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A new application method for nonoscillation criteria of Hille-Wintner type

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Abstract The present paper deals with nonoscillation problem for the Sturm-Liouville half-linear differential equation

$$\big(r(t)\phi_p(x')\big)' + c(t)\phi_p(x) = 0,$$

where $r, c\colon [a,\infty) \to \mathbb{R}$ are continuous functions, $r(t) > 0$ for $t \geq a$, and $\phi_p(z) = |z|^{p-2}z$ with $p > 1$. The purpose of this paper is to show that it is possible to broaden the application range of Hille-Wintner type nonoscillation criteria. To this end, we derive a comparison theorem by means of Riccati's technique. Our result is new even in the linear case that $p = 2$. By the obtained result, we can compare two differential equations having a different power p of the above-mentioned type. To illustrate our comparison theorem, we present two examples of which all non-trivial solutions of the Sturm-Liouville linear differential equation are nonoscillatory even if $\int_a^t \frac{1}{r(s)}ds \int_t^\infty c(s)ds$ or $\int_t^\infty \frac{1}{r(s)}ds \int_a^t c(s)ds$ is less than the lower bound $-3/4$.

Keywords Sturm-Liouville differential equations · Linear differential equations · Half-linear differential equations · Power comparison theorem · Linearization method · Nonoscillation criteria · Riccati's technique

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1 Introduction

As an extension of the second-order linear differential equation

$$\big(r(t)x'\big)' + c(t)x = 0, \tag{1.1}$$

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many researchers have dealt with the half-linear differential equation

$$\big(r(t)\phi_p(x')\big)' + c(t)\phi_p(x) = 0, \tag{1.2}$$

where the prime denotes d/dt; the coefficients r and c are continuous functions on $[a,\infty)$ with $a \geq 0$ and $r(t) > 0$ for $t \geq a$; the real-valued function ϕ_p is defined by

$$\phi_p(z) = |z|^{p-2}z$$

for $z \in \mathbb{R}$ with $p > 1$ a fixed real number. Equations (1.1) and (1.2) are said to be in Sturm-Liouville form or self-adjoint form. As for half-linear differential equations, for example, we can refer the reader to [1, 3, 4, 7, 11, 12] and the references cited therein. A function x is said to be a solution of (1.2) if x is a continuously differentiable function such that $r\phi_p(x')$ is also continuously differentiable and satisfies (1.2) in an unbounded interval $I \subset [a,\infty)$.

It is known that all solutions of (1.2) are unique for given initial conditions and continuable in the future like those of (1.1) (see, for example, [1, 4, 7, 9]). Hence, it is worth while to discuss whether solutions of (1.2) are oscillatory or not. A non-trivial solution x of (1.2) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to infinity such that $x(t_n) = 0$. Otherwise, it is said to be *nonoscillatory*. Hence, a nonoscillatory solution x eventually keeps either positive or negative; that is, there exists a $T \geq a$ such that $x(t) \neq 0$ for $t \geq T$. It is also known that oscillatory solutions and nonoscillatory solutions do not coexist in equation (1.2) as well as equation (1.1). Hence, we may say that equation (1.2) is oscillatory (respectively, nonoscillatory) in case all non-trivial solutions are oscillatory (respectively, nonoscillatory). Riccati's technique is very useful to check that equation (1.2) is nonoscillatory. Results related to Riccati's technique for half-linear differential equations until 2005 were summarized in the book [7] (see also [4]).

Equation (1.1) can be divided into two cases:

$$\int_a^\infty \frac{1}{r(t)}dt \overset{\text{def}}{=} \lim_{t\to\infty}\int_a^t \frac{1}{r(s)}ds = \infty; \tag{1.3}$$

$$\int_a^\infty \frac{1}{r(t)}dt < \infty. \tag{1.4}$$

In the former, it is well-known that by transforming

$$s = \varphi(t) = \int_a^t \frac{1}{r(\tau)}d\tau, \qquad y(s) = y(\varphi(t)) = x(t),$$

Equation (1.1) is reduced to the equation of simple form,

$$\ddot{y} + \tilde{c}(s)y = 0, \tag{1.5}$$

where $\tilde{c}(s) = c(\varphi^{-1}(s))r(\varphi^{-1}(s))$. Note that φ is strictly increasing for $t \geq a$ because $r(t) > 0$ for $t \geq a$. Hence, there exists the inverse function φ^{-1}, which is also increasing for $s \geq 0$. The inverse function diverges to ∞ as s tends to ∞.

Many attempts were made to find nonoscillation criteria for equation (1.5). There is an extensive literature on this topic in the book [18]. Among them, noteworthy is the classical Hille-Wintner comparison theorem and the following result which related (see [13, 14, 19, 20]).

Theorem A *Suppose that*

$$\int_a^\infty \tilde{c}(t)dt \text{ is convergent} \tag{1.6}$$

and

$$-\frac{3}{4} \le t\int_t^\infty \tilde{c}(s)ds \le \frac{1}{4} \quad \text{for } t \ge a.$$

Then equation (1.5) *is nonoscillatory*.

