Simple conditions for parametrically excited oscillation of generalized Mathieu equations

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Abstract

The following equation is considered in this paper:

$$x'' + (-\alpha + \beta\cos(\gamma t))x = 0,$$

where α, β and γ are real parameters and $\gamma > 0$. This equation is referred to as Mathieu's equation when $\gamma = 2$. It is determined by the parameters whether all solutions of this equation are oscillatory or nonoscillatory. Our results provide parametric conditions for oscillation and nonoscillation. There is a feature in which it is very easy to check whether these conditions are satisfied or not. Parametric oscillation region and nonoscillation regions are drawn to help understand the obtained results.

Key words: Mathieu's equation; Oscillation problem; Damped linear differential equations; Variable transformation; Parametric excitation

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1. Introduction

We consider the second-order differential equation

$$x'' + (-\alpha + \beta\cos(\gamma t))x = 0, \tag{1.1}$$

where the prime denotes d/dt; the parameters α, β and γ are real numbers; $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma > 0$. A phenomenon whose amplitude is magnified by varying some parameters is called a *parametric excitation*. Equation (1.1) is a mathematical approximation model to describe the parametric excitation. As a familiar example of the parametric excitation, we can mention children's swing play. When children pump a swing, they move the center of gravity by periodically standing and squatting on the seat of the swing. Movement of the center of gravity amplifies the width of the swing's oscillation. This movement can be considered to cause the parameter variation.

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As a research of parametric excitation phenomenon, Mathieu [18] has studied the vibration of the oval type drum film and derived the special case of (1.1) that $\gamma = 2$; namely,

$$x'' + (-\alpha + \beta\cos(2t))x = 0. \tag{1.2}$$

This case is named Mathieu's equation after him. Mathieu's equation has been applied to many problems in physics and natural sciences. For example, by the transformations from rectangular coordinates (x, y) to elliptic coordinates (ξ, η):

$$x = c\cosh\xi\cos\eta \quad \text{and} \quad y = c\sinh\xi\sin\eta,$$

the two-dimensional Helmholtz equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0$$

becomes

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{c^2k^2}{2}\big(\cosh(2\xi) - \cos(2\eta)\big)V = 0,$$

where $V(\xi, \eta) = U(x, y)$. Putting $V(\xi, \eta) = R(\xi)\Phi(\eta)$, we obtain the Mathieu equation

$$\frac{d^2\Phi}{d\eta^2} + (a - 2q\cos(2\eta))\Phi = 0$$

and the modified Mathieu equation

$$\frac{d^2 R}{d\xi^2} - (a - 2q\cosh(2\xi))R = 0,$$

where a is the separation constant and the parameter $q = c^2k^2/4$. Note that the modified Mathieu equation can be transformed to the Mathieu equation by the mapping $\eta = \pm\sqrt{-1}\,\xi$.

Mathieu's equation is a linearized model of an inverted pendulum where the pivot point oscillates periodically in the vertical direction (see [20]). It is also derived in the study of celestial mechanics (see [3, 4]) and in the vibration of the string whose tension is changed periodically (Melde's experiment). In fluid dynamics, we can find many examples of waves being described by Mathieu's equation. The research of Faraday surface waves is very active (see [2, 6, 9, 21]). About other applications of Mathieu's equation, see McLachan [19].

The purpose of this paper is to give a simple parametric region which guarantees that all nontrivial solutions of the generalized Mathieu equation (1.1) are oscillatory (resp., nonoscillatory) (see Section 2 for the definitions).

Using the method mentioned in the book of McLachan [19, p. 29], we can determine the boundary of the largest parametric oscillation region for Mathieu's equation (1.2). The boundary is described by the infinite continued fraction

$$\alpha = \frac{\beta^2}{2(4+\alpha)-}\,\frac{\beta^2}{2(16+\alpha)-}\,\frac{\beta^2}{2(36+\alpha)-}\,\frac{\beta^2}{2(64+\alpha)-}\cdots\frac{\beta^2}{2(4n^2+\alpha)-}\cdots, \tag{1.3}$$

where $n = 1, 2, \ldots$. To be precise, $\beta^2(\alpha)$ is the smallest positive root of the equation

$$\alpha = \frac{\lambda}{2(4+\alpha)-}\,\frac{\lambda}{2(16+\alpha)-}\,\frac{\lambda}{2(36+\alpha)-}\,\frac{\lambda}{2(64+\alpha)-}\cdots\frac{\lambda}{2(4n^2+\alpha)-}\cdots$$

corresponding to a given value $\alpha > 0$. Let α_0 be any positive number. Then, all nontrivial solutions of (1.2) are oscillatory if $\alpha = \alpha_0$ and $\beta > |\beta(\alpha_0)|$, and those are nonoscillatory if $\alpha = \alpha_0$ and $\beta \le |\beta(\alpha_0)|$. Unfortunately, since the right-hand side of (1.3) has a form of an infinite continued fraction, it is very hard to calculate the exact value $\beta(\alpha_0)$. Let $\beta_n^2(\alpha_0)$ be the the smallest positive root of the equation

$$\alpha_0 = \frac{\lambda}{2(4+\alpha_0)-}\,\frac{\lambda}{2(16+\alpha_0)-}\,\frac{\lambda}{2(36+\alpha_0)-}\,\frac{\lambda}{2(64+\alpha_0)-}\cdots\frac{\lambda}{2(4n^2+\alpha_0)}.$$

Note that the above-mentioned equation is described by using a finite continued fraction. Even if we can find the correct value $\beta_n^2(\alpha_0)$, it is is only an upper approximation value. In other words, $\beta_n^2(\alpha_0)$ approaches the limiting value $\beta^2(\alpha_0)$ from above as $n \to \infty$. Hence, all nontrivial solutions of (1.2) are not always nonoscillatory in the case that $\alpha = \alpha_0$ and $\beta = \beta_n(\alpha_0)$.

