

There is another direction to weaken the negative definiteness of the total derivative. Matrosov [44] has dealt with the general nonautonomous system

$$\mathbf{x}' = \mathbf{F}(t,\mathbf{x})$$

with $\mathbf{F}(t,\mathbf{x})$ being bounded with respect to t. Assuming the existence of a Lyapunov function which is positive definite and decrescent whose total derivative is not greater than a nonpositive and time-invarient function and using an additional auxiliary function, he proved that the zero solution is uniformly asymptotically stable. Matrosov's theorem was extended in various directions by himself and many researchers (for example, see [3, 21, 22, 33, 37, 45, 47, 49, 50]).

About the above-mentioned stability theory via Lyapunov's direct method, we can refer to the book [48] greatly. In any case, it is assumed that the total derivative of Lyapunov function is at least non-positive.

We cannot apply Barbašin and Krasovskiĭ's theorem, because system (1.1) is neither autonomous nor periodic. Let us examine whether Matrosov's theorem is applicable to system (1.1) or not. For this purpose, we adopt

$$V(t,x,y) = \exp(pE(t))\left(|x|^p + \frac{f(t)}{g(t)}|y|^{p^*}\right)$$

as a suitable Lyapunov function, which is regarded as a total energy for system (1.1). Then, taking account of the relations

$$\frac{d}{dz}|z|^q = q\phi_q(z) \quad \text{and} \quad z\phi_q(z) = |z|^q$$

for $q = p$ or $q = p^*$, we get

$$\begin{aligned}
\dot V_{(1.1)}(t,x,y) &= p\,e(t)\exp(pE(t))\left(|x|^p + \frac{f(t)}{g(t)}|y|^{p^*}\right)\\
&\quad + \exp(pE(t))\left\{p\,\phi_p(x)x' + p^*\frac{f(t)}{g(t)}\phi_{p^*}(y)y' + \left(\frac{f(t)}{g(t)}\right)'|y|^{p^*}\right\}\\
&= p\,e(t)\exp(pE(t))\left(|x|^p + \frac{f(t)}{g(t)}|y|^{p^*}\right)\\
&\quad + p\,\exp(pE(t))\big(-e(t)|x|^p + f(t)\phi_p(x)\phi_{p^*}(y)\big)\\
&\quad + p^*(p-1)\exp(pE(t))\left(-f(t)\phi_p(x)\phi_{p^*}(y) - \frac{f(t)h(t)}{g(t)}y\phi_{p^*}(y)\right)\\
&\quad + \exp(pE(t))\left(\frac{f(t)}{g(t)}\right)'\frac{|y|^{p^*}}{p^*}.
\end{aligned}$$

Since $p^*(p-1) = p$ and

$$-\frac{g(t)}{f(t)}\left(\frac{f(t)}{g(t)}\right)' = \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)',$$

we obtain

$$
\begin{aligned}
\dot{V}_{(1.1)}(t,x,y) &= \exp(pE(t))\frac{f(t)}{g(t)}|y|^{p^*}\left\{pe(t)-ph(t)+\frac{g(t)}{f(t)}\left(\frac{f(t)}{g(t)}\right)'\right\}\\
&= -\exp(pE(t))\frac{f(t)}{g(t)}|y|^{p^*}\left\{ph(t)-pe(t)+\frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'\right\}\\
&= -\Psi(t)\exp(pE(t))\frac{f(t)}{g(t)}|y|^{p^*}. \qquad (3.5)
\end{aligned}
$$

We define another Lyapunov function

$$
U(t,x,y) = V(t,x,y)\exp\left(-\int_0^t \Psi_-(s)ds\right).
$$

Then,

$$
\begin{aligned}
\left(e^{-p\alpha}|x|^p+\underline{\beta}|y|^{p^*}\right)\exp\left(-\int_0^\infty \Psi_-(t)dt\right) &\le V(t,x,y)\exp\left(-\int_0^\infty \Psi_-(t)dt\right)\\
&\le U(t,x,y),
\end{aligned}
$$

where α and $\underline{\beta}$ are the numbers given in assumptions (A_1) and (A_2). Hence, we see that

(a) $U(t,x,y)$ is positive definite and radially unbounded if and only if condition (3.4) is satisfied.

Moreover, by (3.5),

$$
\begin{aligned}
\dot{U}_{(1.1)}(t,x,y) &= \left\{\dot{V}_{(1.1)}(t,x,y)-\Psi_-(t)V(t,x,y)\right\}\exp\left(-\int_0^t \Psi_-(s)ds\right)\\
&= -\left\{\Psi_-(t)|x|^p+\Psi_+(t)\frac{f(t)}{g(t)}|y|^{p^*}\right\}\exp\left(pE(t)-\int_0^t \Psi_-(s)ds\right).
\end{aligned}
$$

If condition (3.4) holds, then $\Psi_-(t)$ tends to zero as $t\to\infty$. Hence,

(b) $\dot{U}_{(1.1)}(t,x,y)$ is not negative definite but it less than or equal to zero.