As a simple consequence result of Theorem A, we have the following nonoscillation theorem.

Theorem B *Under the assumption* (1.3)*, if*

$$\int_a^\infty c(t)dt \ \text{ is convergent} \tag{1.7}$$

and

$$-\frac{3}{4} < \liminf_{t\to\infty} A_2(t) \le \limsup_{t\to\infty} A_2(t) < \frac{1}{4}, \tag{1.8}$$

then equation (1.1) *is nonoscillatory, where*

$$A_2(t) = \int_a^t \frac{1}{r(s)}ds \int_t^\infty c(s)ds.$$

Theorem B is a corollary of Theorem 3.1.3 in the book [7, p.86]. Došlý and Řehák have also presented some results of the case (1.4). The following theorem is a corollary of Theorem 3.1.5 in [7, p.88] (see also [3, 4]).

Theorem C *Under the assumption* (1.4)*, if*

$$-\frac{3}{4} < \liminf_{t\to\infty} \overline{A}_2(t) \le \limsup_{t\to\infty} \overline{A}_2(t) < \frac{1}{4}, \tag{1.9}$$

then equation (1.1) *is nonoscillatory, where*

$$\overline{A}_2(t) = \int_t^\infty \frac{1}{r(s)}ds \int_a^t c(s)ds.$$

Theorems B and C have a good balance. There are the upper and lower bounds in Theorems A, B and C. Here, a simple question arises. Will the lower bound for nonoscillation of (1.1) or (1.5) really exist? If the lower bound is necessary for nonoscillation, is it equal to $-3/4$? To explain this question, we consider the Euler differential equation

$$\ddot{y} + \frac{\lambda}{s^2}y = 0 \tag{1.10}$$

as a special case of (1.5). Then it is clear that

(i) if $\lambda \le 1/4$, then equation (1.10) is nonoscillatory;

(ii) if $\lambda > 1/4$, then equation (1.10) is oscillatory.

Of course, equation (1.10) is nonoscillatory even if $\lambda \le 0$. Since $\tilde{c}(s) = \lambda/s^2$ in equation (1.10),

$$t\int_t^\infty \tilde{c}(s)ds = \lambda.$$

Hence, though $1/4$ is the upper bound for nonoscillation of (1.10), no lower bound exists for nonoscillation of (1.10). It is also obvious that equation (1.5) is nonoscillatory provided that $\tilde{c}(s) \le 0$ for s sufficiently large. In this case, even condition (1.6) may not be satisfied. Moreover, it is well-known that

(i) if $-\infty \le \limsup_{s\to\infty} s^2\tilde{c}(s) < 1/4$, then equation (1.5) is nonoscillatory;
(ii) if $1/4 < \liminf_{s\to\infty} s^2\tilde{c}(s) \le \infty$, then equation (1.5) is oscillatory.

This criterion is called Kneser-type (see [15]). In Kneser's criterion, no lower bound exists for nonoscillation of (1.5).

In this paper, we will show that the scopes of Theorems B and C are expandable. To be precise, we give an example that equation (1.1) is nonoscillatory even if

$$\liminf_{t\to\infty} A_2(t) \le -\frac{3}{4}.$$

We also give an example that equation (1.1) is nonoscillatory even if

$$\liminf_{t\to\infty} \overline{A}_2(t) \le -\frac{3}{4}.$$

To check these examples, we present a comparison theorem of nonoscillation for equation (1.2). Half-linear differential equations are described by the power function ϕ_p with $p > 1$. Our result can compare two half-linear differential equations with a different power. Such a result is called a "power comparison theorem". Of course, we can compare two half-linear differential equations with the same power.

2 Equation to be compared

Let b be a nonnegative continuous function on $[a,\infty)$ satisfying

$$b(t) + c(t) > 0 \quad \text{for } t \ge a. \tag{2.1}$$

Such a function exists. For example, let $b(t) = 2\max\{0, -c(t)\} + 1$. Then we see that

$$b(t) + c(t) = 1 + c(t) \ge 1 \quad \text{if } c(t) \ge 0$$

and

$$b(t) + c(t) = -2c(t) + 1 + c(t) = 1 - c(t) \ge 1 \quad \text{if } c(t) \le 0$$

for $t \ge a$; that is, condition (2.1) holds.

For any $\mu > 1$, let $k(\mu)$ be an odd number satisfying

$$k \ge \max\left\{3,\ \frac{\mu-1}{p-1}\right\}.$$

Note that k depends on μ. If $\mu \le 3p-2$, then k may be 3, and if

$$(2n+1)p - 2n < \mu \le (2n+3)p - 2(n+1)$$

for $n \in \mathbb{N}$, then it may be $2n+3$. We denote the conjugate exponents of p and μ by p^* and μ^*, respectively; that is,

$$\frac{1}{p} + \frac{1}{p^*} = 1 \quad \text{and} \quad \frac{1}{\mu} + \frac{1}{\mu^*} = 1.$$

In the calculation of Section 3, it might be more convenient to use $(p-1)(p^*-1) = 1$, $p^* = p/(p-1)$, $p = p^*/(p^*-1)$, $(\mu-1)(\mu^*-1) = 1$, $\mu^* = \mu/(\mu-1)$ and $\mu = \mu^*/(\mu^*-1)$.

