Asymptotic stability of coupled oscillators with time-dependent damping

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Abstract. The present paper is devoted to an investigation on the asymptotic stability for the damped oscillators with multiple degrees of freedom,

$$\mathbf{x}'' + h(t)\mathbf{x}' + A\mathbf{x} = \mathbf{0}$$

and its generalization

$$M\mathbf{x}'' + C(t)\mathbf{x}' + K\mathbf{x} = \mathbf{0},$$

where $h\colon [0,\infty) \to [0,\infty)$ is a function, A, M and K are $n \times n$ real constant matrices. and C is an $n \times n$ matrix whose elements are real-valued functions. The functions h and C correspond to the damping coefficient and the damping matrix, respectively. The origin $(\mathbf{x},\mathbf{x}') = (\mathbf{0},\mathbf{0})$ is the only equilibrium of the above-mentioned damped oscillators. Necessary and sufficient conditions are presented for the equilibrium of these oscillators to be asymptotically stable. The obtained conditions are given by the forms of certain growth conditions concerning the damping h and C, respectively.

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1. Introduction

The damped coupled oscillator

$$\mathbf{x}'' + h(t)\mathbf{x}' + A\mathbf{x} = \mathbf{0} \tag{1.1}$$

is one of very important models that continue to be researched from many angles in a wide range of fields which covers pure science, applied science, and technology. Here, $' = d/dt$, $\mathbf{x}$ is an n-dimensional vector, h is a nonnegative and locally integrable function on $[0,\infty)$, and A is a symmetric $n \times n$ real matrix.

For example, consider a mechanical system consisting of n objects as follows. The masses of the objects are identical. The value is m. The objects are coupled by $n+1$ springs, and the springs at both ends are attached to walls. The stiffness of the

i-th spring from the left is k_i $(i = 1, 2, \ldots, n+1)$. The objects of this system repeat a horizontal reciprocating motion over a flat floor. Rolling friction is caused between the objects and the floor. The friction coefficient of this horizontal surface can be varied and it can be expressed by the function c.

Let x_i $(i = 1, 2, \ldots, n)$ be the displacement of the i-th object from its equilibrium position. Then, the equation of motion of this mechanical system is

$$\begin{cases} mx_1'' + c(t)x_1' + k_1x_1 - k_2(x_2 - x_1) = 0, \\ mx_2'' + c(t)x_2' + k_2(x_2 - x_1) - k_3(x_3 - x_2) = 0, \\ mx_3'' + c(t)x_3' + k_3(x_3 - x_2) - k_4(x_4 - x_3) = 0, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ mx_{n-1}'' + c(t)x_{n-1}' + k_{n-1}(x_{n-1} - x_{n-2}) - k_n(x_n - x_{n-1}) = 0, \\ mx_n'' + c(t)x_n' + k_n(x_n - x_{n-1}) + k_{n+1}x_n = 0. \end{cases}$$

This system can be rewritten to the oscillator (1.1) with $h(t) = c(t)/m$,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \frac{k_1+k_2}{m} & -\frac{k_2}{m} & 0 & \ldots & 0 & 0 \\ -\frac{k_2}{m} & \frac{k_2+k_3}{m} & -\frac{k_3}{m} & \ldots & 0 & 0 \\ 0 & -\frac{k_3}{m} & \frac{k_3+k_4}{m} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{k_{n-1}+k_n}{m} & -\frac{k_n}{m} \\ 0 & 0 & 0 & \ldots & -\frac{k_n}{m} & \frac{k_n+k_{n+1}}{m} \end{pmatrix}.$$

As another application example of (1.1), we can cite the coupled pendulum. The pendulums of the same size are arranged at equal intervals in a row. The number of pendulums is n. The length of thread is ℓ. Each weight is horizontally connected to the next weight with a spring. The number of springs is $n-1$. Natural lengths of springs are equal to the spacing of pendulums. The stiffness of the i-th spring from the left is k_i $(i = 1, 2, \ldots, n-1)$. The friction works at the pivot points of the pendulums. The friction coefficient is expressed by the function c. When the displacements of the weights are small, the equation of motion is approximated to the oscillator (1.1) with $h(t) = c(t)/m$,

$$A = \begin{pmatrix} \frac{g}{\ell} + \frac{k_1}{m} & -\frac{k_1}{m} & 0 & \ldots & 0 & 0 \\ -\frac{k_1}{m} & \frac{g}{\ell} + \frac{k_1+k_2}{m} & -\frac{k_2}{m} & \ldots & 0 & 0 \\ 0 & -\frac{k_2}{m} & \frac{g}{\ell} + \frac{k_2+k_3}{m} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{g}{\ell} + \frac{k_{n-2}+k_{n-1}}{m} & -\frac{k_{n-1}}{m} \\ 0 & 0 & 0 & \ldots & -\frac{k_{n-1}}{m} & \frac{g}{\ell} + \frac{k_{n-1}}{m} \end{pmatrix},$$

where g is the acceleration of gravity. We can find other application examples of (1.1) to electric circuit theory and structural dynamics, etc. For example, refer to the books [7, 15, 17, 23].