If the characteristic function $\Psi(t)$ is not less than a positive value c for all $t\ge 0$, then $\Psi_-(t)\equiv 0$ and $\Psi_+(t)=\Psi(t)\ge c$ for $t\ge 0$. In this case, we might be able to prove that the zero solution of (1.1) is only locally asymptotically stable by use of Matrosov's theorem. However, since

(c) $U(t,x,y)$ is not decrescent;

(d) $\Psi_+(t)$ is allowed to tend to zero as $t\to\infty$;

(e) $e(t)$ and $g(t)$ are not always assumed to be bounded (in addition, the boundedness of $h(t)$ is not assumed in Sect. 5),

it is hard to apply Matrosov's theorem to the proof of Theorem 3.2, in particular, the global attractivity of the zero solution of (1.1).

Before proving that the zero solution of (1.1) is globally attractive, we verify that the zero solution of (1.1) is stable and all solutions of (1.1) are bounded. As mentioned above, if assumptions (A_1), (A_2) and condition (3.4) are satisfied, then the Lyapunov function $U(t,x,y)$

is positive definite and radially unbounded and the total derivative $\dot{U}_{(1.1)}(t,x,y)$ along any solution of (1.1) is nonpositive. Hence, by means of theorems due to Lyapunov [40] and Yoshizawa [54], we conclude that the zero solution of (1.1) is stable and all solutions of (1.1) are bounded, respectively (refer also to Theorem 4.2 in [48, p. 13] or to Theorem 8.7 in [56, pp. 67–68]). To sum up, we obtain the following result.

Proposition 3.3 *Suppose that* (A_1) *and* (A_2) *hold. If* (3.4) *is satisfied, then the zero solution of* (1.1) *is stable and all solutions of* (1.1) *are bounded.*

If we assume that

$$|E(t)| \le \alpha \quad \text{for } t \ge 0, \tag{A_3}$$

instead of (A_1), then

$$\underline{\beta} \le \frac{f(t)}{g(t)} \le \overline{\beta} \quad \text{for } t \ge 0 \tag{A_4}$$

implies assumption (A_2). It also follows from (A_3) that

$$\exp((p-1)E(t)) \le e^{(p-1)\alpha} \quad \text{for } t \ge 0.$$

We therefore conclude that

$$g(t) \text{ has property } (P) \tag{3.6}$$

implies condition (3.1). Hence, we have the following result.

Corollary 3.4 *Let assumptions* (A_3) *and* (A_4) *hold and suppose that* $f(t)/g(t)$ *is differentiable for* $t \ge 0$. *If* (3.2)–(3.4) *and* (3.6) *are satisfied, then the zero solution of* (1.1) *is globally asymptotically stable.*

Note that assumptions (A_3) and (A_4) are stronger than assumptions (A_1) and (A_2). Using Lyapunov-type theorems with $U(t,x,y)$ above, we can easily prove that under the assumptions (A_3) and (A_4) with condition (3.4), the zero solution of (1.1) is uniformly stable and all solutions of (1.1) are uniformly bounded. For the definitions of uniform stability and uniform boundedness, see the books [48, 56].

4 Global attractivity

Let $(x(t),y(t))$ be a solution of (1.1) with the initial time $t_0 \ge 0$ and let

$$v(t) = V(t,x(t),y(t)),$$

where $V(t,x,y)$ is the Lyapunov function given in Sect. 3. Then, from the equality (3.5), it follows that

$$\begin{aligned} v'(t) &= \frac{d}{dt}V(t,x(t),y(t)) = \dot{V}_{(1.1)}(t,x,y)\Big|_{(x,y)=(x(t),y(t))} \\ &= -\Psi(t)\exp(pE(t))\frac{f(t)}{g(t)}|y(t)|^{p^*} \le \Psi_-(t)v(t) \end{aligned}$$

for $t \ge t_0$. Since $\Psi_-(t)$ satisfies condition (3.4), as in the proof of Lemma 5.2 in [53], we see that $v'(t)$ is absolutely integrable, and therefore, $v(t)$ has a limiting value $v_0 \ge 0$.

We are now able to demonstrate our main result stated in the first half of Sect. 3.

Proof of Theorem 3.2 By virtue of Proposition 3.3, we conclude that the zero solution of (1.1) is stable. Thus, we have only to prove that the zero solution of (1.1) is globally attractive; that is, every solution of (1.1) approaches the origin.

From assumptions (A_1) and (A_2) it turns out that

$$e^{-p\alpha}|x(t)|^p + \underline{\beta}|y(t)|^{p^*} \le v(t)$$

for $t \ge t_0$. Hence, if the limiting value v_0 of $v(t)$ is zero, then both $x(t)$ and $y(t)$ tend to zero as $t \to \infty$. This means that the solution $(x(t), y(t))$ approaches the origin $(0,0)$ as time t increases. This completes the proof. Hereafter, we consider only the case in which v_0 is positive and we show that this case cannot happen.