Let f be a continuously differentiable function. Define

$$C(t) = 2k(p-1)\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*+1/p}\big(r(t)\big)^{(k-1)/p} + 2^{\mu^*}k(p-1)\left(\frac{p-1}{b(t)+c(t)}\right)^{(\mu^*-1)k/p^*-1/p}\left(\frac{1}{r(t)}\right)^{(\mu^*-1)k/p+1/p}|f(t)|^{\mu^*} - f'(t).$$

Let $f(t) \equiv 0$. Then $|f(t)|^{\mu^*} \equiv 0$ and $f'(t) \equiv 0$. Hence, by (2.1) we have

$$C(t) = 2k(p-1)\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*+1/p}\big(r(t)\big)^{(k-1)/p} > 0 \quad \text{for } t \geq a.$$

Of course, depending on b, c and r, we may choose a continuously differentiable function f that diverges to infinity as $t \to \infty$.

To compare with (1.2), we consider the Sturm-Liouville half-linear differential equation

$$\big(R(t)\phi_\mu(y')\big)' + C(t)\phi_\mu(y) = 0, \tag{2.2}$$

where

$$R(t) = \left(\frac{\mu-1}{2^{\mu^*}k(p-1)}\right)^{\mu-1}\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*-(\mu-1)/p}\big(r(t)\big)^{(k+\mu-1)/p}.$$

Then we can obtain the following comparison theorem concerning nonoscillation.

Theorem 2.1 *Suppose that* (2.1) *holds. If equation* (2.2) *is nonoscillatory, then equation* (1.2) *is also nonoscillatory.*

As mentioned above, we may choose 3 as the odd number k in the special case that $p = \mu = 2$. Hence, the functions $C(t)$ and $R(t)$ become

$$\widetilde{C}(t) = 6\big(b(t)+c(t)\big)^2 r(t) + \frac{12 f^2(t)}{\big(b(t)+c(t)\big)r^2(t)} - f'(t)$$

and

$$\widetilde{R}(t) = \frac{\big(b(t)+c(t)\big)r^2(t)}{12},$$

respectively; and therefore, we have the following result.

Corollary 2.2 *Suppose that* (2.1) *holds. If equation*

$$\big(\widetilde{R}(t)y'\big)' + \widetilde{C}(t)y = 0 \tag{2.3}$$

is nonoscillatory, then equation (1.1) *is also nonoscillatory.*

Remark 2.1 Needless to say, ϕ_p is the power function with real exponent p. Comparison theorems are divided into two types. In the one type, half-linear differential equations with the same power are compared. Corollary 2.2 belongs to this type. In another type, equation (1.2) is compared with the half-linear differential equation with a different power $\beta > 1$,

$$\big(q(t)\phi_\beta(x')\big)' + d(t)\phi_\beta(x) = 0.$$

Such results can be found in [2, 5, 10, 16, 17]. In those results, the authors have assumed only one of either $\beta > p$ or $\beta < p$. Theorem 2.1 is a power comparison theorem concerning nonoscillation without these assumptions. This point is a distinctive feature of Theorem 2.1.

3 Proof of Theorem 2.1

We have only to show that the Riccati differential inequality

$$w' + c(t) + (p-1)\frac{|w|^{p^*}}{(r(t))^{p^*-1}} \le 0 \tag{3.1}$$

has a solution w for t sufficiently large.

The Riccati differential inequality corresponding to equation (2.2) is

$$v' + C(t) + (\mu-1)\frac{|v|^{\mu^*}}{(R(t))^{\mu^*-1}} \le 0. \tag{3.2}$$

Since equation (2.2) has a nonoscillatory solution y, there exists a $T \ge a$ such that $y(t) > 0$ or $y(t) < 0$ for $t \ge T$. We may assume without loss of generality that $y(t) > 0$ for $t \ge T$. If necessary, we may choose more large $T \ge a$. Let

$$v(t) = \frac{R(t)\phi_\mu(y'(t))}{\phi_\mu(y(t))}.$$

Since y is a solution of (2.2), the denominator $\phi_\mu(y)$ and the numerator $R\phi_\mu(y')$ are continuously differentiable functions. Hence, v is also a continuously differentiable function. In addition, the function v satisfies the inequality (3.2) for $t \ge T$.