The oscillator (1.1) has the only equilibrium $(\mathbf{x}, \mathbf{x}') = (\mathbf{0}, \mathbf{0})$. The purpose of this paper is to give a necessary and sufficient condition for the equilibrium of (1.1) to be asymptotically stable. The *asymptotic stability* of the equilibrium referred to here is that every solution $\mathbf{x}$ of (1.1) satisfies

$$\lim_{t\to\infty}\mathbf{x}(t) = \lim_{t\to\infty}\mathbf{x}'(t) = \mathbf{0}. \tag{1.2}$$

Strictly speaking, the above-mentioned explanation is the definition in which the equilibrium of (1.1) is *attractive*. As known well, however, the attractivity implies the asymptotic stability because the oscillator (1.1) is linear. Needless to say, the asymptotic stability for the oscillator (1.1) is a global properties of the equilibrium. About the definitions of stability and attractivity, refer to the books [4, 6, 16, 18, 24] for example. The study of the (global) asymptotic stability is one of main themes in the qualitative theory of differential equations.

To describe our result concerning the asymptotic stability of (1.1), we introduce two concepts as follows. A symmetric $n\times n$ real matrix P is said to be *positive definite* if $\mathbf{v}^{\mathrm{T}}P\mathbf{v}$ is positive for every nonzero column vector $\mathbf{v}$ of n real numbers. Here, $\mathbf{v}^{\mathrm{T}}$ denotes the transpose of $\mathbf{v}$. The damping coefficient h is said to belong to $\mathscr{F}_{[\mathrm{WIP}]}$ if

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying

$$\liminf_{n\to\infty}(\sigma_n - \tau_n) > 0 \quad \text{and} \quad 0 < \limsup_{n\to\infty}(\tau_{n+1} - \sigma_n) < \infty.$$

The concept of the positive definite matrix is known well in linear algebra and its applications. The concept of the weak integral positivity was first published in Hatvani [8]. It is clear that if h has a positive lower bound, then h belongs to $\mathscr{F}_{[\mathrm{WIP}]}$. There is a possibility that h belongs to $\mathscr{F}_{[\mathrm{WIP}]}$ even if $\liminf_{t\to\infty} h(t) = 0$. For example, $1/(1+t) \in \mathscr{F}_{[\mathrm{WIP}]}$ and $\sin^2 t/(1+t) \in \mathscr{F}_{[\mathrm{WIP}]}$ (for the proof, see [21, Proposition 2.1]). The following result is our main theorem.

Theorem 1.1 *Suppose that there exist an* $\varepsilon_0 > 0$ *and a* $\delta_0 > 0$ *such that* $|h(t) - h(s)| < \varepsilon_0$ *for all* $t \ge 0$ *and* $s \ge 0$ *with* $|t-s| < \delta_0$ *and suppose that* h *belongs to* $\mathscr{F}_{[\mathrm{WIP}]}$. *Then the equilibrium of* (1.1) *is asymptotically stable if and only if* A *is a positive definite matrix and*

$$\int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt = \infty, \tag{1.3}$$

where

$$H(t) = \int_0^t h(s)ds.$$

Remark 1.1 If h is uniformly continuous on $[0,\infty)$; namely, for any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $|h(t) - h(s)| < \varepsilon$ for all $t \ge 0$ and $s \ge 0$ with $|t-s| < \delta$, then the first assumption in Theorem 1.1 is satisfied with respect to any $\varepsilon_0 > 0$ and $\delta_0 = \delta(\varepsilon_0)$. Of course, the converse is not necessarily true.

Remark 1.2 If there exists an $\overline{h} > 0$ such that $0 \le h(t) \le \overline{h}$ for $t \ge 0$, then $|h(t) - h(s)| \le |h(t)| + |h(s)| \le 2\overline{h}$ for all $t \ge 0$ and $s \ge 0$. Hence, the first assumption of Theorem 1.1 is satisfied with respect to $\varepsilon_0 = 2\overline{h}$ and any $\delta_0 > 0$.

Remark 1.3 The first assumption of Theorem 1.1 may be satisfied even if h is a discontinuous function. For example, if

$$h(t) = \begin{cases} a & \text{if } t \in I_n, \\ b & \text{if } t \notin I_n \end{cases}$$

for $n \in \mathbb{N}$, where $0 < a < b$ and $\{I_n\}$ is a sequence of bounded intervals such that $I_i \cap I_j = \phi$ $(i \ne j)$, then the first assumption holds. This step function h belongs to $\mathscr{F}_{[\mathrm{WIP}]}$.

Remark 1.4 Condition (1.3) is the so-called growth condition on the damping coefficient h. Since

$$\int_0^\infty \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}}\, dt = \int_0^\infty \int_0^t e^{-(H(t)-H(s))} ds\, dt,$$

condition (1.3) can be expressed in the double integral. This double integral (1.3) was given for the first time by Smith [19]. He discussed the asymptotic stability of the equilibrium of a single degree of freedom system under the strong restriction condition that there exists an $\underline{h} > 0$ such that $h(t) \ge \underline{h}$ for $t \ge 0$. Because of this restriction, Smith's result cannot be applied to the case of $\liminf_{t\to\infty} h(t) = 0$. For this reason, many attempts were carried out to weaken this restriction. One of the attempts is a setting of the family of functions named $\mathscr{F}_{[\mathrm{WIP}]}$. The historical development of this research is concisely summarized in [10, 20].

Remark 1.5 It is known that the equilibrium of dynamical systems with one degree of freedom can be not become an asymptotically stable when the damping coefficient of the system increases fast or it decreases fast. In Theorem 1.1, the assumption that $h \in \mathscr{F}_{[\mathrm{WIP}]}$ prohibits too rapidly decline of h. On the other hand, condition (1.3) prohibits too rapidly growth of h. For example, if h is bounded or $h(t) = t$, then condition (1.3) holds; if $h(t) = t^2$, then condition (1.3) fails to hold. Zheng and the present author [26] discussed the issue what the upper limit of the growth rate which can guarantee that condition (1.3) is satisfied is.

