42. Continuation Results for Differential Equations without Uniqueness by Two Liapunov Functions

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1. Introduction. In [1] Bernfeld investigated the continuability of every solution $x(t; t_0, x_0)$ of a non-unique system

(1.1) $$x' = F(t, x)$$

where $F: R^1 \times R^d \to R^d$ is continuous. He used a Liapunov function which is radially unbounded for fixed t. In [2] the authors used two Liapunov functions which are not radially unbounded for fixed t to investigate the continuability of solutions on $[t_0, \infty)$ (existence in the future) under the assumption that solutions are unique. In this paper we combine the results in [1], [2] and extend our results in [2] to non-unique systems.

2. Main results. We consider a system

(2.1) $$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned}$$

where x, y are n and m-vectors respectively, and $f(t, x, y)$, $g(t, x, y)$ are continuous on $[0, \infty) \times R^n \times R^m$. For $K > 0$ let $S_K = \{y \in R^m : \|y\| \leqq K\}$ and $\Omega_K = \{(x, y) : \|x\| \leqq K,\ y \in R^m\}$.

Let $V(t, x, y)$ be a continuous scalar function satisfying a local Lipschitz condition. We define

$$\dot{V}_{(2.1)}(t, x, y) = \limsup_{h \to 0} (1/h)\{V(t+h, x+hf(t, x, y), y+hg(t, x, y)) - V(t, x, y)\}.$$

We say that a continuous scalar function $\phi: [0, \infty) \times R^1 \to R^1$ is of class $\mathcal{G}$ if, for every real $t_0 \geqq 0$ and u_0, the maximal solution $u(t; t_0, u_0)$ of the equation $u' = \phi(t, u)$ exists in the future.

We now state our main results.

Theorem 2.1. *Let $V: [0, \infty) \times R^n \times R^m \to R^1$ be locally Lipschitzian, satisfying*

(2.2) $V(t, x, y) \to \infty$ *as $\|y\| \to \infty$ uniformly in x for each fixed t,*

and there exists ϕ of class $\mathcal{G}$ such that

(2.3) $$\dot{V}_{(2.1)}(t, x, y) \leqq \phi(t, V(t, x, y)).$$

Moreover, suppose that for each $K > 0$ and $T > 0$, there exists $W: [0, T] \times R^n \times S_K \to R^1$, locally Lipschitzian, satisfying

(2.4) $W(t, x, y) \to \infty$ *as $\|x\| \to \infty$ for each fixed (t, y),*

and there exists ψ of class $\mathcal{G}$ such that

(2.5) $$\dot{W}_{(2.1)}(t, x, y) \leqq \psi(t, W(t, x, y)).$$

Then every solution of (2.1) *exists in the future.*

Let $D_{MT}=\{(x, y): V(T, x, y) \leqq M\}$ for each $M>0$ and $T>0$.

Theorem 2.2. *Let* $V:[0, \infty) \times \boldsymbol{R}^n \times \boldsymbol{R}^m \rightarrow \boldsymbol{R}^1$ *be locally Lipschitzian, satisfying*

(2.6) $$V(t, x, y) \rightarrow \infty \quad \text{as } \|y\| \rightarrow \infty \text{ for each fixed } (t, x),$$

and there exists ϕ *of class* $\mathcal{G}$ *such that*

(2.7) $$\dot{V}_{(2.1)}(t, x, y) \leqq \phi(t, V(t, x, y)).$$

Moreover, assume that D_{MT} *is unbounded for each* $T>0$ *and sufficiently large* $M>0$, *and there exists* $W:[0, T] \times D_{MT} \rightarrow \boldsymbol{R}^1$, *locally Lipschitzian, satisfying*

(2.8) $$W(t, x, y) \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty \text{ uniformly in } y \text{ for each fixed } t,$$

and there exists ψ *of class* $\mathcal{G}$ *such that*

(2.9) $$\dot{W}_{(2.1)}(t, x, y) \leqq \psi(t, W(t, x, y)).$$

Then every solution of (2.1) *exists in the future.*

Remark. If D_{MT} is bounded, then

(2.10) $$V(t, x, y) \rightarrow \infty \quad \text{as } \|x\|+\|y\| \rightarrow \infty \text{ for each fixed } t.$$

In this case (2.7) and (2.10) assure the global existence of every solution of (2.1) ([1], [3]). Thus we consider the case that D_{MT} is unbounded for each $T>0$ and sufficiently large $M>0$.

We give the following lemmas (Lemmas 3.1 and 3.2 in [2]) which are used in the proof of main results.