Since k is an odd number that is larger than 3, we can define

$$w(t) = \sqrt[k]{v(t) - f(t)}$$

for $t \ge T$. From (2.1) we see that $C(t) + f'(t)$ and $R(t)$ are positive for $t \ge T$. Hence, we have

$$(v(t) - f(t))' \le -C(t) - (\mu-1)\frac{|v(t)|^{\mu^*}}{(R(t))^{\mu^*-1}} - f'(t) < 0 \tag{3.3}$$

for $t \ge T$. It turns out from this inequality that $v - f$ has only one zero at most. This means that w is continuously differentiable for t sufficiently large, because the functions v and f are continuously differentiable. Since

$$v(t) = w^k(t) + f(t),$$

we have

$$v'(t) = k\,w^{k-1}(t)w'(t) + f'(t)$$

for t sufficiently large. Hence,

$$\begin{aligned}
&k\,w^{k-1}(t)\left(w'(t) + c(t) + (p-1)\frac{|w(t)|^{p^*}}{(r(t))^{p^*-1}}\right)\\
&\quad = v'(t) - f'(t) + k\,c(t)|w(t)|^{k-1} + k\,(p-1)\frac{|w(t)|^{p^*+k-1}}{(r(t))^{p^*-1}}\\
&\quad \le v'(t) + k\,(p-1)\left(\frac{|w(t)|^{p^*+k-1}}{(r(t))^{p^*-1}} + \frac{b(t)+c(t)}{p-1}|w(t)|^{k-1}\right) - f'(t)\\
&\quad = v'(t) + 2k\,(p-1)\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*+1/p}\big(r(t)\big)^{(k-1)/p} - f'(t)\\
&\qquad + k\,(p-1)\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*+1/p}\big(r(t)\big)^{(k-1)/p}z(t)
\end{aligned}$$

for t sufficiently large, where

$$z(t) = \left(\frac{p-1}{b(t)+c(t)}\right)^{k/p^*+1/p}\left(\frac{1}{r(t)}\right)^{(p^*+k-1)/p}|w(t)|^{p^*+k-1} + \left(\frac{p-1}{b(t)+c(t)}\right)^{(k-1)/p^*}\left(\frac{1}{r(t)}\right)^{(k-1)/p}|w(t)|^{k-1} - 2.$$

For simplicity, let $q = 1 + p^*/(k-1)$ and

$$u(t) = \left(\frac{p-1}{b(t)+c(t)}\right)^{(k-1)/p^*}\left(\frac{1}{r(t)}\right)^{(k-1)/p}|w(t)|^{k-1}.$$

From (2.1), it is clear that $u(t) \geq 0$ for $t \geq T$. Since

$$\frac{k-1}{p^*}\,q = \frac{k}{p^*} + \frac{1}{p}, \quad \frac{k-1}{p}\,q = \frac{p^*+k-1}{p} \quad \text{and} \quad (k-1)\,q = p^* + k - 1,$$

it follows that

$$z(t) = \big(u(t)\big)^q + u(t) - 2$$

for t sufficiently large. Taking into account that

$$k \geq \frac{\mu-1}{p-1} = \frac{p^*-1}{\mu^*-1},$$

it turns out that

$$\frac{\mu^* k}{k-1} \geq 1 + \frac{p^*}{k-1} = q.$$

Hence, we see that

$$u^q + u - 2 < 2u^{\mu^* k/(k-1)} \quad \text{for } u \geq 0.$$

It follows from this estimation that

$$\begin{aligned} z(t) &< 2\left(\frac{p-1}{b(t)+c(t)}\right)^{\mu^* k/p^*}\left(\frac{1}{r(t)}\right)^{\mu^* k/p}|w(t)|^{\mu^* k} \\ &= 2\left(\frac{p-1}{b(t)+c(t)}\right)^{\mu^* k/p^*}\left(\frac{1}{r(t)}\right)^{\mu^* k/p}|v(t)-f(t)|^{\mu^*} \\ &\leq 2\left(\frac{p-1}{b(t)+c(t)}\right)^{\mu^* k/p^*}\left(\frac{1}{r(t)}\right)^{\mu^* k/p}\big(|v(t)|+|f(t)|\big)^{\mu^*} \\ &\leq 2^{\mu^*}\left(\frac{p-1}{b(t)+c(t)}\right)^{\mu^* k/p^*}\left(\frac{1}{r(t)}\right)^{\mu^* k/p}\Big(|v(t)|^{\mu^*}+|f(t)|^{\mu^*}\Big) \end{aligned}$$