As seen immediately from the definition of $\mathscr{F}_{[\mathrm{WIP}]}$, if h belongs to $\mathscr{F}_{[\mathrm{WIP}]}$, then

$$\lim_{t\to\infty} H(t) = \infty. \tag{1.4}$$

Hatvani et al. [11] proved that condition (1.3) is equivalent to

$$\sum_{n=1}^{\infty} \left(H^{-1}(n) - H^{-1}(n-1)\right)^2 = \infty$$

under the assumption (1.4), where

$$H^{-1}(r) = \min\{t \in \mathbb{R} : H(t) \ge r\}.$$

Note that the integral H is not necessarily strictly increasing because $h(t)$ is allowed to become zero at a certain t. For this reason, the inverse function H^{-1} may be discontinuous but it is strictly increasing on $[0,\infty)$.

Using their method, we can prove the following equivalence relation (we omit the proof).

Proposition 1.2 *Under the assumption* (1.4), *condition* (1.3) *holds if and only if*

$$\int_0^\infty \frac{\int_0^t e^{\rho H(s)}ds}{e^{\rho H(t)}}\,dt = \infty$$

for any $\rho > 0$.

2. Damped single oscillator

Consider the damped linear oscillator

$$x'' + h(t)x' + \omega^2 x = 0, \tag{2.1}$$

where ω is a positive constant and h is the same function given in Eq. (1.1). Needless to say, the only equilibrium of (2.1) is $(x,x') = (0,0)$.

The present author [20] has recently obtained a necessary and sufficient condition which guarantees that the equilibrium of damped nonlinear oscillators including Eq. (2.1) is globally asymptotically stable. By applying this result to Eq. (2.1), we can derive the following result.

Theorem A *Suppose that h is uniformly continuous, and it belongs to* $\mathscr{F}_{[\mathrm{WIP}]}$. *Then the equilibrium of* (2.1) *is asymptotically stable if and only if condition* (1.3) *holds.*

Theorem A contains many results of previous researches concerning the asymptotic stability for Eq. (2.1). We will try a further extension.

Theorem 2.1 *Suppose that there exist an* $\varepsilon_0 > 0$ *and a* $\delta_0 > 0$ *such that* $|h(t) - h(s)| < \varepsilon_0$ *for all* $t \geq 0$ *and* $s \geq 0$ *with* $|t-s| < \delta_0$ *and suppose that h belongs to* $\mathscr{F}_{[\mathrm{WIP}]}$. *Then the equilibrium of* (2.1) *is asymptotically stable if and only if condition* (1.3) *holds.*

Remark 2.1 To prove the necessity of Theorem 2.1, we will show that there exists a solution of (2.2) which does not approach the origin provided that

$$\int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt < \infty.$$

The necessity was proved in Smith [19, Theorem 1]. This method was given by Wintner [22]. Wintner's method was generalized so that nonlinear differential equations including Eq. (2.1) could be applied (see [20, Theorems 2.1 and 3.2]). We will prove the necessity by using another method.

Proof of Theorem 2.1 By putting $y = x'/\omega$ as a new variable, Eq. (2.1) becomes the planar system

$$\begin{aligned} x' &= \omega y, \\ y' &= -\omega x - h(t)y. \end{aligned} \tag{2.2}$$

System (2.2) has the zero solution $(x,y) \equiv (0,0)$, which corresponds to the equilibrium of (2.1). Hence, to prove Theorem 2.1, we have only to show that under the assumptions concerning h, every solution (x,y) of (2.2) approaches the origin $(0,0)$ as t tends to ∞ if and only if condition (1.3) holds.

Necessity. We can choose a $T \geq 0$ so large that

$$\int_T^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt < \frac{1}{2\omega^2}. \tag{2.3}$$

Consider the solution $(\tilde{x},\tilde{y})$ of (2.2) that passes through $(1,0)$ at $t = T$. Since $\tilde{x}'(T) = \omega\tilde{y}(T) = 0$ and $\tilde{y}'(T) = -\omega\tilde{x}(T) - h(t)\tilde{y}(T) = -\omega < 0$, it turns out that $(\tilde{x},\tilde{y})$ enters the fourth quadrant

$$Q_4 \stackrel{\text{def}}{=} \{(x,y) : x > 0 \text{ and } y < 0\}$$

in a right-hand neighborhood of $t = T$. Taking account of the vector field in Q_4, we see that $(\tilde{x},\tilde{y})$ does not move to the first quadrant

$$Q_1 \stackrel{\text{def}}{=} \{(x,y) : x > 0 \text{ and } y > 0\}$$

from Q_4 directly as t increases. Also, we see that $0 \leq \tilde{x}(t) < 1$ as long as $(\tilde{x},\tilde{y})$ is in Q_4.

Suppose that there exists a $T^* > T$ such that $\tilde{x}(T^*) = 1/2$ and $\tilde{x}(t) > 1/2$ for $T \leq t < T^*$. Since

$$\tilde{y}'(t) + h(t)\tilde{y}(t) = -\omega\tilde{x}(t) \geq -\omega$$

for $T \leq t < T^*$, it follows that

$$\left(e^{H(t)}\tilde{y}(t)\right)' \geq -\omega e^{H(t)} \quad \text{for } T \leq t < T^*.$$

Integrate both sides of this inequality from T to $t < T^*$ to obtain

$$e^{H(t)}\tilde{y}(t) \geq e^{H(T)}\tilde{y}(T) - \omega\int_T^t e^{H(s)}ds = -\omega\int_T^t e^{H(s)}ds.$$