In order to avoid such an inconvenience of (1.3), we will provide an easy-to-use parametric condition for oscillation (resp., nonoscillation) of (1.1).

Equations (1.1) is a simple example of the more general Hill's differential equation

$$x'' + c(t)x = 0, \tag{1.4}$$

where c is a periodic function with period $T > 0$ (refer to [11, 17, 19]). The oscillation problem for equation (1.4) has been widely studied in many books (for example, see [7, 25]). It is well-known that if $c(t) \le 0$ for t sufficiently large, then all nontrivial solutions of (1.4) are nonoscillatory even if c is not a periodic function (see [25, p. 45]). Hence, it is clear that if $\alpha \ge |\beta|$, then all nontrivial solutions of (1.1) are nonoscillatory. If the periodic function c is of *mean value zero*; that is, c is not identically zero and

$$\int^T c(t)dt = 0,$$

then all nontrivial solutions of (1.4) are oscillatory (for the proof, see [7, p. 25]). Hence, if $\alpha = 0$ and $\beta \neq 0$, then the coefficient of (1.1) is periodic of mean value zero, and therefore, all nontrivial solutions of (1.1) are oscillatory. From Sturm's comparison theorem, we see that if $\alpha < 0$ and $\beta \neq 0$, then all nontrivial solutions of (1.1) are oscillatory. It is obvious that if $\alpha < 0$ and $\beta = 0$, then all nontrivial solutions of (1.1) are oscillatory. Thus, we should consider only the case that $0 < \alpha < |\beta|$. Our results are as follows.

Theorem 1.1. *If*

$$\alpha > 0 \quad \textit{and} \quad |\beta| \ge \gamma\sqrt{2\alpha} + \alpha, \tag{1.5}$$

then all nontrivial solutions of (1.1) *are oscillatory.*

Theorem 1.2. *If*

$$\alpha > 0 \quad \textit{and} \quad |\beta| \leq \frac{\gamma\sqrt{2\alpha}}{2} + \alpha, \tag{1.6}$$

then all nontrivial solutions of (1.1) *are nonoscillatory*.

It is very easy to calculate a value of β satisfying conditions (1.5) and (1.6) for a given value $\alpha > 0$. Conversely, it is also not difficult to obtain a positive value of α satisfying conditions (1.5) and (1.6) for a given value $\beta \in \mathbb{R}$. We can get it by hand calculation, because only solving a quadratic equation about α. A parametric condition for oscillation of (1.2) has been already given by El-Sayed [8]. Theorem 1.1 includes his result (see Section 4 for the details). On the other hand, a parametric condition for nonoscillation of (1.1) has not been reported until now.

Let

$$\tilde{\gamma}s = \gamma t + \frac{\pi}{2} \quad \text{and} \quad y(s) = x(t).$$

Then equation (1.1) can be rewritten as follows:

$$\frac{d^2z}{ds^2} + (-\tilde{\alpha} + \tilde{\beta}\sin(\tilde{\gamma}s))z = 0, \tag{1.7}$$

where $\tilde{\alpha} = \alpha(\tilde{\gamma}/\gamma)^2$ and $\tilde{\beta} = \beta(\tilde{\gamma}/\gamma)^2$. From this variable transformation, we see that all nontrivial solutions of (1.1) are oscillatory if and only if those of (1.7) are oscillatory.

Sun et al. [24] applied their result to a forced Mathieu equation, and obtained a condition which guarantees that all nontrivial solutions are oscillatory. Their condition has the advantage that it can be applied to the forced case. Unfortunately, however, their condition does not improve El-Sayed's result in the unforced case (1.7). In Section 4, we will mention the relation between the result of El-Sayed (and our condition (1.5)) and that of Sun et al. In addition, using our result which is equivalent to Theorem 1.2, we disprove mathematically a conjecture given by Sun et al. [24].

Leighton [16] considered equation (1.7) with $\tilde{\gamma} = 1$ and presented an oscillation criteria which has a close relation with the infinite continued fraction (1.3) (refer also to [15]). We compare our results with that of Leighton in Section 4.

In Section 5, we expand Theorem 1.2 in order to be able to apply to a Mathieu equation with a non-periodic coefficient.

2. Linear differential equations with periodic coefficients

We consider the second-order differential equation

$$y'' + a(t)y' + b(t)y = 0, \tag{2.1}$$

where a, $b\colon [0,\infty) \to \mathbb{R}$ are continuous and periodic functions with period $T > 0$. As is well known, all solutions of (2.1) exist in the future. Hence, it is worthwhile to discuss whether solutions of (2.1) are oscillatory or not. A nontrivial solution $y(t)$ of (2.1) is said to be *oscillatory* if it has an infinite number of zeros on $0 < t < \infty$. Otherwise, the solution is said to be *nonoscillatory*. This means that if x is a nonoscillatory solution of (2.1), then it is eventually positive or eventually negative.

As one of very important models, equation (2.1) appears in a wide range of field which covers pure science, applied science, and technology. For this reason, numerous papers have been devoted to find some conditions which guarantee that all nontrivial solutions of (2.1) (and more general nonlinear equations) are oscillatory (resp., nonoscillatory). For example, see [1, 5, 10, 27, 28, 29, 30, 31] and the references cited therein. These results are called "oscillation theorem" and "nonoscillation theorem", respectively.

We can cite the following result which were given in [23] as an oscillation theorem that has a close relation to this paper.

Theorem A. *Suppose that a is periodic of mean value zero. Let B be an indefinite integral of b and let*

$$E(t) = \exp\int_0^t (a(\tau) - 2B(\tau))d\tau.$$

If B is periodic of mean value zero and satisfies

$$\int_0^T E(t)(B(t) - a(t))B(t)dt > 0, \tag{2.2}$$

then all nontrivial solutions of (2.1) *are oscillatory.*

Remark 2.1. In Theorem A, the integral E is a periodic function with period T, because a and B are periodic of mean value zero.