for $t \geq T$. Hence, we obtain

$$
\begin{aligned}
& k w^{k-1}(t)\left(w'(t)+c(t)+(p-1)\frac{|w(t)|^{p^*}}{(r(t))^{p^*-1}}\right) \\
&< v'(t)+2k(p-1)\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*+1/p}\big(r(t)\big)^{(k-1)/p}-f'(t) \\
&\quad +2^{\mu^*}k(p-1)\left(\frac{p-1}{b(t)+c(t)}\right)^{(\mu^*-1)k/p^*-1/p}\left(\frac{1}{r(t)}\right)^{(\mu^*-1)k/p+1/p}\big(|v(t)|^{\mu^*}+|f(t)|^{\mu^*}\big) \\
&= v'(t)+2k(p-1)\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*+1/p}\big(r(t)\big)^{(k-1)/p} \\
&\quad +2^{\mu^*}k(p-1)\left(\frac{p-1}{b(t)+c(t)}\right)^{(\mu^*-1)k/p^*-1/p}\left(\frac{1}{r(t)}\right)^{(\mu^*-1)k/p+1/p}|f(t)|^{\mu^*}-f'(t) \\
&\quad +2^{\mu^*}k(p-1)\left(\frac{p-1}{b(t)+c(t)}\right)^{(\mu^*-1)k/p^*-1/p}\left(\frac{1}{r(t)}\right)^{(\mu^*-1)k/p+1/p}|v(t)|^{\mu^*} \\
&= v'(t)+C(t)+(\mu-1)\frac{|v(t)|^{\mu^*}}{(R(t))^{\mu^*-1}}
\end{aligned}
$$

for t sufficiently large. From (3.3) and the fact that $w^{k-1}(t) > 0$ for t sufficiently large, it turns out that the function w is a solution of (3.1) for t sufficiently large. Hence, by Riccati's technique, equation (1.2) is nonoscillatory.

The proof of Theorem 2.1 is now complete. □

We can choose 2 as the power μ. Then μ^* is also 2 and equation (2.2) becomes a linear differential equation of Sturm-Liouville type. Since $k(2)$ is an odd number satisfying

$$k \geq \max\left\{3,\ \frac{1}{p-1}\right\},$$

the number k may be 3 if $p \geq 4/3$, and it may be $2n+3$ if

$$\frac{2n+4}{2n+3} \leq p < \frac{2n+2}{2n+1}$$

for $n \in \mathbb{N}$. In addition, the coefficients R and C of (2.2) become slightly easy as follows:

$$R(t) = \frac{1}{4k(p-1)}\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*-1/p}\big(r(t)\big)^{(k+1)/p};$$

$$
\begin{aligned}
C(t) &= 2k(p-1)\left(\frac{b(t)+c(t)}{p-1}\right)^{k/p^*+1/p}\big(r(t)\big)^{(k-1)/p} \\
&\quad +4k(p-1)\left(\frac{p-1}{b(t)+c(t)}\right)^{k/p^*-1/p}\left(\frac{1}{r(t)}\right)^{k/p+1/p}f^2(t)-f'(t).
\end{aligned}
$$

Thus, Theorem 2.1 has a deep relationship with the linearization method (or technique). As for this method, see [6, 8] for example.

4 Discussion

To answer our question that was raised in Section 1, we give an example. For this purpose, we first define g as follows:

$$g(t) = \frac{4\sin(\ln t) + 3\cos(\ln t)}{5}$$

for $t \geq 1$. Let $\theta = \mathrm{Tan}^{-1}(3/4)$. Then it is easy to check that $|g(t)| = |\sin(\ln t + \theta)| \leq 1$,

$$g(t) \geq 0 \quad \text{for } t_{2i-2} \leq t \leq t_{2i-1}$$

and

$$g(t) < 0 \quad \text{for } t_{2i-1} < t < t_{2i}$$

with $i \in \mathbb{N}$, where $t_0 = 1$ and $t_n = e^{n\pi-\theta}$ for $n \in \mathbb{N}$. For any $\alpha > 0$, let

$$r(t) = \frac{6}{\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}}\, t^{\frac{7}{3}} > 0 \tag{4.1}$$

for $t \geq 1$ and

$$c(t) = \begin{cases} \dfrac{1}{6}\left(\sqrt[3]{\dfrac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}} g(t) & \text{if } t_{2i-2} \leq t \leq t_{2i-1}, \\ \dfrac{5(2^4\alpha+1)}{24}\left(\sqrt[3]{\dfrac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}} g(t) & \text{if } t_{2i-1} < t < t_{2i} \end{cases} \tag{4.2}$$

with $i \in \mathbb{N}$. Note that

$$\frac{1}{6} < \frac{5}{24} < \frac{5(2^4\alpha+1)}{24}$$

for $\alpha > 0$, and the sign of the function c changes infinitely many times and the relative maxima of $|c(t)|$ diverge to infinity as $t \to \infty$ (see Figure 1 below).

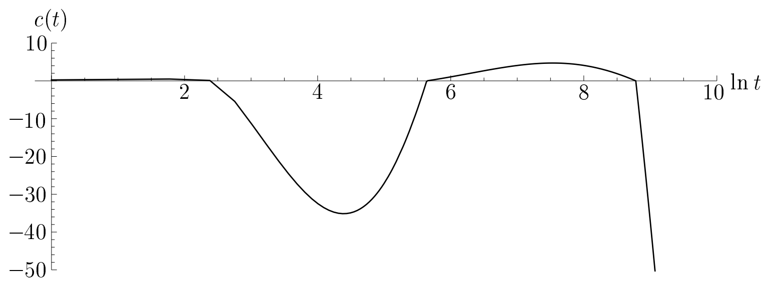

Fig. 1 The graph of the function $c(t)$ defined by (4.2)

Example 4.1 Equation (1.1) with (4.1) and (4.2) is nonoscillatory if $\alpha > 0$.