For the sake of convenience, let

$$u(t) = \exp(pE(t))\frac{f(t)}{g(t)}|y(t)|^{p^*}.$$

Then, using (A_2), again, we obtain

$$\underline{\beta}|y(t)|^{p^*} \le u(t) \le \overline{\beta}|y(t)|^{p^*} \tag{4.1}$$

for $t \ge t_0$. From Proposition 3.3, we see that $(x(t), y(t))$ is bounded for $t \ge t_0$. Hence, it follows from (4.1) that $u(t)$ has an inferior limit and a superior limit. First, we shall show that the inferior limit of $u(t)$ is zero, and we shall then show that the superior limit of $u(t)$ is also zero.

Suppose that $\liminf_{t\to\infty} u(t) > 0$. Then, there exist a $\lambda > 0$ and a $T_1 \ge t_0$ such that $u(t) > \lambda$ for $t \ge T_1$. Since

$$v'(t) = -\Psi(t)u(t) \quad \text{for } t \ge t_0, \tag{4.2}$$

it follows that

$$\int_{t_0}^{\infty}|v'(t)|dt = \int_{t_0}^{\infty}|\Psi(t)|u(t)dt \ge \int_{T_1}^{\infty}\Psi_+(t)u(t)dt > \lambda\int_{T_1}^{\infty}\Psi_+(t)dt.$$

Hence, from the fact that $v'(t)$ is absolutely integrable, it turns out that

$$\int_{T_1}^{\infty}\Psi_+(t)dt < \infty.$$

On the other hand, from (3.3), we see that

$$\int_{T_1}^{\infty}\Psi_+(t)dt = \infty.$$

This is a contradiction. We therefore conclude that $\liminf_{t\to\infty} u(t) = 0$.

Suppose that $\limsup_{t\to\infty} u(t) > 0$. Let $\upsilon = \limsup_{t\to\infty} u(t)$. From (A_2), it follows that

$$\exp(E(t))|f(t)| \le \frac{\overline{\beta}|g(t)|}{\exp((p-1)E(t))} \quad \text{for } t \ge 0.$$

Since $g(t)/\exp((p-1)E(t))$ has property (P), it is bounded for $t \ge 0$. Hence, there exists a $\gamma > 0$ satisfying

$$\exp(E(t))|f(t)| \le \gamma \quad \text{for } t \ge 0. \tag{4.3}$$

It also follows from (3.2) that there exists a $\overline{h} > 0$ with

$$|h(t)| \le \overline{h} \quad \text{for } t \ge 0. \tag{4.4}$$

Since $v(t)$ tends to a positive value v_0 as $t \to \infty$, there exists $T_2 \ge t_0$ such that

$$0 < \frac{1}{p} v_0 < v(t) < \frac{2p-1}{p} v_0 \quad \text{for } t \ge T_2. \tag{4.5}$$

Note that $2p-1 > 1$. By (3.1), we can choose positive numbers δ, ω, d and a positive sequence $\{t_m\}$ satisfying

$$\frac{|g(t)|}{\exp((p-1)E(t))} \ge \delta \quad \text{for } (m-1)\omega + t_m \le t \le (m-1)\omega + t_m + d.$$

Let ε be so small that

$$\delta \left(\frac{v_0 - p\varepsilon}{p} \right)^{1/p^*} > \left(\overline{h} + \frac{2}{(p-1)d} \right) \left(\frac{\varepsilon}{\underline{\beta}} \right)^{1/p^*}. \tag{4.6}$$

It is possible to find such a positive number ε, because the left-hand side approaches a positive number but the right-hand side approaches zero as $\varepsilon \to 0$. We may assume without loss of generality that $\varepsilon < \min\{\upsilon/2, v_0/p\}$.

Since $\liminf_{t\to\infty} u(t) = 0 < \upsilon = \limsup_{t\to\infty} u(t)$, we can select three divergent sequences τ_n, ρ_n and σ_n with $T_2 < \tau_n < \rho_n < \sigma_n < \tau_{n+1}$ such that $u(\tau_n) = u(\sigma_n) = \varepsilon$, $u(\rho_n) > 3\upsilon/4$,

$$u(t) \ge \varepsilon \quad \text{for } \tau_n < t < \sigma_n \tag{4.7}$$

and

$$u(t) \le \varepsilon \quad \text{for } \sigma_n < t < \tau_{n+1}. \tag{4.8}$$

Since $\varepsilon < \upsilon/2$ and $u(\rho_n) > 3\upsilon/4$, we also find two sequences t_n and s_n with $\tau_n < t_n < s_n < \rho_n$ such that $u(t_n) = \upsilon/2$, $u(s_n) = 3\upsilon/4$ and

$$\frac{1}{2}\upsilon < u(t) < \frac{3}{4}\upsilon \quad \text{for } t_n < t < s_n. \tag{4.9}$$