Kwong and Wong [14] have already given a nonoscillation counterpart of Theorem A as follows.

Theorem B. *Suppose that b is periodic of mean value zero. If there exists an indefinite B of b such that*

$$(B(t) - a(t))B(t) \le 0 \quad \textit{for } 0 \le t \le T, \tag{2.3}$$

then all nontrivial solutions of (2.1) *are nonoscillatory.*

Theorems A and B are available only for second-order linear differential equations with periodic coefficients. These theorems do not give us any information about oscillation or nonoscillation even if the coefficient $c(t)$ of (1.4), or the coefficient $a(t)$ or $b(t)$ of (2.1) is almost periodic or quasi-periodic. Theorem C below is applicable even to a Mathieu equation with a non-periodic coefficient (see [22] for the proof).

Theorem C. *Let S be a bounded, closed and convex set in the region*

$$R = \left\{(u,v) : u \ge 0 \ \textit{ and } \ 0 \le v \le u^2/4\right\}.$$

Suppose that

$$(a(t), b(t)) \in S \quad \textit{for } t \ge T, \tag{2.4}$$

with T sufficiently large. Then all nontrivial solutions of (2.1) *are nonoscillatory.*

3. Proof of the main theorems

Using Theorems A and B, we will prove our main theorems which were presented in Section 1 (Theorem C will be used in Section 5). To this end, we consider equation (2.1) with

$$a(t) = 2\sqrt{2\alpha}\,\sin\left(\frac{\gamma}{2}t\right) \quad \text{and} \quad b(t) = \frac{\gamma\sqrt{2\alpha}}{2}\cos\left(\frac{\gamma}{2}t\right) + (\beta-\alpha)\cos(\gamma t). \tag{3.1}$$

Define

$$x = y\exp\left(\frac{1}{2}\int_0^t a(\tau)d\tau\right).$$

Since

$$\begin{aligned}\frac{1}{4}a^2(t) + \frac{1}{2}a'(t) - \alpha + \beta\cos(\gamma t) &= 2\alpha\sin^2\left(\frac{\gamma}{2}t\right) + \frac{\gamma\sqrt{2\alpha}}{2}\cos\left(\frac{\gamma}{2}t\right) - \alpha + \beta\cos(\gamma t)\\ &= -\alpha\cos(\gamma t) + \frac{\gamma\sqrt{2\alpha}}{2}\cos\left(\frac{\gamma}{2}t\right) + \beta\cos(\gamma t) = b(t),\end{aligned}$$

we see that

$$\begin{aligned}x'' + (-\alpha+\beta\cos(\gamma t))x &= \left(y'' + a(t)y' + \left(\frac{1}{4}a^2(t) + \frac{1}{2}a'(t) - \alpha + \beta\cos(\gamma t)\right)y\right)\\ &\quad\times\exp\left(\frac{1}{2}\int_0^t a(\tau)d\tau\right)\\ &= \left(y'' + a(t)y' + b(t)y\right)\exp\left(\frac{1}{2}\int_0^t a(\tau)d\tau\right).\end{aligned}$$

This means that all nontrivial solutions of (1.1) are oscillatory if and only if those of (2.1) are oscillatory under the assumption (3.1).

Proof of Theorem 1.1. Let

$$s = t - \pi/\gamma \quad \text{and} \quad z(s) = x(t).$$

Then, this variable transformation changes equation (1.1) to

$$\frac{d^2z}{ds^2} + (-\alpha - \beta\cos(\gamma s))z = 0$$

which has the same form as equation (1.1). Hence, we have only to cope with the case that $\beta \ge 0$.

As an indefinite integral of b, we choose B defined by

$$B(t) = \sqrt{2\alpha}\,\sin\left(\frac{\gamma}{2}t\right) + \frac{\beta-\alpha}{\gamma}\sin(\gamma t).$$

It is clear that a and B are periodic of mean value zero. Those periods are $4\pi/\gamma$. Since

$$a(t) - 2B(t) = -\frac{2(\beta - \alpha)}{\gamma}\sin(\gamma t),$$

we have

$$E(t) = \exp\int_0^t (a(\tau) - 2B(\tau))d\tau = \exp\left(\frac{2(\beta-\alpha)}{\gamma^2}(\cos(\gamma t) - 1)\right).$$

From (1.5) it follows that $\beta - \alpha \geq \gamma\sqrt{2\alpha} > 0$. Hence, we see that $E(0) = E(T/2) = E(T) = 1$ and $E(T/4) = E(3T/4) = \exp(-4(\beta-\alpha)/\gamma^2)$, where $T = 4\pi/\gamma$; and E is strictly decreasing on the intervals $[0, T/4]$ and $[T/2, 3T/4]$, and strictly increasing on the intervals $[T/4, T/2]$ and $[3T/4, T]$; that is, the maximum and minimum values of E are 1 and $\exp(-4(\beta-\alpha)/\gamma^2)$, respectively. Also, we see that the graph of E is line-symmetrical with respect to the vertical lines $t = T/4$, $t = T/2$ and $t = 3T/4$. Moreover, we have

$$\begin{aligned}(B(t) - a(t))B(t) &= \frac{(\beta-\alpha)^2}{\gamma^2}\sin^2(\gamma t) - 2\alpha\sin^2\left(\frac{\gamma}{2}t\right)\\ &= \frac{4(\beta-\alpha)^2}{\gamma^2}\sin^2\left(\frac{\gamma}{2}t\right)\left(\cos^2\left(\frac{\gamma}{2}t\right) - \frac{\alpha\gamma^2}{2(\beta-\alpha)^2}\right).\end{aligned}$$