Hence, by (2.2) we have

$$\tilde{x}'(t) = \omega\tilde{y}(t) \geq -\omega^2\frac{\int_T^t e^{H(s)}ds}{e^{H(t)}}$$

for $T \leq t < T^*$. From this estimation and (2.3) it follows that

$$\tilde{x}(T^*) \geq \tilde{x}(T) - \omega^2\int_T^{T^*}\frac{\int_T^t e^{H(s)}ds}{e^{H(t)}}\,dt \geq 1 - \omega^2\int_T^\infty\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt > \frac{1}{2}.$$

This contradicts the assumption that $\tilde{x}(T^*) = 1/2$. Hence, such T^* does not exist. This fact means that the solution $(\tilde{x},\tilde{y})$ of (2.2) stays in the region

$$\{(x,y) : 1/2 < x \leq 1 \text{ and } y \leq 0\}$$

for $t \geq T$. Thus, $(\tilde{x},\tilde{y})$ does not approach the origin.

Sufficiency. Let (x,y) be any solution of (2.1) with the initial time $t_0 \geq 0$ and define

$$v(t) = \frac{1}{2}\left(x^2(t) + y^2(t)\right). \tag{2.4}$$

Then, we have

$$v'(t) = x(t)x'(t) + y(t)y'(t) = -h(t)y^2(t) \leq 0$$

for $t \geq t_0$. Hence, v is a decreasing function on $[0,\infty)$. Since $v(t) \geq 0$ for $t \geq t_0$, there exists a limiting value $v^* \geq 0$. If $v^* = 0$, then it follows from (2.4) that the solution (x,y) of (2.2) tends to the origin as $t \to \infty$. This is our desired conclusion. Thus, we have only to show that the case in which $v^* > 0$ does not occur. By way of contradiction, we suppose that v^* is positive. Then, there exists a $T_1 \geq t_0$ such that

$$0 < v^* \leq v(t) \leq 2v^* \quad \text{for } t \geq T_1. \tag{2.5}$$

Hereafter, we will complete the proof of sufficiency in two steps. In the first step, we show that y approaches zero as $t \to \infty$. If $\lim_{t\to\infty} y(t) = 0$, then from (2.4) we see that $\lim_{t\to\infty} x(t) = \sqrt{2v^*} > 0$ or $\lim_{t\to\infty} x(t) = -\sqrt{2v^*} < 0$. In the second step, we will lead a contradiction.

Since $|y|$ is bounded, it has finite lower and upper limits. In the first step, we show that the inferior limit is zero, and then show that the superior limit is also zero. In the second step, we examine the movement of (x,y) in the whole x-y plane in details.

Step (1): We first suppose that $\liminf_{t\to\infty} |y(t)| > 0$. Then, we can choose a $\gamma > 0$ and a $T_2 \geq t_0$ such that $|y(t)| > \gamma$ for $t \geq T_2$. Hence, we have

$$v'(t) = -h(t)y^2(t) \leq -\gamma^2 h(t)$$

for $t \geq T_2$. Integrating this inequality from t_0 to t, we obtain

$$-v(t_0) < v^* - v(t_0) \leq v(t) - v(t_0) = \int_{t_0}^{t} v'(s)ds \leq -\gamma^2 \int_{T_2}^{t} h(s)ds.$$

However, the integral of h diverges to ∞ as t tends to ∞, because h belongs to $\mathscr{F}_{[\mathrm{WIP}]}$. Hence, this inequality does not hold. Thus, we conclude that $\liminf_{t\to\infty} |y(t)| = 0$.

Next, we suppose that $\limsup_{t\to\infty} |y(t)| \overset{\text{def}}{=} \mu > 0$. Let ε be so small enough as to satisfy the inequalities $0 < \varepsilon < \min\{\mu/2, \sqrt{v^*/2}\}$,

$$\frac{4\varepsilon}{\delta_0} + 2(1+2\varepsilon_0)\varepsilon < \sqrt{2\left(v^* - 2\varepsilon^2\right)}\,\omega. \tag{2.6}$$

Note that the left-hand side of this inequality approaches 0 and the right-hand side of this inequality approaches $\sqrt{2v^*}\omega$ as $\varepsilon \to 0$.. Hence, we can find a positive number ε which satisfies (2.6).

We can choose three sequences $\{s_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ with $T_1 < \tau_n < s_n < \sigma_n \leq \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ such that $|y(s_n)| = 2\varepsilon$, $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$ and

$$|y(t)| \geq \varepsilon \quad \text{for } \tau_n < t < \sigma_n, \tag{2.7}$$

$$|y(t)| \leq 2\varepsilon \quad \text{for } \sigma_n < t < \tau_{n+1}, \tag{2.8}$$

$$\varepsilon < |y(t)| < 2\varepsilon \quad \text{for } \tau_n < t < s_n. \tag{2.9}$$