Needless to say, the intervals $[\tau_n, \sigma_n]$ and $[t_n, s_n]$ have the inclusion relation that $[t_n, s_n] \subset [\tau_n, \sigma_n]$. Taking into account that

$$\exp(pE(t))|x(t)|^p = v(t) - u(t)$$

and using (4.5) and (4.8), we obtain

$$\exp((p-1)E(t))|x(t)|^{p-1} > \left(\frac{v_0 - p\varepsilon}{p} \right)^{(p-1)/p} = \left(\frac{v_0 - p\varepsilon}{p} \right)^{1/p^*} > 0 \tag{4.10}$$

for $\sigma_n < t < \tau_{n+1}$. From (4.1) and (4.8), we see that

$$|y(t)| \le \left(\frac{u(t)}{\underline{\beta}} \right)^{1/p^*} \le \left(\frac{\varepsilon}{\underline{\beta}} \right)^{1/p^*} \quad \text{for } \sigma_n < t < \tau_{n+1}. \tag{4.11}$$

We shall show that the distance between intervals $[\tau_n, \sigma_n]$ and $[\tau_{n+1}, \sigma_{n+1}]$ does not unlimitedly grow as $n \to \infty$.

Claim. The sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfy $\tau_{n+1} - \sigma_n \leq 2\omega$ for $n \in \mathbb{N}$ sufficiently large.

Suppose that there exists a sufficiently large $n_0 \in \mathbb{N}$ such that $\tau_{n_0+1} - \sigma_{n_0} > 2\omega$. We can choose an $m_0 \in \mathbb{N}$ such that $(m_0-2)\omega < \sigma_{n_0} \leq (m_0-1)\omega$. Since

$$\tau_{n_0+1} > \sigma_{n_0} + 2\omega > (m_0-2)\omega + 2\omega = m_0\omega,$$

we see that $[(m_0-1)\omega, m_0\omega] \subset [\sigma_{n_0}, \tau_{n_0+1}]$. Let us turn our attention to the interval

$$[(m_0-1)\omega, m_0\omega].$$

Note that $|g(t)|/\exp((p-1)E(t)) \geq \delta$ for $(m_0-1)\omega + t_{m_0} \leq t \leq (m_0-1)\omega + t_{m_0} + d$. Hence, from (4.4), (4.10) and (4.11) and the second equation of (1.1), we can estimate that

$$\begin{aligned}
|y'(t)| &\geq (p-1)|g(t)||\phi_p(x(t))| - (p-1)|h(t)||y(t)| \\
&= (p-1)\left\{|g(t)||x(t)|^{p-1} - |h(t)||y(t)|\right\} \\
&\geq (p-1)\left\{\delta \exp((p-1)E(t))|x(t)|^{p-1} - \overline{h}\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*}\right\} \\
&> (p-1)\left\{\delta\left(\frac{v_0 - p\,\varepsilon}{p}\right)^{1/p^*} - \overline{h}\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*}\right\}
\end{aligned}$$

for $(m_0-1)\omega + t_{m_0} \leq t \leq (m_0-1)\omega + t_{m_0} + d$. From (4.6), we see that

$$|y'(t)| > \frac{2}{d}\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*} > 0$$

in this interval. Integrating this inequality, we obtain

$$\begin{aligned}
|y((m_0-1)\omega + t_{m_0} + d)| + |y((m_0-1)\omega + t_{m_0})| &\geq \left|\int_{(m_0-1)\omega+t_{m_0}}^{(m_0-1)\omega+t_{m_0}+d} y'(t)dt\right| \\
&= \int_{(m_0-1)\omega+t_{m_0}}^{(m_0-1)\omega+t_{m_0}+d} |y'(t)|dt \\
&> 2\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*},
\end{aligned}$$

which contradicts (4.11). Thus, the claim is proved.

Let $I = \displaystyle\bigcup_{n=1}^{\infty}[\tau_n, \sigma_n]$. Then, it follows from (4.2) and (4.7) that

$$\int_{t_0}^{\infty}|v'(t)|dt = \int_{t_0}^{\infty}|\Psi(t)|u(t)dt \geq \int_{t_0}^{\infty}\Psi_+(t)u(t)dt \geq \varepsilon\int_I \Psi_+(t)dt.$$

Since $v'(t)$ is absolutely integrable, it turns out that

$$\int_I \Psi_+(t)dt < \infty. \tag{4.12}$$

Now, suppose that there exists a $\delta > 0$ such that $\sigma_n - \tau_n > \delta$ for each $n \in \mathbb{N}$. By the Claim, there exists a $\Delta \geq 2\omega$ such that $\tau_{n+1} - \sigma_n \leq \Delta$ for any $n \in \mathbb{N}$. Hence, by (3.3),

$$\int_I \Psi_+(t)dt = \infty.$$

This contradicts (4.12). We therefore conclude that $\liminf_{n\to\infty}(\sigma_n - \tau_n) = 0$. Since $[t_n, s_n] \subset [\tau_n, \sigma_n]$, it follows that

$$\liminf_{n\to\infty}(s_n - t_n) = 0. \tag{4.13}$$

By (4.5) and (4.9), we obtain

$$0 \leq \exp(pE(t))|x(t)|^p = v(t) - u(t) < \frac{2p-1}{p}v_0 - \frac{1}{2}\upsilon,$$