Since $\beta - \alpha \geq \gamma\sqrt{2\alpha}$, it follows that

$$0 < \frac{\alpha\gamma^2}{2(\beta-\alpha)^2} \leq \frac{1}{4}.$$

Hence, the function $(B-a)B$ becomes zero at 0, t_1, t_2, $T/2$, t_3, t_4 and T, where

$$\begin{aligned}\frac{T}{6} &< t_1 = \mathrm{Cos}^{-1}\frac{\gamma\sqrt{\alpha}}{\sqrt{2}(\beta-\alpha)} < \frac{T}{4},\\ \frac{T}{4} &< t_2 = \mathrm{Cos}^{-1}\frac{-\gamma\sqrt{\alpha}}{\sqrt{2}(\beta-\alpha)} < \frac{T}{3},\end{aligned}$$

$t_3 = t_1 + T/2$ and $t_4 = b + T/2$. It is easy to check that $(B-a)B$ is positive on the intervals $(0, t_1)$, $(t_2, T/2)$, $(T/2, t_3)$ and (t_4, T), and negative on the intervals (t_1, t_2) and (t_3, t_4); and the graph of $(B-a)B$ is line-symmetrical with respect to the vertical lines $t = T/4$, $t = T/2$ and $t = 3T/4$.

Note that $T/4 - t_1 = t_2 - T/4$ and $3T/4 - t_3 = t_4 - 3T/4$. As mentioned above, the graph of E has the axial symmetry. Hence, it turns out that $E(t_1) = E(t_2) = E(t_3) = E(t_4)$. For the sake of simplicity, let $\delta = E(t_1)$. From the property of E, we see that $E(t) > \delta$ for $0 < t < t_1$, $t_2 < t < T/2$, $T/2 < t < t_3$ and $t_4 < t < T$, and $0 < \exp(-4(\beta-\alpha)/\gamma^2) \leq$

$E(t) < \delta$ for $t_1 < t < t_2$ and $t_3 < t < t_4$. We therefore conclude that

$$\begin{aligned}
\int_0^T E(t)(B(t)-a(t))B(t)dt > \delta\int_0^{t_1}(B(t)-a(t))B(t)dt + \delta\int_{t_1}^{t_2}(B(t)-a(t))B(t)dt \\
+\delta\int_{t_2}^{T/2}(B(t)-a(t))B(t)dt + \delta\int_{T/2}^{t_3}(B(t)-a(t))B(t)dt \\
+\delta\int_{t_3}^{t_4}(B(t)-a(t))B(t)dt + \delta\int_{t_4}^{T}(B(t)-a(t))B(t)dt \\
= \delta\int_0^T(B(t)-a(t))B(t)dt \\
= \frac{(\beta-\alpha)^2\delta}{\gamma^2}\int_0^T \sin^2(\gamma t)dt - 2\alpha\delta\int_0^T \sin^2\left(\frac{\gamma}{2}t\right)dt \\
= \frac{2\delta\pi}{\gamma}\left(\frac{(\beta-\alpha)^2}{\gamma^2} - 2\alpha\right) \geq 0.
\end{aligned}$$

Hence, condition (2.2) is satisfied.

Thus, by Thereom A, all nontrivial solutions of (2.1) are oscillatory under the assumption (3.1), and therefore, those of (1.1) are oscillatory. □

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we have only to consider the case that $\beta \geq 0$. If $\alpha \geq \beta$, then

$$-\alpha + \beta\cos(\gamma t) \leq -\alpha + \beta \leq 0$$

for $t \in \mathbb{R}$. Hence, by virtue of Sturm's comparison theorem, all nontrivial solutions of (1.1) are nonoscillatory. Thus, the only remaining case is that $0 < \alpha < \beta$.

It is clear that the function b which was defined in (3.1) is periodic of mean value zero. From (1.6) it follows that

$$0 < \beta - \alpha \leq \frac{\gamma\sqrt{2\alpha}}{2}.$$

Hence, we obtain

$$\begin{aligned}
(B(t)-a(t))B(t) &= \frac{4(\beta-\alpha)^2}{\gamma^2}\sin^2\left(\frac{\gamma}{2}t\right)\left(\cos^2\left(\frac{\gamma}{2}t\right) - \frac{\alpha\gamma^2}{2(\beta-\alpha)^2}\right) \\
&\leq \frac{4(\beta-\alpha)^2}{\gamma^2}\sin^2\left(\frac{\gamma}{2}t\right)\left(\cos^2\left(\frac{\gamma}{2}t\right) - 1\right) \leq 0
\end{aligned}$$

for $t \in \mathbb{R}$; that is, condition (2.3) is satisfied for $T = 4\pi/\gamma$.

Thus, by Theorem B, all nontrivial solutions of (2.1) are nonoscillatory under the assumption (3.1), and therefore, those of (1.1) are nonoscillatory. □

4. Comparison with previous studies

El-Sayed [8] has presented an oscillation criterion for a second-order forced linear differential equation and dealt with Mathieu's equation (1.2) as an application of his result. Needless to say, equation (1.2) coincides with equation (1.1) when $\gamma = 2$. Note that all nontrivial solutions of (1.2) are oscillatory provided that $\alpha \leq 0$. Hence, we have only to discuss the case that $\alpha > 0$. El-Sayed showed that if $\alpha > 0$ and $|\beta| \geq 2\alpha + 2$, then all nontrivial solutions of (1.2) are oscillatory (see also [13, Remark 4] and [26]). Theorem 1.1 shows that if $\alpha > 0$ and $|\beta| \geq 2\sqrt{2\alpha} + \alpha$, then all nontrivial solutions of (1.2) are oscillatory.

Let us consider the straight lines $\beta = 2\alpha + 2$ and $\beta = -2\alpha - 2$, the convex curve $\beta = 2\sqrt{2\alpha} + \alpha$ and the concave curve $\beta = -2\sqrt{2\alpha} - \alpha$ in the half-plane

$$\left\{(\alpha, \beta) : \alpha > \ 0 \text{ and } \beta \in \ \mathbb{R}\right\}.$$

As shown in Figure 1, the line $\beta = 2\alpha + 2$ is the tangent of the curve $\beta = 2\sqrt{2\alpha} + \alpha$ at the point $(2, 6)$. Similarly, the line $\beta = -2\alpha - 2$ touches the curve $\beta = -2\sqrt{2\alpha} - \alpha$ at the point $(2, -6)$. Hence, Theorem 1.1 completely contains El-Sayed's result. All nontrivial solutions of (1.2) are oscillatory if a pair of (α, β) is contained in the shadow part of Figure 1. The dark shadow part is the parametric oscillation region given by El-Sayed [8].