Let $E_{MT}=\{(x, y): W(T, x, y) \leqq M\}$ for each $M>0$ and $T>0$.

Lemma 2.1. *Let* $W:[0, T] \times \boldsymbol{R}^n \times S_K \rightarrow \boldsymbol{R}^1$ *be a continuous function, satisfying* (2.4) *in Theorem* 2.1. *Then for any* $M>0$ *the set* E_{MT} *is bounded.*

Lemma 2.2. *Let* $V:[0, \infty) \times \boldsymbol{R}^n \times \boldsymbol{R}^m \rightarrow \boldsymbol{R}^1$ *be a continuous function, satisfying* (2.6) *in Theorem* 2.2. *Then for any* $M>0$, $T>0$ *and* $K>0$, *the set* $D_{MT} \cap \Omega_K$ *is bounded.*

3. Proof. The following lemma is a generalized continuous dependence result for non-unique systems ([4], Theorems 3.6 and 5.1).

Lemma 3.1. *Let* F *be continuous on an open set* $D \subset \boldsymbol{R}^1 \times \boldsymbol{R}^d$. *Let* $(\tau_0, \xi_0) \in D$ *and suppose all solutions of* (1.1) *through* (τ_0, ξ_0) *exist on* $[a, b]$, $\tau_0 \in [a, b]$. *Then for each* $\varepsilon>0$, *there exists* $\delta(\varepsilon)>0$, *such that if* $d((\tau, \xi),(\tau_0, \xi_0))<\delta$, *then for each solution* $x(\cdot\,; \tau, \xi)$ *of* (1.1), *there exists a solution of* (1.1) *through* (τ_0, ξ_0), $x(\cdot\,; \tau_0, \xi_0)$, *such that* $\|x(t ; \tau, \xi)-x(t ; \tau_0, \xi_0)\|<\varepsilon$ *for all* $t \in [a, b]$.

Let $S_\alpha(t_0, z_0)$ be the intersection of the solution funnel through (t_0, z_0) and the hyperplane $t=\alpha$.

Proof of Theorem 2.1. Suppose that there exist $(\tau, \zeta) \in [0, \infty) \times \boldsymbol{R}^{n+m}$, $\omega>\tau$ and a solution $z(\cdot\,; \tau, \zeta)=(x(\cdot\,; \tau, \zeta), y(\cdot\,; \tau, \zeta))$ of (2.1) such that

$$\|z(t ; \tau, \zeta)\|=\|x(t ; \tau, \zeta)\|+\|y(t ; \tau, \zeta)\| \rightarrow \infty \qquad \text{as } t \rightarrow \omega^-,$$

and $z(\cdot\,;\tau,\zeta)$ is defined on $[\tau,\omega)$. There exists (t_0, z_0) such that $\tau<t_0<\omega$ and all solutions through (t_0, z_0) exist on $[t_0, \omega]$ by an application of local existence.

Define $z_1=z(t_0\,;\tau,\zeta)$ and $L=\{z_\lambda=\lambda z_1+(1-\lambda)z_0 : 0\leqq\lambda\leqq 1\}$, and let $\lambda_*=\sup\{\lambda : z(\omega\,; t_0, z_\mu)$ is finite for all $\mu : 0\leqq\mu<\lambda$ and for all solutions $z(\cdot)$ through $(t_0, z_\mu)\}$. By an application of Lemma 3.1, we see that $0<\lambda_*\leqq 1$.

We first claim that not all solutions through (t_0, z_{λ_*}) exist up to $t=\omega$. If they do, then by Lemma 3.1, there would exist a neighborhood of z_{λ_*} such that all solutions passing through that neighborhood exist at $t=\omega$, contradicting the definition of λ_*. This establishes the claim. Define $\omega_*=\sup\{T : \text{all solutions through } (t_0, z_{\lambda_*}) \text{ exist on } [t_0, T)\}$.

We next claim that $B=\bigcup_{0\leqq\lambda\leqq\lambda_*} S_{\omega_*}(t_0, z_\lambda)$ is unbounded. In fact, if $\bigcup_{0\leqq\lambda\leqq\lambda_*} S_{\omega_*}(t_0, z_\lambda)$ is bounded, $S_{\omega_*}(t_0, z_{\lambda_*})$ is unbounded by the same argument in [1].

Hence we can choose a sequence of solutions $\{z_n(\omega_*\,; t_0, z_{\lambda_n})\}$ such that

(3.1) $\quad \|z_n(\omega_*\,; t_0, z_{\lambda_n})\|=\|x_n(\omega_*\,; t_0, z_{\lambda_n})\|+\|y_n(\omega_*\,; t_0, z_{\lambda_n})\|\to\infty \quad \text{as } n\to\infty,$

where $0\leqq\lambda_n\leqq\lambda_*$, $\lambda_n\to\lambda_*$.