namely,

$$\exp((p-1)E(t))|x(t)|^{p-1} < \left(\frac{2p-1}{p}v_0 - \frac{1}{2}\upsilon\right)^{1/p^*}$$

for $t_n \leq t \leq s_n$. Hence, together with (4.2) and (4.3) and the relations

$$\frac{d}{dz}|z|^p = p\phi_p(z) \quad \text{and} \quad z\phi_p(z) = |z|^p,$$

it turns out that

$$\begin{aligned}
u'(t) &= v'(t) - \big(\exp(pE(t))|x(t)|^p\big)' \\
&= -\Psi(t)u(t) - p\exp(pE(t))\big\{e(t)|x(t)|^p + \phi_p(x(t))x'(t)\big\} \\
&= -\Psi(t)u(t) - p\exp(pE(t))f(t)\phi_p(x(t))\phi_{p^*}(y(t)) \\
&\leq -\Psi(t)u(t) + p\exp(E(t))|f(t)|\exp((p-1)E(t))|x(t)|^{p-1}|y(t)|^{p^*-1} \\
&\leq -\Psi(t)u(t) + p\gamma\left(\frac{2p-1}{p}v_0 - \frac{1}{2}\upsilon\right)^{1/p^*}|y(t)|^{p^*-1}
\end{aligned}$$

for $t_n \leq t \leq s_n$. Noticing that $\Psi_-(t) + \Psi(t) = \Psi_+(t) \geq 0$ for $t \geq t_0$, we get

$$\begin{aligned}
\left(\exp\left(-\int_{t_0}^t \Psi_-(s)ds\right)u(t)\right)' &= \exp\left(-\int_{t_0}^t \Psi_-(s)ds\right)\big\{-\Psi_-(t)u(t) + u'(t)\big\} \\
&\leq \exp\left(-\int_{t_0}^t \Psi_-(s)ds\right)\bigg\{-\Psi_-(t)u(t) - \Psi(t)u(t) \\
&\qquad + p\gamma\left(\frac{2p-1}{p}v_0 - \frac{1}{2}\upsilon\right)^{1/p^*}|y(t)|^{p^*-1}\bigg\} \\
&\leq p\gamma\left(\frac{2p-1}{p}v_0 - \frac{1}{2}\upsilon\right)^{1/p^*}|y(t)|^{p^*-1}
\end{aligned}$$

for $t_n \leq t \leq s_n$. Since $(x(t), y(t))$ is bounded for $t \geq t_0$, there exists a $\mu > 0$ such that

$$\left(\exp\left(-\int_{t_0}^t \Psi_-(s)ds\right)u(t)\right)' < \mu \quad \text{for } t_n \leq t \leq s_n.$$

Integrate this inequality from t_n to s_n to obtain

$$\exp\left(-\int_{t_0}^{s_n}\Psi_-(t)dt\right)u(s_n)-\exp\left(-\int_{t_0}^{t_n}\Psi_-(t)dt\right)u(t_n)\le\mu(s_n-t_n).$$

Hence,

$$\begin{aligned}\exp\left(-\int_{t_n}^{\infty}\Psi_-(t)dt\right)u(s_n)-u(t_n)&\le\exp\left(-\int_{t_n}^{s_n}\Psi_-(t)dt\right)u(s_n)-u(t_n)\\&\le\mu\exp\left(\int_0^{t_n}\Psi_-(t)dt\right)(s_n-t_n)\\&\le\mu\exp\left(\int_0^{\infty}\Psi_-(t)dt\right)(s_n-t_n).\end{aligned}$$

Recall that $t_n\to\infty$, $u(t_n)=\upsilon/2$ and $u(s_n)=3\upsilon/4$. From (3.4), it follows that

$$\exp\left(-\int_{t_n}^{\infty}\Psi_-(t)dt\right)u(s_n)-u(t_n)\to\frac{\upsilon}{4}\quad\text{as }\ n\to\infty.$$

On the other hand, by (3.4) and (4.13),

$$\mu\exp\left(\int_0^{\infty}\Psi_-(t)dt\right)(s_n-t_n)\to 0\quad\text{as }\ n\to\infty.$$

This is a contradiction. Consequently, $\limsup_{t\to\infty}u(t)=\upsilon=0$.

As proved above, $u(t)$ tends to zero as $t\to\infty$. Hence, there exists a $T_3\ge t_0$ such that

$$u(t)<\varepsilon\quad\text{for }\ t\ge T_3,\tag{4.14}$$

where ε is a positive number given in (4.6). Let m_1 be an integer satisfying

$$(m_1-1)\omega>T_3.$$

Using (4.14) instead of (4.8) and following the same process as in the proof of the Claim, we can estimate that

$$2\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*}\ge|y((m_1-1)\omega+t_{m_1}+d)|+|y((m_1-1)\omega+t_{m_1})|\ge\left|\int_{(m_1-1)\omega+t_{m_1}}^{(m_1-1)\omega+t_{m_1}+d}y'(t)dt\right|.$$

Since

$$|y'(t)|>\frac{2}{d}\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*}>0$$

for $(m_1-1)\omega+t_{m_1}\le t\le(m_1-1)\omega+t_{m_1}+d$, it follows that

$$2\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*}\ge\left|\int_{(m_1-1)\omega+t_{m_1}}^{(m_1-1)\omega+t_{m_1}+d}y'(t)dt\right|=\int_{(m_1-1)\omega+t_{m_1}}^{(m_1-1)\omega+t_{m_1}+d}|y'(t)|dt>2\left(\frac{\varepsilon}{\underline{\beta}}\right)^{1/p^*}.$$

This is a contradiction. Thus, the case of $v_0>0$ does not occur.