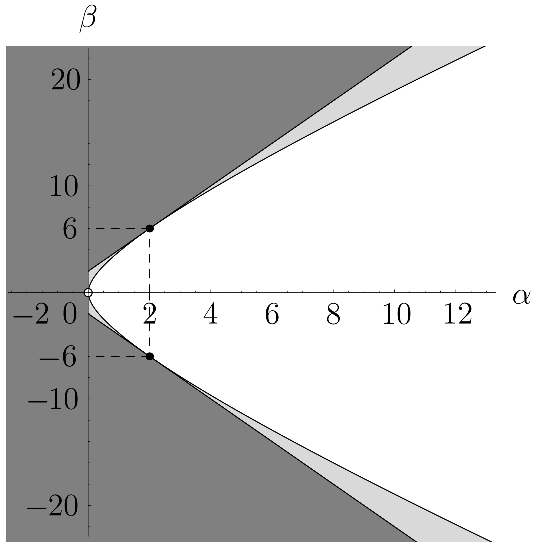

Figure 1: Parametric oscillation region given by condition (1.5) in the case that $\gamma = 2$

As mentioned in Section 1, equation (1.1) is equivalent to equation (1.7). Hence, we can rewrite Theorems 1.1 and 1.2 as follows.

Theorem 4.1. *If*

$$\tilde{\alpha} > 0 \quad \textit{and} \quad |\tilde{\beta}| \geq \tilde{\gamma}\sqrt{2\tilde{\alpha}} + \tilde{\alpha}, \tag{4.1}$$

then all nontrivial solutions of (1.7) *are oscillatory*.

Theorem 4.2. *If*

$$\tilde{\alpha} > 0 \quad \textit{and} \quad |\tilde{\beta}| \le \frac{\tilde{\gamma}\sqrt{2\tilde{\alpha}}}{2} + \tilde{\alpha}, \tag{4.2}$$

then all nontrivial solutions of (1.7) *are nonoscillatory*.

Sun et al. [24] gave interval oscillation theorems for the forced linear differential equation of the form,

$$\left(p(t)x'\right)' + q(t)x = r(t),$$

where p, q and r are continuous functions, and p is positive and continuously differentiable on $(0,\infty)$. Also, they dealt with the forced Mathieu equation

$$\frac{d^2z}{ds^2} + (-\tilde{\alpha} + \tilde{\beta}\sin s)z = ks^{\ell}\cos s, \tag{4.3}$$

in order to show how to use their result. Here, $\ell > 0$ and $k \ge 0$. They showed that if $\tilde{\alpha} = 1$ and $\tilde{\beta} \ge \beta^* = ((27+\sqrt{1097})/46)^2 + 1 \approx 2.70819162\cdots$, then all nontrivial solutions of (4.3) are oscillatory. In the unforced case of (4.3); namely,

$$\frac{d^2z}{ds^2} + (-\tilde{\alpha} + \tilde{\beta}\sin s)z = 0 \tag{4.4}$$

coincides with equation (1.2) in the case that $\alpha = 4\tilde{\alpha}$ and $\beta = 4\tilde{\beta}$. Hence, the above result of El-Sayed [8] assures that if $\tilde{\alpha} = 1$ and $|\tilde{\beta}| \ge 2.5$, then all nontrivial solutions of (4.4) are oscillatory. Equation (1.7) coincides with equation (4.4) in the case that $\tilde{\gamma} = 1$. Hence, from Theorem 4.1, we can guarantee that if $\tilde{\alpha} = 1$ and $|\tilde{\beta}| \ge \beta_* = \sqrt{2}+1 \approx 2.41421356\cdots$, then all nontrivial solutions of (4.4) are oscillatory (see Figure 2).

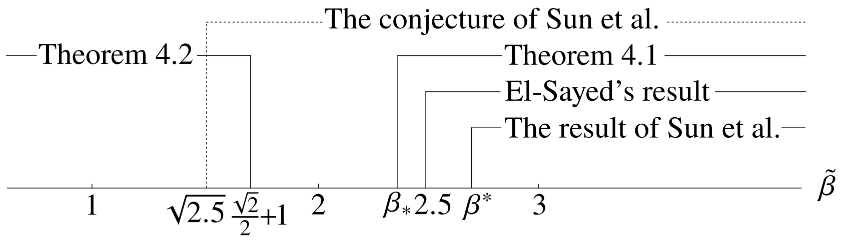

Figure 2: Relation between our theorems and previous results concerning equation (4.4) when $\tilde{\alpha} = 1$

Komkov [12] had examined equation (4.4) before the research of Sun et al. [24] and stated that if $\tilde{\alpha} > 0$ and $|\tilde{\beta}| > \sqrt{2\tilde{\alpha}}$, then all nontrivial solutions are oscillatory. However, his statement is obviously not true. In fact, from Sturm's comparison theorem, we see that if $|\tilde{\beta}| \le \tilde{\alpha}$, then all nontrivial solutions of (4.4) are nonoscillatory. Since $\sqrt{2\tilde{\alpha}} < \tilde{\alpha}$ for $\tilde{\alpha} > 2$, we can choose a $\tilde{\beta}$ such that

$$\sqrt{2\tilde{\alpha}} < |\tilde{\beta}| \le \tilde{\alpha}.$$

In such a pair of $(\tilde{\alpha}, \tilde{\beta})$, all nontrivial solutions of (4.4) are nonoscillatory. This contradicts his statement.