In fact, since the inferior limit of $|y(t)|$ is zero, there exists a $t_* > T_1$ such that $|y(t_*)| < \varepsilon$. Because

$$\limsup_{t\to\infty} |y(t)| = \mu > 2\varepsilon,$$

we can choose numbers s_1, τ_1 and σ_1 such that $s_1 = \inf\{t > t_* : |y(t)| > 2\varepsilon\}$, $\tau_1 = \sup\{t < s_1 : |y(t)| < \varepsilon\}$ and $\sigma_1 = \inf\{t > s_1 : |y(t)| < \varepsilon\}$. It is clear that $|y(s_1)| = 2\varepsilon$, $|y(\tau_1)| = |y(\sigma_1)| = \varepsilon$ and $|y(t)| \ge \varepsilon$ for $\tau_1 < t < \sigma_1$. Using σ_1 instead of t_*, we define τ_2 and σ_2 similarly to τ_1 and σ_1, and so on. Then, we obtain three sequences $\{s_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ with $n \in \mathbb{N}$ such that $s_n = \inf\{t > \sigma_{n-1} : |y(t)| > 2\varepsilon\}$, $\tau_n = \sup\{t < s_n : |y(t)| < \varepsilon\}$ and $\sigma_n = \inf\{t > s_n : |y(t)| < \varepsilon\}$. It is also clear that $|y(s_n)| = 2\varepsilon$, $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$,

$$|y(t)| \ge \varepsilon \quad \text{for } \tau_n < t < \sigma_n,$$
$$|y(t)| \le 2\varepsilon \quad \text{for } \sigma_n < t < \tau_{n+1}$$

and

$$\varepsilon < |y(t)| < 2\varepsilon \quad \text{for } \tau_n < t < s_n.$$

Hence, the inequalities (2.7)–(2.9) are satisfied.

Using (2.9) and the second equality of (2.2), we can estimate that

$$\begin{aligned} 3\varepsilon^2 = y^2(s_n) - y^2(\tau_n) &= 2\int_{\tau_n}^{s_n} y(t)y'(t)dt \\ &= -2\omega\int_{\tau_n}^{s_n} x(t)y(t)dt - 2\int_{\tau_n}^{s_n} h(t)y^2(t)dt \\ &\le 2\omega\int_{\tau_n}^{s_n} |x(t)||y(t)|dt \le 4\varepsilon\omega\int_{\tau_n}^{s_n}|x(t)|dt. \end{aligned}$$

By (2.5), we have

$$|x(t)| \le \sqrt{2v(t)} \le 2\sqrt{v^*}$$

for $t \ge T_1$. Hence, we obtain

$$3\varepsilon^2 \le 4\varepsilon\omega\int_{\tau_n}^{s_n}|x(t)|dt \le 8\sqrt{v^*}\varepsilon\omega(s_n - \tau_n);$$

namely,

$$s_n - \tau_n \ge \frac{3\varepsilon}{8\sqrt{v^*}\omega} \stackrel{\text{def}}{=} m > 0$$

for each $n \in \mathbb{N}$. It is clear that the positive number m is independent of $n \in \mathbb{N}$. Since $[\tau_n, s_n] \subsetneq [\tau_n, \sigma_n]$, we see that $\liminf_{n\to\infty}(\sigma_n - \tau_n) \ge m > 0$.

From the assumption of $h(t)$ it follows that

$$|h(t) - h(\sigma_n)| < \varepsilon_0 \quad \text{for } \sigma_n - \delta_0 < t < \sigma_n + \delta_0. \tag{2.10}$$

Let us examine the value of $h(t)$ at $t = \sigma_n$ for each $n \in \mathbb{N}$. Define

$$S = \{n \in \mathbb{N} : h(\sigma_n) \ge 1 + \varepsilon_0\}.$$

We will show that the number of elements in the set S is finite. Suppose that the number of elements is infinite. Let card S denote the cardinal number of the set S. As shown above, $\tau_n + m < \sigma_n$ for each $n \in \mathbb{N}$. Let $\ell = \min\{\delta_0, m\}$. Then, from (2.7) and (2.10) it follows that

$$|y(t)| \ge \varepsilon \quad \text{for } \sigma_n - \ell \le t \le \sigma_n,$$

and that $n \in S$ implies

$$h(t) \geq 1 \quad \text{for } \sigma_n - \ell \leq t \leq \sigma_n.$$

Hence, we obtain

$$\int_{\sigma_n-\ell}^{\sigma_n} h(t)y^2(t)dt \geq \ell\varepsilon^2 \quad \text{if } n \in S.$$

Using this inequality, we get

$$\begin{aligned} v^* - v(t_0) \leq v(t) - v(t_0) &= \int_{t_0}^{t} v'(s)ds = -\int_{t_0}^{t} h(s)y^2(s)ds \\ &\leq -\sum_{n\in S}\int_{\sigma_n-\ell}^{\sigma_n} h(t)y^2(t)dt = -\ell\varepsilon^2 \operatorname{card} S = -\infty. \end{aligned}$$

This is a contradiction.

Since the number of elements in the set S is finite, we can find an $N \in \mathbb{N}$ such that

$$h(\sigma_n) < 1 + \varepsilon_0 \quad \text{for } n \geq N. \tag{2.11}$$

We next show that $\tau_{n+1} - \sigma_n \leq \delta_0$ for $n \geq N$. Suppose there exists an $n_0 \geq N$ such that

$$\sigma_{n_0} + \delta_0 < \tau_{n_0+1}. \tag{2.12}$$

From (2.4), (2.5) and (2.8), we obtain

$$\frac{1}{2}x^2(t) = v(t) - \frac{1}{2}y^2(t) \geq v^* - 2\varepsilon^2 \stackrel{\text{def}}{=} w^*$$

for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$. Note that w^* is positive because $0 < \varepsilon < \sqrt{2v^*}$. We proceed the proof by dividing into two cases: (a) $x(t) \geq \sqrt{2w^*} > 0$ for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$; (b) $x(t) \leq -\sqrt{2w^*} < 0$ for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$. Note that

$$h(t) < \varepsilon_0 + h(\sigma_{n_0}) < 1 + 2\varepsilon_0 \quad \text{for } \sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta_0$$

because of (2.10) and (2.11). In the former case, using (2.6) and (2.8) with the second equation of (2.2), we get