The proof of Theorem 3.2 is thus complete. □

5 Generalization

In Theorem 3.2, the coefficient $h(t)$ is assumed to be bounded for all $t \geq 0$. We can somewhat weaken this assumption. In fact, if there exists a $\overline{h} > 0$ such that

$$|h(t)| \leq \overline{h} \quad \text{for} \;\; (m-1)\omega + t_m \leq t \leq (m-1)\omega + t_m + d, \tag{5.1}$$

then we can proceed with the same argument as in the proof of Theorem 3.2. We only have to use assumption (5.1) instead of inequality (4.4) (we entrust the detailed proof to the reader). Hence, we have the following generalization of Theorem 3.2.

Theorem 5.1 *Let assumptions* (A_1) *and* (A_2) *hold and suppose that* $f(t)/g(t)$ *is differentiable for* $t \geq 0$. *If conditions* (3.1), (3.3), (3.4) *and* (5.1) *are satisfied, then the zero solution of* (1.1) *is globally asymptotically stable.*

Condition (5.1) implies that $\liminf_{t\to\infty} |h(t)| < \infty$. Of course, it is permitted that

$$\limsup_{t\to\infty} |h(t)| = \infty.$$

However, under the assumptions (A_1) and (A_2) and conditions (3.1), (3.3) and (3.4), if

$$\lim_{t\to\infty} |h(t)| = \infty,$$

then the zero solution of (1.1) is not always globally asymptotically stable.

To see this fact, the following equation is often cited:

$$x'' + (2 + e^t)x' + x = 0.$$

For example, see the paper [5, 26] and the books [20, p. 326] and [48, pp. 39–41]. This equation can be transformed into the system

$$\begin{aligned} x' &= y, \\ y' &= -x - (2 + e^t)y, \end{aligned}$$

which has a nontrivial solution $(x(t), y(t)) = (c(1 + e^{-t}), -ce^{-t})$ for each constant $c \neq 0$. Note that the solution $(x(t), y(t))$ approaches a point other than the origin. This phenomenon is called ‘overdamping’. The phenomenon of overdamping is caused because of a rapid increase of the damping coefficient.

To decide the limit of the damping coefficient in which this phenomenon is not caused, Hatvani *et al*. [27] have considered the damped linear oscillator

$$x'' + h(t)x' + k^2x = 0,$$

where k is a positive constant. They presented a necessary and sufficient condition for the equilibrium to be (globally) asymptotically stable. This is a so-called growth condition on $h(t)$ and it can be checked with comparative ease (see also [26]).

In the next section, we will confirm that the phenomenon of overdamping occurs even by the half-linear differential system (1.1).

6 Examples

To illustrate our theorems, we give some examples. As mentioned in Sect. 3, it is easy to deal with the case that $\Psi(t)$ is not less than a positive value c for all $t \geq 0$. For this reason, we cite other cases. In the first example, $\Psi(t)$ is allowed to be zero at a time sequence $\{t_n\}$, but it is positive otherwise.

Example 6.1 Consider system (1.1) with

$$e(t) = \frac{1}{1+t}, \quad f(t) = \frac{\sin t}{1+t}, \quad g(t) = (1+t)^{p-1}\left(\sin t + \frac{1}{2}\sin 3t\right) \quad \text{and} \quad h(t) = \frac{4\sqrt{5}}{5p},$$

where $p > 1$. Then the zero solution is globally asymptotically stable.

Since $E(t) = \log(1+t)$ and

$$\begin{aligned}\exp(pE(t))\frac{f(t)}{g(t)} &= e^{p\log(1+t)}\frac{\sin t}{(1+t)^p(\sin t + \sin 3t/2)} \\ &= \frac{\sin t}{\sin t + (3\sin t - 4\sin^3 t)/2} = \frac{2}{5-4\sin^2 t}\end{aligned}$$

for $t \geq 0$, assumption (A_1) is satisfied with $\alpha = 1$ and assumption (A_2) is satisfied with $\underline{\beta} = 2/5$ and $\overline{\beta} = 2$, respectively. Condition (3.1) is also satisfied, because

$$\frac{g(t)}{\exp((p-1)E(t))} = \frac{5}{2}\sin t - 2\sin^3 t$$

which is a periodic function with period 2π. To be precise, $g(t)/\exp((p-1)E(t))$ has property (P) with $\delta = 1/2$, $\omega = 2\pi$, $d = 5\pi/6$ and $t_m = \pi/12$ for all $m \in \mathbb{N}$. It is clear that condition (3.2) holds. To confirm conditions (3.3) and (3.4), we examine the increase and decrease of the periodic function $k(t) = \sin 2t/(5-4\sin^2 t)$. The period of $k(t)$ is π and