Sun et al. [24] has already pointed out the mistake of Komkov's result. Also, judging from Komkov's result, they estimated that if $\tilde{\alpha} > 0$ and $\tilde{\beta} \geq \sqrt{2.5\tilde{\alpha}}$, then all nontrivial solutions of (4.4) (or (4.3)) would be oscillatory. If their estimation is correct, then it is better than El-Sayed's result, because $\sqrt{2.5} < 2.5$. Unfortunately, however, this conjecture is not true. In fact, it is clear that $\sqrt{2.5\tilde{\alpha}} \leq \sqrt{2\tilde{\alpha}}/2 + \tilde{\alpha}$ for $\tilde{\alpha} \geq 3 - \sqrt{5} \approx 0.76393202\cdots$. Hence, by Theorem 4.2, all nontrivial solutions of (4.4) are nonoscillatory provided that $\tilde{\alpha} \geq 3 - \sqrt{5}$ and $\tilde{\beta} \geq \sqrt{2.5\tilde{\alpha}}$.

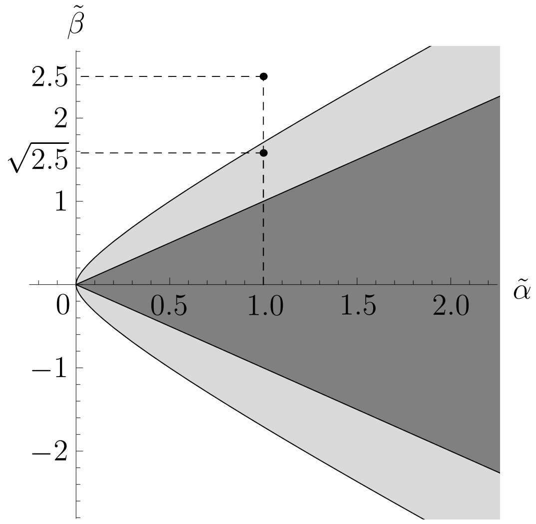

Figure 3: Parametric nonoscillation region given by condition (4.2) in the case that $\tilde{\gamma} = 1$

All nontrivial solutions of (4.4) are nonoscillatory if a pair of $(\tilde{\alpha}, \tilde{\beta})$ is contained in the shadow part of Figure 3. The dark shadow part is the region $\left\{(\tilde{\alpha}, \tilde{\beta}) : \tilde{\alpha} > 0 \text{ and } |\tilde{\beta}| \leq \tilde{\alpha}\right\}$. The point $(1.0, 2.5)$ is outside of the shadow part of Figure 3. As already mentioned, from Theorem 4.1 (or El-Sayed's result), we see that all nontrivial solutions of (4.4) are oscillatory if $\tilde{\alpha} = 1.0$ and $\tilde{\beta} = 2.5$. The point $(1.0, \sqrt{2.5})$ is inside of the shadow part of Figure 3. By Theorem 4.2, all nontrivial solutions of (4.4) are nonoscillatory if $\tilde{\alpha} = 1.0$ and $\tilde{\beta} = \sqrt{2.5}$.

According to McLachan [19, p. 29], the infinite continued fraction for the Mathieu equation

$$x'' + (a - 2q\cos(2t))x = 0$$

is as follows:

$$a = \frac{-q^2/2}{1 - a/4-}\,\frac{q^2/64}{1 - a/16-}\,\frac{q^2/576}{1 - a/36-}\,\frac{q^2/2304}{1 - a/64-}\cdots\frac{q^2/(16n^2(n-1)^2)}{1 - a/(4n^2)-}\cdots, \tag{4.5}$$

where $n = 2, 3, \ldots$. Substituting $a = -4\tilde{\alpha}$ and $q = 2\tilde{\beta}$ in (4.5), Leighton [16] showed that the infinite continued fraction for equation (4.4) is

$$\tilde{\alpha} = \frac{\tilde{\beta}^2}{2(1+\tilde{\alpha})-}\,\frac{\tilde{\beta}^2}{2(4+\tilde{\alpha})-}\,\frac{\tilde{\beta}^2}{2(9+\tilde{\alpha})-}\,\frac{\tilde{\beta}^2}{2(16+\tilde{\alpha})-}\cdots\frac{\tilde{\beta}^2}{2(n^2+\tilde{\alpha})-}\cdots, \tag{4.6}$$

where $n = 1, 2, \ldots$. Since it is hard to analyze (4.6), he provided a recurrence formula which is equivalent to (4.6). His recurrence formula yields approximating curves C_n $(n = 1, 2, \ldots)$ in the $(\tilde{\alpha}, \tilde{\beta})$-plane. For example, the curves C_1, C_2, C_3 and C_4 are given by

$$\begin{cases} \tilde{\beta}^2 = 2\tilde{\alpha}(\tilde{\alpha}+1) & \text{if } n = 1, \\ \tilde{\beta}^2 = \dfrac{4\tilde{\alpha}(\tilde{\alpha}+1)(\tilde{\alpha}+4)}{3\tilde{\alpha}+8} & \text{if } n = 2, \\ \tilde{\beta}^4 - (8\tilde{\alpha}+3)(\tilde{\alpha}+6)\tilde{\beta}^2 + 8\tilde{\alpha}(\tilde{\alpha}+1)(\tilde{\alpha}+4)(\tilde{\alpha}+9) = 0 & \text{if } n = 3, \\ 5(\tilde{\alpha}+8)\tilde{\beta}^4 - 4(5\tilde{\alpha}^3 + 105\tilde{\alpha}^2 + 652\tilde{\alpha} + 1152)\tilde{\beta}^2 & \\ \qquad +16\tilde{\alpha}(\tilde{\alpha}+1)(\tilde{\alpha}+4)(\tilde{\alpha}+9)(\tilde{\alpha}+16) = 0 & \text{if } n = 4, \end{cases}$$

respectively. He proved that all nontrivial solutions of (4.4) are oscillatory if a pair of $(\tilde{\alpha}, \tilde{\beta})$ is on or outside the curve C_n $(n = 1, 2, \ldots)$. To state the case that $n = 4$ more precisely, we put

$$\begin{aligned} p_1(\lambda) &= 5(\lambda+8) \\ p_2(\lambda) &= 2(5\lambda^3 + 105\lambda^2 + 652\lambda + 1152) \\ p_3(\lambda) &= 16\lambda(\lambda+1)(\lambda+4)(\lambda+9)(\lambda+16). \end{aligned}$$