By the continuity of V, there exists $v_0>0$ such that

$$V(t_0, z_\lambda)\leqq v_0 \qquad \text{for all } 0\leqq\lambda\leqq 1.$$

Using the comparison theorem, we have from (2.3) that

$$V(\omega_*, x_n(\omega_*\,; t_0, z_{\lambda_n}), y_n(\omega_*\,; t_0, z_{\lambda_n}))\leqq v(\omega_*\,; t_0, v_0)$$

for all n, where $v(t\,; t_0, v_0)$ is the maximal solution of $v'=\phi(t, v)$ through (t_0, v_0). Hence it follows from (2.2) that there exists $K>0$ such that

$$\|y_n(\omega_*\,; t_0, z_{\lambda_n})\|\leqq K \qquad \text{for all } n.$$

Choose $w_0>0$ such that

$$W(t_0, z_\lambda)\leqq w_0 \qquad \text{for all } 0\leqq\lambda\leqq 1.$$

Let $w(t\,; t_0, w_0)$ be the maximal solution of $w'=\psi(t, w)$ through (t_0, w_0). Then from the differential inequality (2.5) we conclude that

$$W(\omega_*, x_n(\omega_*\,; t_0, z_{\lambda_n}), y_n(\omega_*\,; t_0, z_{\lambda_n}))\leqq N \qquad \text{for all } n$$

where $N=|w(\omega_*\,; t_0, w_0)|$, that is,

$$(x_n(\omega_*\,; t_0, z_{\lambda_n}), y_n(\omega_*\,; t_0, z_{\lambda_n}))\in E_{N\omega_*} \qquad \text{for all } n.$$

While $E_{N\omega_*}$ is bounded by Lemma 2.1. Hence $\|x_n(\omega_*\,; t_0, z_{\lambda_n})\|$ are bounded for all n. This contradicts (3.1) and the proof is completed.

The proof of Theorem 2.2 is similar to that of Theorem 2.1, and is omitted (cf. [2]).

4. Application. Consider the forced generalized Liénard equation

(4.1) $$x''+f(t, x, x')+g(x)h(x')=e(t)$$

or an equivalent system

(4.2) $$x'=y, \qquad y'=-f(t, x, y)-g(x)h(y)+e(t)$$

where $f:[0,\infty)\times \boldsymbol{R}^2\to\boldsymbol{R}^1$, $g:\boldsymbol{R}^1\to\boldsymbol{R}^1$, $h:\boldsymbol{R}^1\to(0,\infty)$ and $e:[0,\infty)\to\boldsymbol{R}^1$ are continuous. We assume the following conditions:

(i) $\qquad yf(t,x,y)\geqq 0 \qquad \text{for } (t,x,y)\in[0,\infty)\times\boldsymbol{R}^2,$

(ii) there exist $M>0$ and $m\geqq 0$ such that

$$\frac{|y|}{h(\eta)}\leqq m+M\int_0^y \frac{\eta}{h(\eta)}d\eta \qquad \text{for } y\in\boldsymbol{R}^1.$$

Under these conditions, we have

Theorem 4.1. *Suppose that there exists a positive number P such that*

$$\int_0^x g(\xi)d\xi\geqq -P \qquad \text{for } x\in\boldsymbol{R}^1, \tag{4.3}$$

$$\int_0^y \frac{\eta}{h(\eta)}d\eta\to\infty \qquad \text{as } |y|\to\infty. \tag{4.4}$$

Then every solution of (4.2) *exists in the future.*

Proof. Let

$$V(x,y)=\int_0^y \frac{\eta}{h(\eta)}d\eta+\int_0^x g(\xi)d\xi+P$$

and $W(x,y)=|x|$, then these satisfy (2.2) and (2.4) in Theorem 2.1, and we obtain

$$\begin{aligned} V_{(4.2)}(t,x,y)&=-yf(t,x,y)/h(y)+e(t)y/h(y)\\ &\leqq\left(m+M\int_0^y\frac{\eta}{h(\eta)}d\eta\right)|e(t)|\\ &\leqq M|e(t)|V+m|e(t)|, \end{aligned}$$

therefore (2.3) holds. Let $(t,x,y)\in[0,T]\times\boldsymbol{R}^1\times S_K$, then

$$\dot{W}_{(4.2)}(t,x,y)\leqq|y|\leqq K,$$

thus (2.5) is satisfied. Hence every solution of (4.2) exists in the future.

References

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