$$\begin{aligned}k'(t) &= \frac{1}{(5-4\sin^2 t)^2}\left\{2\cos 2t\,(5-4\sin^2 t) + 8\sin 2t\sin t\cos t\right\} \\ &= \frac{2}{(5-4\sin^2 t)^2}\left\{(2\cos^2 t - 1)(1+4\cos^2 t) + 8(1-\cos^2 t)\cos^2 t\right\} \\ &= \frac{2(6\cos^2 t - 1)}{(5-4\sin^2 t)^2}\end{aligned}$$

for $t \geq 0$. Hence, $k(t)$ is increasing for $t \in [0,t^*] \cup [\pi - t^*, \pi]$ and it is decreasing for $t \in [t^*, \pi - t^*]$, where $t^* = \arccos 1/\sqrt{6}$, and therefore, $k(t)$ has the maximum value $1/\sqrt{5}$ at $t = \pi(n-1) + t^*$ and the minimum value $-1/\sqrt{5}$ at $t = \pi n - t^*$ with $n \in \mathbb{N}$. We obtain

$$\begin{aligned}\Psi(t) &= ph(t) - pe(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)' \\ &= \frac{4\sqrt{5}}{5} - \frac{p}{1+t} + \frac{p(1+t)^{p-1}(5-4\sin^2 t) - 8(1+t)^p\sin t\cos t}{(1+t)^p(5-4\sin^2 t)} \\ &= \frac{4\sqrt{5}}{5} - \frac{p}{1+t} + \frac{p}{1+t} - \frac{4\sin 2t}{5-4\sin^2 t} = \frac{4\sqrt{5}}{5} - 4k(t),\end{aligned}$$

which is nonnegative and periodic. Hence, it turns out that $\Psi_+(t) = \Psi(t)$ and $\Psi_-(t) \equiv 0$, and conditions (3.3) and (3.4) hold. Thus, by means of Theorem 3.2, we conclude that the zero solution is globally asymptotically stable.

In Example 6.1, since the characteristic function $\Psi(t)$ is a nonnegative periodic function with period π, it fails to have a limit. In the next example, we consider the case that $\lim_{t\to\infty}\Psi(t) = 0$. This case is harder to deal with.

Example 6.2 Consider system (1.1) with

$$e(t) = \sin t, \quad f(t) = \frac{1+t}{2+t}\sin t, \quad g(t) = \frac{2+t}{1+t}\sin t \quad \text{and} \quad h(t) = \frac{1}{1+t} + \sin t.$$

Then the zero solution is globally asymptotically stable.

Since $E(t) = 1-\cos t$, assumption (A_3) is satisfied with $\alpha = 2$. Assumption (A_4) is also satisfied with $\underline{\beta} = 1/4$ and $\overline{\beta} = 1$, because

$$\frac{f(t)}{g(t)} = \left(1 - \frac{1}{2+t}\right)^2 \nearrow 1 \quad \text{as } t \to \infty.$$

Note that $g(t)$ is asymptotically periodic and $|g(t)| \ge |\sin t|$ for $t \ge 0$. Hence, $g(t)$ has property (P) with $\delta = 1/2$, $\omega = 2\pi$, $d = 2\pi/3$ and $t_m = \pi/6$ for all $m \in \mathbb{N}$; that is, condition (3.6) is satisfied. It is obvious that condition (3.2) holds. Since

$$\begin{aligned}\Psi(t) &= ph(t) - pe(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)' \\ &= \frac{p}{1+t} - \frac{2}{(1+t)(2+t)} \\ &= \frac{1}{1+t}\left(p - \frac{2}{2+t}\right) \ge \frac{p-1}{1+t}\end{aligned}$$

for $t \ge 0$, it follows that $\Psi_+(t) = \Psi(t) \ge (p-1)/(1+t)$ and $\Psi_-(t) \equiv 0$. Hence, conditions (3.3) and (3.4) are satisfied. Thus, by virtue of Corollary 3.4, we conclude that the zero solution is globally asymptotically stable.

In Examples 6.1 and 6.2, the coefficient $h(t)$ is bounded for all $t \ge 0$. We next consider the case that $h(t)$ is unbounded.

Example 6.3 Consider system (1.1) with

$$e(t) = -\frac{1}{(1+t)^2}, \quad f(t) = g(t) = \sin t \quad \text{and} \quad h(t) = \frac{1}{1+t} + t\,(|\sin t| - \sin t).$$

Then the zero solution is globally asymptotically stable.