Then, if $\tilde{\alpha} > 0$ and

$$|\tilde{\beta}| \geq \sqrt{\frac{1}{p_1(\tilde{\alpha})}\left\{p_2(\tilde{\alpha}) - \sqrt{p_2^2(\tilde{\alpha}) - p_1(\tilde{\alpha})p_3(\tilde{\alpha})}\right\}}, \tag{4.7}$$

than all nontrivial solutions of (4.4) are oscillatory.

Since p_1, p_2 and p_3 are a linear polynomial, a quadratic polynomial and a cubic polynomial of λ, respectively, the right-hand side of (4.7) is considerably complicated. Hence, we cannot seek $\tilde{\alpha}$ satisfying (4.7) by a hand calculation when we choose $\tilde{\beta}$ arbitrarily. Although the estimation (4.7) seems to be a very sharp sufficient condition, it is not a necessary condition for all solutions of (4.4) to be oscillatory. In fact,

$$\sqrt{2\tilde{\alpha}} + \tilde{\alpha} < \sqrt{\frac{1}{p_1(\tilde{\alpha})}\left\{p_2(\tilde{\alpha}) - \sqrt{p_2^2(\tilde{\alpha}) - p_1(\tilde{\alpha})p_3(\tilde{\alpha})}\right\}}$$

for $\tilde{\alpha}$ sufficiently large. For example, we can check that

$$\sqrt{2\times 700} + 700 < 738 < \sqrt{\frac{1}{p_1(700)}\left\{p_2(700) - \sqrt{p_2^2(700) - p_1(700)p_3(700)}\right\}}.$$

This means that if $\tilde{\alpha} = 700$ and $\tilde{\beta} = 738$, then the inequality (4.7) is not satisfied. Hence, the result of Leighton [16] is inapplicable for equation (4.4) with $\tilde{\alpha} = 700$ and $\tilde{\beta} = 738$. On the other hand, condition (4.1) is satisfied for $\tilde{\alpha} = 700$ and $\tilde{\beta} = 738$. Hence, we conclude that all nontrivial solutions of (4.4) are oscillatory.

5. Nonoscillation theorem for a Mathieu-type differential equation

As a generalization of (1.1), we consider the second-order differential equation

$$x'' + (-\alpha + \beta\varphi'(t)\cos(\gamma\varphi(t)))x = 0, \tag{5.1}$$

where φ is continuously differentiable on $(0,\infty)$ and satisfies $|\varphi'(t)| \le 1$ for $t > 0$, and give a nonoscillation theorem for equation (5.1). For example, $\varphi(t) = \sin t$ and $\varphi(t) = t(\sin(\ln t) + \cos(\ln t))/2$ for $t > 0$. Consider an inverted simple pendulum whose pivot point oscillates up and down. Let $\sin(\gamma\varphi(t))$ be the movement speed of the pivot point. Assume that the friction at the pivot point can be ignored. Then, the equation of motion for the vertically driven pendulum can be written as

$$x'' + (-\alpha + \beta\varphi'(t)\cos(\gamma\varphi(t)))\sin x = 0.$$

Equation (5.1) is a linearized model of this motion equation. Needless to say, $\varphi'\cos(\gamma\varphi)$ is not always periodic. Hence, Theorem B is inapplicable to equation (5.1). Using Theorem C, we prove the following result.

Theorem 5.1. *If*

$$\alpha > 0 \quad \textit{and} \quad |\beta| \le \frac{\gamma\sqrt{2\alpha}}{2} + \alpha, \tag{5.2}$$

then all nontrivial solutions of (5.1) *are nonoscillatory*.

Proof. As mentioned in the proof of Theorems 1.1 and 1.2, we have only to consider the case that $0 < \alpha < \beta$. For the sake of simplicity, let

$$h = \frac{\beta - \alpha}{\gamma} + \sqrt{\frac{(\beta-\alpha)^2}{\gamma^2} + 2\alpha}. \tag{5.3}$$

Then, h is a positive constant depending on parameters α, β and γ. We choose a and b defined by

$$\begin{aligned} a(t) &= 2h - \frac{2(\beta-\alpha)}{\gamma}\sin(\gamma\varphi(t)), \\ b(t) &= \frac{2h(\beta-\alpha)}{\gamma}\big(1 - \sin(\gamma\varphi(t))\big) + \alpha\big(1 + \varphi'(t)\cos(\gamma\varphi(t))\big) \\ &\quad + \frac{(\beta-\alpha)^2}{\gamma^2}\sin^2(\gamma\varphi(t)). \end{aligned} \tag{5.4}$$

Then by a straightforward calculation, we can check that

$$b(t)-\frac{1}{4}a^2(t)-\frac{1}{2}a'(t)=-\alpha+\beta\varphi'(t)\cos(\gamma\varphi(t)).$$

Hence, we conclude that equation (5.1) is equivalent to equation (2.1) with (5.4).