Using Corollary 2.2, we will verify Example 4.1. Let us make equation (2.3) which is compared with equation (1.1). Define

$$b(t)=\frac{1}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}}-c(t).$$

If $t_{2i-2}\le t\le t_{2i-1}$, then $0\le g(t)\le 1$ and hence,

$$b(t)=\frac{1}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}}\big(1-g(t)\big)\ge 0.$$

On the other hand, if $t_{2i-1}<t<t_{2i}$, then $-1\le g(t)<0$ and hence,

$$b(t)=\frac{1}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}}-\frac{5(2^4\alpha+1)}{24}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}}g(t)>0.$$

In either case, the function b is a nonnegative continuous function on $[1,\infty)$ satisfying

$$b(t)+c(t)=\frac{1}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}}>0$$

for $t\ge 1$; namely, condition (2.1). From the functions b, c and r given above, it follows that

$$\begin{aligned}\widetilde{R}(t)&=\frac{(b(t)+c(t))\,r^2(t)}{12}\\&=\frac{1}{12}\,\frac{1}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}}\,36\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^{-2} t^{\frac{14}{3}}=\frac{t^5}{2}.\end{aligned}$$

Let $f(t)=t^4$. Then we have

$$\begin{aligned}\widetilde{C}(t)&=6\,(b(t)+c(t))^2\,r(t)+\frac{12\,f^2(t)}{(b(t)+c(t))\,r^2(t)}-f'(t)\\&=\frac{1}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^4 t^{\frac{2}{3}}\,\frac{6}{\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}}\,t^{\frac{7}{3}}+\frac{6\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 12\,t^8}{\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}}\,36\,t^{\frac{14}{3}}}-4t^3\\&=\frac{2^6\alpha}{2^4\alpha+1}\,t^3+2t^3-4t^3=\frac{2^5\alpha-2}{2^4\alpha+1}\,t^3.\end{aligned}$$

Thus, we obtain the linear differential equation

$$\left(\frac{t^5}{2}y'\right)'+\frac{2^5\alpha-2}{2^4\alpha+1}\,t^3 y=0 \tag{4.3}$$

which is of equation (2.3) type.

To show that equation (4.3) is nonoscillatory, we will utilize Theorem C that was mentioned in Section 1. Since

$$\int_1^\infty \frac{1}{\widetilde{R}(t)}dt = \int_1^\infty \frac{2}{t^5}\,dt = \frac{1}{2} < \infty,$$

we see that condition (1.4) is satisfied with $a = 1$. In addition, we obtain

$$\begin{aligned}\overline{A}_2(t) &= \int_t^\infty \frac{1}{\widetilde{R}(s)}ds \int_1^t \widetilde{C}(s)ds = \lim_{T\to\infty}\left[-\frac{1}{2s^4}\right]_t^T \frac{2^5\alpha-2}{2^4\alpha+1}\int_1^t s^3 ds\\ &= \frac{1}{4}\left(\frac{2^4\alpha-1}{2^4\alpha+1}\right)\left(1-\frac{1}{t^4}\right)\end{aligned}$$

for $t \geq 1$. Taking into account that $\alpha > 0$, we see that

$$-\frac{1}{4} < \liminf_{t\to\infty} \overline{A}_2(t) = \limsup_{t\to\infty} \overline{A}_2(t) = \frac{1}{4}\left(\frac{2^4\alpha-1}{2^4\alpha+1}\right) < \frac{1}{4}.$$

Hence, condition (1.9) is also satisfied. From Theorem C it turns out that equation (4.3) is nonoscillatory for $\alpha > 0$.

By means of our comparison theorem (Corollary 2.2), we can conclude that equation (1.1) with (4.1) and (4.2) is nonoscillatory (see Figure 2 below).

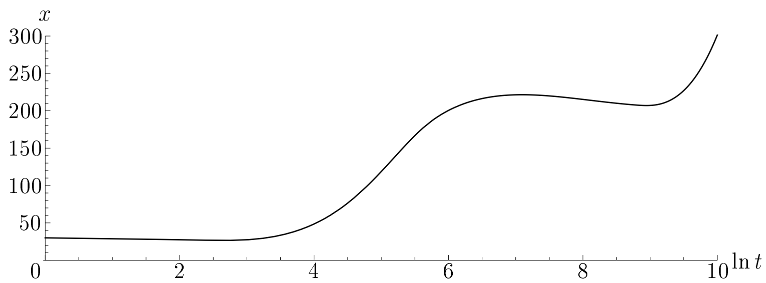

Fig. 2 The solution of (1.1) with (4.1) and (4.2) satisfying the initial condition that $x(1) = 30$ and $x'(1) = 0$

However, we cannot apply Theorem C directly to equation (1.1) under the assumptions (4.1) and (4.2). In other words, we cannot confirm Example 4.1 without Corollary 2.2. We will show this fact below.