It is clear that assumption (A_3) is satisfied with $\alpha = 1$ and assumption (A_4) is satisfied with $\underline{\beta} = \overline{\beta} = 1$. It is also clear that $g(t)$ has property (P) with $\delta = 1/2$, $\omega = 2\pi$, $d = 2\pi/3$ and $t_m = \pi/6$ for all $m \in \mathbb{N}$; that is, condition (3.6) is satisfied. Recall that assumption (A_3)

implies assumption (A_1); under the assumption (A_3), assumption (A_4) implies assumption (A_2) and condition (3.6) implies condition (3.1). Moreover, we obtain

$$\begin{aligned}\Psi(t) &= ph(t) - pe(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)' \\ &= \frac{p}{1+t} + \frac{p}{(1+t)^2} + pt\,(|\sin t| - \sin t) \geq \frac{p}{1+t}\end{aligned}$$

for $t \geq 0$. Hence, it turns out that conditions (3.3) and (3.4) hold. Note that

$$|h(t)| \leq \frac{1}{1+t} \leq 1 \quad \text{for } 2\pi(m-1) \leq t \leq \pi(2m-1).$$

Since

$$(m-1)\omega + t_m = 2\pi(m-1) + \frac{1}{6}\pi \geq 2\pi(m-1)$$

and

$$(m-1)\omega + t_m + d = 2\pi(m-1) + \frac{5}{6}\pi \leq \pi(2m-1),$$

condition (5.1) is satisfied with $\bar{h} = 1$. Thus, all of the assumptions in Theorem 5.1 can be confirmed, and therefore, the zero solution is globally asymptotically stable.

Although the coefficient $e(t)$ is bounded in Examples 6.1–6.3, the boundedness of $e(t)$ is not essential for global asymptotic stability of the zero solution. For example, let

$$e(t) = t\,(|\sin t| - \sin t).$$

Then it is clear that $e(t)$ is unbounded. By a straightforward calculation, we obtain

$$E(t) = \begin{cases} 2\pi(m-1)(2m-1) & \text{if } 2\pi(m-1) \leq t \leq \pi(2m-1) \\ 2\big(\pi m(2m-1) - \sin t + t\cos t\big) & \text{if } \pi(2m-1) \leq t \leq 2\pi m \end{cases} \tag{6.1}$$

with $m \in \mathbb{N}$. Let

$$f(t) = \frac{\sin t}{\exp(pE(t))}, \quad g(t) = \exp((p-1)E(t))\sin t \quad \text{and} \quad h(t) = \frac{1}{1+t},$$

where $E(t)$ is the function given in (6.1). Then,

$$\begin{aligned}\Psi(t) &= ph(t) - pe(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)' \\ &= ph(t) - pe(t) + pe(t) = \frac{p}{1+t}\end{aligned}$$

for $t \geq 0$. It is easy to confirm that all of the assumptions in Theorem 3.2 is satisfied. Thus, the zero solution is globally asymptotically stable.

As mentioned in the last paragraph in Sect. 5, we cannot drop condition (5.1) in Theorem 5.1. If we remove condition (5.1) from Theorem 5.1, then the phenomenon of overdamping may well happen. For example, consider the half-linear differential system

$$\begin{aligned}x' &= \phi_{p^*}(y), \\ y' &= -(p-1)\phi_p(x) - (p-1)\left((2+t)(1+t)^{2p-3} + \frac{2}{1+t}\right)y.\end{aligned} \tag{6.2}$$

System (6.2) has a solution

$$(x(t),y(t)) = \left(\frac{2+t}{1+t}, -\frac{1}{(1+t)^{2(p-1)}}\right), \tag{6.3}$$

which satisfies the initial condition $(x(0),y(0)) = (2,-1)$. In fact, since

$$e(t) = 0, \quad f(t) = g(t) = 1 \quad \text{and} \quad h(t) = (2+t)(1+t)^{2p-3} + \frac{2}{1+t} \tag{6.4}$$

in system (6.2), we obtain

$$-e(t)x(t) + f(t)\phi_{p^*}(y(t)) = -\frac{1}{(1+t)^2} = x'(t)$$

and

$$\begin{aligned} &-(p-1)g(t)\phi_p(x(t)) - (p-1)h(t)y(t) \\ &\quad = -(p-1)\frac{2+t}{1+t} + \frac{p-1}{(1+t)^{2(p-1)}}\left\{(2+t)(1+t)^{2p-3} + \frac{2}{1+t}\right\} \\ &\quad = \frac{2(p-1)}{(1+t)^{2p-1}} = y'(t). \end{aligned}$$

The solution $(x(t),y(t))$ given by (6.3) approaches the point $(1,0)$ as $t \to \infty$. Hence, the zero solution of (6.2) is not globally asymptotically stable.

From (6.4), it is clear that assumptions (A_1) and (A_2) and condition (3.1) are satisfied. Since $\Psi(t) = 2h(t)$ for $t \geq 0$, it is easy to verify that conditions (3.3) and (3.4) are also satisfied. However, condition (5.1) does not hold, because $\lim_{t\to\infty} |h(t)| = \infty$. Thus, all of the assumptions in Theorem 5.1 are satisfied except that $h(t)$ satisfies condition (5.1). This means that condition (5.1) cannot be removed from Theorem 5.1.

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