We will examine whether that $(a(t),b(t))$ given by (5.4) satisfies condition (2.4). From (5.3) it follows that

$$\begin{aligned}h_* \overset{\text{def}}{=} 2\sqrt{\frac{(\beta-\alpha)^2}{\gamma^2}+2\alpha}=2h-\frac{2(\beta-\alpha)}{\gamma}\le a(t)\\ \le 2h+\frac{2(\beta-\alpha)}{\gamma}=\frac{4(\beta-\alpha)}{\gamma}+2\sqrt{\frac{(\beta-\alpha)^2}{\gamma^2}+2\alpha}\overset{\text{def}}{=} h^*\end{aligned}$$

and

$$b(t)\ge\frac{(\beta-\alpha)^2}{\gamma^2}\sin^2(\gamma\varphi(t))\ge 0$$

for $t>0$. Also, we see that

$$\begin{aligned}\frac{1}{4}a^2(t)-b(t)&=h^2-\frac{2h(\beta-\alpha)}{\gamma}-\alpha\big(1+\varphi'(t)\cos(\gamma\varphi(t))\big)\\&=\alpha\big(1-\varphi'(t)\cos(\gamma\varphi(t))\big)\ge\alpha\big(1-\cos(\gamma\varphi(t))\big)\ge 0\end{aligned}$$

for $t>0$. Hence, we conclude that $(a(t),b(t))\in R=\big\{(u,v)\colon u\ge 0 \text{ and } 0\le v\le u^2/4\big\}$ for $t>0$.

Put $u=a(t)$ and $v=b(t)$. Then, by (5.4) we have

$$2h-\frac{2(\beta-\alpha)}{\gamma}\le u\le 2h+\frac{2(\beta-\alpha)}{\gamma},$$

$$\sin(\gamma\varphi(t))=\frac{\gamma(2h-u)}{2(\beta-\alpha)}\quad\text{and}\quad\cos(\gamma\varphi(t))=\pm\sqrt{1-\frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2}.$$

Hence, we obtain

$$v=\frac{2h(\beta-\alpha)}{\gamma}-h(2h-u)+\alpha\pm\alpha\varphi'(t)\sqrt{1-\frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2}+\frac{(2h-u)^2}{4}.$$

Define two functions f_+ and f_- as follows:

$$\begin{aligned}f_+(u)&=\frac{2h(\beta-\alpha)}{\gamma}+\alpha-h(2h-u)+\frac{(2h-u)^2}{4}+\alpha\sqrt{1-\frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2};\\ f_-(u)&=\frac{2h(\beta-\alpha)}{\gamma}+\alpha-h(2h-u)+\frac{(2h-u)^2}{4}-\alpha\sqrt{1-\frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2}\end{aligned}$$

for $h_* \le u \le h^*$. Let

$$S = \left\{(u,v)\colon h_* \le u \le h^* \text{ and } f_-(u) \le v \le f_+(u)\right\}.$$

It is clear that S is a bounded and closed set. Since $|\varphi'(t)| \le 1$ for $t > 0$, we see that $(a(t), b(t)) \in S$ for $t > 0$.

We will show that S is a convex set contained in R. Using (5.3), we can rewrite f_+ and f_- as

$$f_+(u) = \frac{u^2}{4} - \alpha\left(1 - \sqrt{1 - \frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2}\right)$$

and

$$f_-(u) = \frac{u^2}{4} - \alpha\left(1 + \sqrt{1 - \frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2}\right),$$

respectively. It is clear that

$$f_+(u) \le \frac{u^2}{4} \quad \text{and} \quad f_-(u) \ge \frac{(\beta-\alpha)^2}{\gamma^2} + 2\alpha - \alpha(1+1) = \frac{(\beta-\alpha)^2}{\gamma^2} > 0$$

for $h_* \le u \le h^*$. By a direct calculation, we have

$$\frac{d}{du} f_\pm(u) = \frac{u}{2} \pm \frac{\alpha\gamma^2(2h-u)}{4(\beta-\alpha)^2\sqrt{1 - \frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2}}$$

and

$$\frac{d^2}{du^2} f_\pm(u) = \frac{1}{2} \mp \frac{\alpha\gamma^2}{4(\beta-\alpha)^2\left\{1 - \frac{\gamma^2}{4}\left(\frac{2h-u}{\beta-\alpha}\right)^2\right\}^{3/2}}$$

for $h_* \le u \le h^*$. Hence, by (5.2) we obtain

$$\frac{d^2}{du^2} f_+(u) \le \frac{1}{2} - \frac{\alpha\gamma^2}{4(\beta-\alpha)^2} \le 0$$

and

$$\frac{d^2}{du^2} f_-(u) \ge \frac{1}{2} + \frac{\alpha\gamma^2}{4(\beta-\alpha)^2} > \frac{1}{2}$$

for $h_* \le u \le h^*$. We therefore conclude that the set S is convex and $S \subset R$. Also, we see that the curves $v = u^2/4$ and $v = f_+(u)$ has a unique common tangent at $(2h, h^2)$. The common tangent is given by $v = hu - h^2$ (see Figure 4).

Thus, by means of Theorem C, all nontrivial solutions of (5.1) are nonoscillatory. $\square$

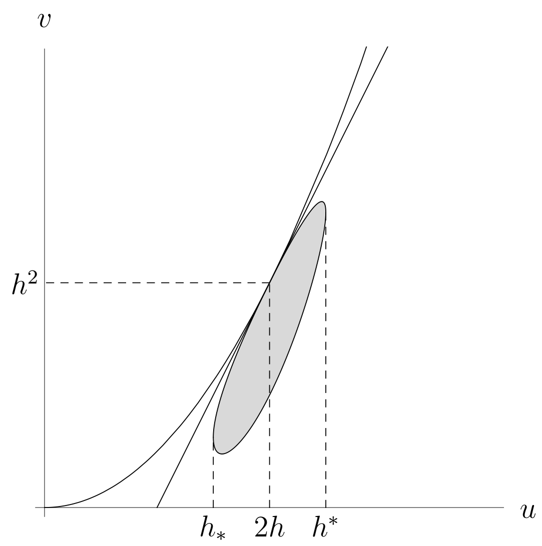

Figure 4: The parametric set S and the common tangent in the case that $\alpha = 1$, $\beta = 1.5$, $\gamma = 1$ and $h = 2$

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