It follows from (4.1) that

$$\int_1^\infty \frac{1}{r(t)}dt = \frac{1}{6}\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\int_1^\infty t^{-\frac{7}{3}}\,dt = \frac{1}{8}\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}} < \infty.$$

Since $g(t) \geq 0$ for $t_{2i-2} \leq t \leq t_{2i-1}$ and

$$\frac{1}{6} < \frac{5}{24} < \frac{5(2^4\alpha+1)}{24}$$

for $\alpha > 0$, we see that

$$c(t) < \frac{5(2^4\alpha+1)}{24}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 t^{\frac{1}{3}} g(t)$$

for $t_{2i-2} \le t \le t_{2i-1}$. Hence, by (4.2) we obtain

$$\begin{aligned}\overline{A}_2(t) &= \int_t^\infty \frac{1}{r(s)} ds \int_1^t c(s)ds = \lim_{T\to\infty}\left[-\frac{\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}}{8s^{\frac{4}{3}}}\right]_t^T \int_1^t c(s)ds \\ &< \frac{\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}}{8t^{\frac{4}{3}}} \frac{5(2^4\alpha+1)}{24}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 \int_1^t s^{\frac{1}{3}} g(s)ds \\ &= \frac{5\alpha}{3t^{\frac{4}{3}}}\int_1^t s^{\frac{1}{3}}\frac{4\sin(\ln s)+3\cos(\ln s)}{5}ds \\ &= \frac{\alpha}{3t^{\frac{4}{3}}}\int_1^t \frac{d}{ds}\left(3s^{\frac{4}{3}}\sin(\ln s)\right)ds = \alpha\sin(\ln t)\end{aligned}$$

for $t \ge 1$. From this estimation it turns out that $\liminf_{t\to\infty} \overline{A}_2(t) = -\alpha$. Thus, condition (1.4) is satisfied with $a = 1$. However, condition (1.9) is not satisfied provided that $\alpha \ge 3/4$. Hence, Theorem C cannot be applied to Example 4.1 directly.

As was mentioned in Section 1, equation (1.1) is divided into two cases (1.3) and (1.4). Example 4.1 corresponds to the case (1.4). Also, we can take an example corresponding to the case (1.3). Let

$$g(t) = \frac{3\cos(\ln t) - 16\sin(\ln t)}{\sqrt{265}}$$

for $t \ge 1$. Then it is clear that $|g(t)| = |\sin(\ln t + \theta)| \le 1$, where $\theta = \pi - \mathrm{Tan}^{-1}(3/16)$. We choose a sequence $\{t_n\}$ such that $t_0 = 1$ and $t_n = e^{n\pi-\theta}$ for $n \in \mathbb{N}$. Then it follows that

$$\begin{aligned} g(t) &\ge 0 \quad \text{for } t_{2i-2} \le t \le t_{2i-1}; \\ g(t) &< 0 \quad \text{for } t_{2i-1} < t < t_{2i} \end{aligned}$$

with $i \in \mathbb{N}$. Let α be any positive number. We define

$$r(t) = \frac{6}{\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\, t^{\frac{13}{3}}} > 0 \tag{4.4}$$

for $t \ge 1$ and

$$c(t) = \begin{cases} \dfrac{1}{6}\left(\sqrt[3]{\dfrac{2^6\alpha}{2^4\alpha+1}}\right)^2 \dfrac{g(t)}{t^{\frac{19}{3}}} & \text{if } t_{2i-2} \le t \le t_{2i-1}, \\ \dfrac{\sqrt{265}\,(2^4\alpha+1)}{6}\left(\sqrt[3]{\dfrac{2^6\alpha}{2^4\alpha+1}}\right)^2 \dfrac{g(t)}{t^{\frac{19}{3}}} & \text{if } t_{2i-1} < t < t_{2i} \end{cases} \tag{4.5}$$

with $i \in \mathbb{N}$. It is obvious that the function r satisfies condition (1.3) with $a = 1$ and the sign of the function c changes infinitely many times and the relative maxima of $|c(t)|$ converge to zero as $t \to \infty$ (see Figure 3 below).

Example 4.2 Equation (1.1) with (4.4) and (4.5) is nonoscillatory if $\alpha > 0$.