$$\begin{aligned} y'(t) &= -\omega x(t) - h(t)y(t) \leq -\sqrt{2w^*}\omega + h(t)|y(t)| \\ &\leq -\sqrt{2w^*}\omega + 2(1+2\varepsilon_0)\varepsilon < -\frac{4\varepsilon}{\delta_0} \end{aligned}$$

for $\sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta_0$. In the latter case, we get

$$\begin{aligned} y'(t) &= -\omega x(t) - h(t)y(t) \geq \sqrt{2w^*}\omega - h(t)|y(t)| \\ &\geq \sqrt{2w^*}\omega - 2(1+2\varepsilon_0)\varepsilon > \frac{4\varepsilon}{\delta_0} \end{aligned}$$

for $\sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta_0$. Thus, in either case, we have

$$|y'(t)| > \frac{4\varepsilon}{\delta_0} \quad \text{for } \sigma_{n_0} \leq t \leq \tau_{n_0+1}.$$

Taking (2.12) into account and integrating this inequality from σ_{n_0} to $\sigma_{n_0}+\delta_0$, we obtain

$$|y(\sigma_{n_0}+\delta_0)|+|y(\sigma_{n_0})| \ge \left|\int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta_0} y'(t)dt\right| = \int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta_0} |y'(t)|dt > 4\varepsilon.$$

However, it follows from (2.8) that

$$|y(\sigma_{n_0}+\delta_0)|+|y(\sigma_{n_0})| \le 4\varepsilon.$$

This is a contradiction. We therefore conclude that $\limsup_{n\to\infty}(\tau_{n+1}-\sigma_n) \le \delta_0 < \infty$.

From how to choose sequences $\{\tau_n\}$ and $\{\sigma_n\}$, we see that

$$0 < \limsup_{n\to\infty}(\tau_{n+1}-\sigma_n).$$

Recall that $\liminf_{n\to\infty}(\sigma_n-\tau_n) \ge m > 0$. Since h belongs to $\mathscr{F}_{[\mathrm{WIP}]}$, we conclude that

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt = \infty. \tag{2.13}$$

On the other hand, it follows from (2.7) that

$$\int_{t_0}^{\infty} v'(t)dt = -\int_{t_0}^{\infty} h(t)y^2(t)dt \le -\varepsilon^2\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt.$$

Since

$$\int_{t_0}^{\infty} v'(t)dt = \lim_{t\to\infty} v(t) - v(t_0) = v^* - v(t_0) < 0,$$

we obtain

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt \le \frac{v(t_0)-v^*}{\varepsilon^2} < \infty.$$

This contradicts (2.13). Thus, we conclude that $\limsup_{t\to\infty}|y(t)| = \mu = 0$. The proof of Step (1) is now complete.

Step (2): From the conclusion of Step (1) it follows that $\lim_{t\to\infty}x(t) = \sqrt{2v^*} > 0$ or $\lim_{t\to\infty}x(t) = -\sqrt{2v^*} < 0$. Taking into account of the vector field of (2.2), we see that the solution (x,y) has to approach the point $\left(\sqrt{2v^*},0\right)$ or the point $\left(-\sqrt{2v^*},0\right)$ by passing through the region

$$\left\{(x,y) : x > \sqrt{2v^*} \text{ and } y < 0\right\}$$

or the region

$$\left\{(x,y) : x < -\sqrt{2v^*} \text{ and } y > 0\right\}$$

ultimately. Hence, we can find a $T_3 \ge t_0$ such that

$$x(t) > \sqrt{2v^*} \quad \text{and} \quad y(t) < 0 \quad \text{for } t \ge T_3 \tag{2.14}$$

or

$$x(t) < -\sqrt{2v^*} \quad \text{and} \quad y(t) > 0 \quad \text{for } t \ge T_3. \tag{2.15}$$

We consider only the former, because the latter is carried out in the same way by using (2.15) instead of (2.14). In the former, by (2.14) we have

$$y'(t) + h(t)y(t) = -\omega x(t) < -\sqrt{2v^*}\omega < 0$$

for $t \geq T_3$. Hence, by (2.14) again, we get

$$y(t) < y(t) - e^{(H(T_3)-H(t))}y(T_3) < -\sqrt{2v^*}\,\omega\,\frac{\int_{T_3}^t e^{H(s)}ds}{e^{H(t)}}$$

for $t \geq T_3$. From this inequality it follows that

$$x'(t) = \omega y(t) < -\sqrt{2v^*}\,\omega^2\,\frac{\int_{T_3}^t e^{H(s)}ds}{e^{H(t)}}$$

for $t \geq T_3$. Integrating this inequality from T_3 to t, we obtain

$$x(t) < -\sqrt{2v^*}\,\omega^2\int_{T_3}^t \frac{\int_{T_3}^s e^{H(\tau)}d\tau}{e^{H(s)}}\,ds + x(T_3).$$

Since $h(t) \geq 0$ for $t \geq 0$, it is clear that

$$\int_0^\infty e^{H(t)}dt = \infty.$$

Hence, there exists a $T_4 \geq T_3$ such that

$$\int_{T_3}^s e^{H(\tau)}d\tau > \frac{1}{2}\int_0^s e^{H(\tau)}d\tau \quad \text{for } s \geq T_4.$$

Using this inequality, we can evaluate that

$$\sqrt{2v^*} < x(t) < -\sqrt{\frac{v^*}{2}}\,\omega^2\int_{T_4}^t \frac{\int_0^s e^{H(\tau)}d\tau}{e^{H(s)}}\,ds - \sqrt{2v^*}\,\omega^2\int_{T_3}^{T_4} \frac{\int_{T_3}^s e^{H(\tau)}d\tau}{e^{H(s)}}\,ds + x(T_3)$$

for $t \geq T_4$. This contradicts condition (1.3). The proof of Step (2) is now complete.
Theorem 2.1 is thus proved. □

Remark 2.2 The damping coefficient h is said to belong to $\mathscr{F}_{[\mathrm{IP}]}$ if

$$\sum_{n=1}^\infty \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying

$$\liminf_{n\to\infty}(\sigma_n - \tau_n) > 0.$$

The integral positivity was introduced by Matrosov [14]. The concept of the integral positivity is quite strong than that of the weak integral positivity. For example, the functions $1/(1+t)$ and $\sin^2 t/(1+t)$ belong to $\mathscr{F}_{[\mathrm{WIP}]}$, but these functions do not belong to $\mathscr{F}_{[\mathrm{IP}]}$. In the step (1) of the sufficiency, we proved that $\lim_{t\to\infty}|y(t)| = 0$. If h belongs to $\mathscr{F}_{[\mathrm{IP}]}$, the convergence of $y(t)$ is shown even in more general mechanical systems (for example, see [9, 20]).

The term $\omega^2 x$ in Eq. (1.1) expresses the restoring force. Since the restoring force is power that returns the object to the original position when the object is displaced slightly from the equilibrium, we assumed the spring constant (named ω^2) to be positive in the argument above. Of course, from a mathematical interest, we may treat the case that the spring constant is negative.

Proposition 2.2 *Consider the equation*

$$x'' + h(t)x' + \lambda x = 0. \tag{2.16}$$

Then the equilibrium is not asymptotically stable if $\lambda < 0$.

Proof Let φ be any solution of (2.16) with the initial time $t_0 \geq 0$. Consider the case that $(\varphi(t_0), \varphi'(t_0))$ is in the first quadrant Q_1. We can easily confirm that Q_1 is a positively invariant set for the system

$$\begin{aligned} x' &= y, \\ y' &= -\lambda x - h(t)y \end{aligned} \tag{2.17}$$

which is equivalent to Eq. (2.16). Let (x,y) be a solution of (2.17), which corresponds to the solution φ. Then, from the vector field of (2.17), we see that the solution curve of (x,y) moves from the left to the right through Q_1. Hence, it turns out that $x(t) > x(t_0)$ and $y(t) > 0$ for $t > t_0$, and therefore, the solution (x,y) does not approach the origin as t tends to ∞. This means that the equilibrium of (2.16) is not asymptotically stable. □

Remark 2.3 In the proof of Proposition 2.2, we do not use the assumption that h is a nonnegative function on $[0,\infty)$.

Remark 2.4 All nontrivial solutions (x,y) of (2.17) are not necessarily unbounded. There is a possibility that a solution curve of (2.17) approaches a certain point other than the origin on the x-axis.

Remark 2.5 In the case that $\lambda = 0$, the equilibrium of (2.16) is not asymptotically stable.

3. Proof of the main result

Since A is a symmetric $n \times n$ real matrix, all eigenvalues of A are real numbers. As a basic knowledge of linear algebra, it is well-known that A is a positive definite matrix if and only if all of its eigenvalues are positive.

Proof of Theorem 1.1 For $i = 1, 2, \ldots, n$, let λ_i be an eigenvalue of A and $\mathbf{v}_i$ be an eigenvector for A corresponding to the eigenvalue λ_i. Let B be the $n \times n$ matrix such that

$$B = (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n)^{\mathrm{T}}.$$

Then, it is clear that $\det B \neq 0$. Multiplying Eq. (1.1) by the matrix B from the left, we get

$$B\mathbf{x}'' + h(t)B\mathbf{x}' + BA\mathbf{x} = \mathbf{0}. \tag{3.1}$$

Taking into account that $(A\mathbf{v}_i)^{\mathrm{T}} = \mathbf{v}_i^{\mathrm{T}} A^{\mathrm{T}}$ and $(A\mathbf{v}_i)^{\mathrm{T}} = (\lambda_i \mathbf{v}_i)^{\mathrm{T}} = \lambda_i \mathbf{v}_i^{\mathrm{T}}$ for $i = 1, 2, \dots,$ n, we see that

$$BA = BA^{\mathrm{T}} = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}} \end{pmatrix} A^{\mathrm{T}} = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}} A^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} A^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}} A^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \mathbf{v}_1^{\mathrm{T}} \\ \lambda_2 \mathbf{v}_2^{\mathrm{T}} \\ \vdots \\ \lambda_n \mathbf{v}_n^{\mathrm{T}} \end{pmatrix}.$$

Let $y_i = \mathbf{v}_i^{\mathrm{T}} \mathbf{x}$. Then, Eq. (3.1) becomes the isolated system of second-order differential equations

$$\begin{cases} y_1'' + h(t) y_1' + \lambda_1 y_1 = 0, \\ y_2'' + h(t) y_2' + \lambda_2 y_2 = 0, \\ \dots\dots\dots\dots\dots \\ y_n'' + h(t) y_n' + \lambda_n y_n = 0, \end{cases} \tag{3.2}$$

because

$$B\mathbf{x}' = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}} \mathbf{x}' \\ \mathbf{v}_2^{\mathrm{T}} \mathbf{x}' \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}} \mathbf{x}' \end{pmatrix} = \begin{pmatrix} (\mathbf{v}_1^{\mathrm{T}} \mathbf{x})' \\ (\mathbf{v}_2^{\mathrm{T}} \mathbf{x})' \\ \vdots \\ (\mathbf{v}_n^{\mathrm{T}} \mathbf{x})' \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix},$$

and

$$B\mathbf{x}'' = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ \vdots \\ x_n'' \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}} \mathbf{x}'' \\ \mathbf{v}_2^{\mathrm{T}} \mathbf{x}'' \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}} \mathbf{x}'' \end{pmatrix} = \begin{pmatrix} (\mathbf{v}_1^{\mathrm{T}} \mathbf{x})'' \\ (\mathbf{v}_2^{\mathrm{T}} \mathbf{x})'' \\ \vdots \\ (\mathbf{v}_n^{\mathrm{T}} \mathbf{x})'' \end{pmatrix} = \begin{pmatrix} y_1'' \\ y_2'' \\ \vdots \\ y_n'' \end{pmatrix}.$$