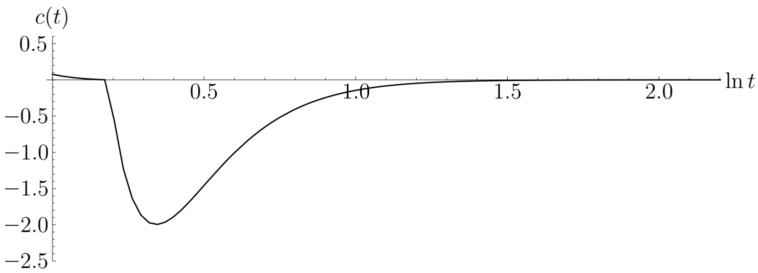

Fig. 3 The graph of the function $c(t)$ defined by (4.5)

Combining Corollary 2.2 with Theorem B, we can verify Example 4.2. Since we can discuss by the same manner as in Example 4.1, we omit the detailed calculation process and leave it to the reader. We define

$$b(t) = \frac{1}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2 \frac{1}{t^{\frac{19}{3}}} - c(t)$$

and $f(t) = 1/t^{16}$. Then we see that the function b continuous function on $[1,\infty)$ and condition (2.1) holds. By a straightforward computation, we can obtain

$$\widetilde{R}(t) = \frac{(b(t)+c(t))\,r^2(t)}{12} = \frac{1}{2t^{15}}$$

and

$$\begin{aligned}\widetilde{C}(t) &= 6\,(b(t)+c(t))^2\,r(t) + \frac{12f^2(t)}{(b(t)+c(t))\,r^2(t)} - f'(t)\\ &= \left(\frac{2^6\alpha}{2^4\alpha+1}+18\right)\frac{1}{t^{17}}.\end{aligned}$$

Hence, the coefficients $\widetilde{R}$ and $\widetilde{C}$ of (2.3) satisfy conditions (1.3) and (1.7) with $a = 1$, respectively. In addition, we obtain

$$A_2(t) = \int_1^t \frac{1}{\widetilde{R}(s)}ds\int_t^\infty \widetilde{C}(s)ds = \frac{1}{128}\left(\frac{2^6\alpha}{2^4\alpha+1}+18\right)\left(1-\frac{1}{t^{16}}\right)$$

for $t \geq 1$. Hence,

$$\liminf_{t\to\infty} A_2(t) = \limsup_{t\to\infty} A_2(t) = \frac{1}{128}\left(\frac{2^6\alpha}{2^4\alpha+1}+18\right).$$

Since

$$\frac{9}{64} < \frac{1}{128}\left(\frac{2^6\alpha}{2^4\alpha+1}+18\right) < \frac{11}{64} < \frac{1}{4},$$

condition (1.8) is also satisfied. Thus, it turns out from Theorem B that the linear differential equation

$$\left(\frac{y'}{2t^{15}}\right)' + \left(\frac{2^6\alpha}{2^4\alpha+1}+18\right)\frac{y}{t^{17}} = 0 \qquad (4.6)$$

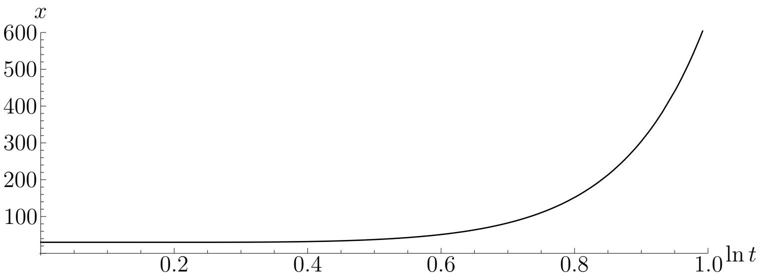

Fig. 4 The solution of (1.1) with (4.4) and (4.5) satisfying the initial condition that $x(1) = 30$ and $x'(1) = 0$

is nonoscillatory for $\alpha > 0$. Using Corollary 2.2 and comparing equation (1.1) with equation (4.6) under the assumptions (4.4) and (4.5), we can verify Example 4.2 (see Figure 4).

However, Example 4.2 cannot be confirmed using only Theorem B. In fact,

$$\int_1^t \frac{1}{r(s)}ds = \frac{1}{32}\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\,(t^{\frac{16}{3}}-1)$$

and

$$\int_t^\infty c(s)ds < \frac{\sqrt{265}\,(2^4\alpha+1)}{6}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2\int_t^\infty \frac{g(s)}{s^{\frac{19}{3}}}ds$$
$$= -\frac{2^4\alpha+1}{2}\left(\sqrt[3]{\frac{2^6\alpha}{2^4\alpha+1}}\right)^2\frac{\sin(\ln t)}{t^{\frac{16}{3}}}$$

for $t \geq 1$. Hence, we obtain

$$A_2(t) = \int_1^t \frac{1}{r(s)}ds\int_t^\infty c(s)ds < -\alpha\left(1-\frac{1}{t^{\frac{16}{3}}}\right)\sin(\ln t)$$

for $t \geq 1$. This means that $\liminf_{t\to\infty} A_2(t) \leq -\alpha$. Thus, condition (1.8) is not satisfied provided that $\alpha \geq 3/4$. We therefore conclude that Theorem B cannot be applied to Example 4.2 directly.

Conditions (1.8) and (1.9) give the upper and lower bounds for nonoscillation of (1.1) in Theorems B and C, respectively. These upper and lower bounds have been thought to be reasonable and proper by a lot of reports. However, Examples 4.1 and 4.2 illustrate that there is room for the improvement in the lower bound.

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