Note that the system (3.2) consists of N oscillators which are not coupled to each other. Let $\mathbf{y} = (y_1, y_2, \dots, y_n)^{\mathrm{T}}$. Then, we can rewrite system (3.2) as

$$\mathbf{y}'' + h(t)\mathbf{y}' + D\mathbf{y} = \mathbf{0}, \tag{3.3}$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Sufficiency. Since A is a positive definite matrix, all eigenvalues are positive. Hence, we may denote the eigenvalues by $\omega_i^2 > 0$ instead of λ_i. By virtue of Theorem 2.1, we conclude that every solution y_i of the single oscillator

$$y_i'' + h(t) y_i' + \omega_i^2 y_i = 0, \qquad i = 1, 2, \dots, n \tag{3.4}$$

and its derivative y_i' approach zero as t tends to ∞. Hence, every solution $\mathbf{y}$ of (3.3) satisfies

$$\lim_{t\to\infty} \mathbf{y}(t) = \lim_{t\to\infty} \mathbf{y}'(t) = 0.$$

Since

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}}\mathbf{x} \\ \mathbf{v}_2^{\mathrm{T}}\mathbf{x} \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}}\mathbf{x} \end{pmatrix} = (\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_n)^{\mathrm{T}}\mathbf{x} = B\mathbf{x},$$

it follows that $\mathbf{x} = B^{-1}\mathbf{y}$. Hence, every solution $\mathbf{x}(t)$ of (3.1) satisfies condition (1.2). We therefore conclude that the equilibrium of (1.1) is asymptotically stable.

Necessity. If A is not a positive definite matrix though it is symmetric, then we can find an integer j with $1 \le j \le n$ such that $\lambda_j \le 0$. Hence, from Proposition 2.2 and Remark 2.3, it turns out that a solution $y_j(t)$ of the equation

$$y_j'' + h(t)y_j' + \lambda_j y_j = 0$$

or its derivative $y_j'(t)$ do not approach zero as $t \to \infty$. Since $\mathbf{y} = (y_1, y_2, \ldots, \ldots, y_n)^{\mathrm{T}} = B\mathbf{x}$ and $\det B \neq 0$, we can choose a solution $\mathbf{x}(t)$ of (3.1) which does not satisfy condition (1.2). Thus, the equilibrium of (1.1) is not asymptotically stable.

If A is a positive definite matrix, then all the eigenvalues of A is real and positive. Hence, we may denote the eigenvalues by $\omega_i^2 > 0$ $(i = 1, 2, \ldots, n)$. If condition (1.3) does not hold, then by means of Theorem 2.1, we see that the equilibrium of (3.4) is not asymptotically stable for all i. Hence, the equilibrium of (3.3) is also not asymptotically stable. Recall that Eq. (3.1) is equivalent to Eq. (3.3), because $\mathbf{y} = B\mathbf{x}$ and $BA = DB$. Since $\det B \neq 0$, we conclude that the equilibrium of (1.1) is not asymptotically stable. □

4. Generalization to proportional viscous damping systems

The equation of motion of dynamical systems with multiple degrees of freedom can be written in matrix form as:

$$M\mathbf{x}'' + C\mathbf{x}' + K\mathbf{x} = \mathbf{0},$$

where M, C and K are $n \times n$ real constant matrices. In mechanical, civil, architectural, and other fields of engineering, the matrices M, C and K are called the mass, damping and stiffness matrices, respectively. For example, refer to [3, 12, 13, 23, 25]. It is often assumed that these matrices have the relation that

$$C = \alpha M + \beta K,$$

where α and β are positive numbers. When the damping matrix C is represented by such a linear combination of the mass matrix M and the stiffness matrix K, this system is said to be a proportional viscous damping model. As to proportional viscous damping, see [1, 2, 5] for example.

In this section, we will attempt to extend Theorem 1.1 to be able to apply to the time-varying system

$$M\mathbf{x}'' + C(t)\mathbf{x}' + K\mathbf{x} = \mathbf{0}, \tag{4.1}$$

in which M is an $n \times n$ real regular matrix, $M^{-1}K$ is a symmetric $n \times n$ real matrix and C is an $n \times n$ matrix whose elements are real-valued functions. We assume that

$$C(t) = f(t)M + g(t)K \quad \text{for} \quad t \ge 0, \tag{4.2}$$

where $f, g\colon [0,\infty) \to [0,\infty)$ is locally integrable functions. Let

$$\tilde{h}(t) = f(t) + g(t) \quad \text{and} \quad \widetilde{H}(t) = \int_0^t \tilde{h}(s)ds$$

for $t \geq 0$. Then, we have the following result.

Theorem 4.1 *Suppose that there exist an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that $|f(t) - f(s)| < \varepsilon_0$ and $|g(t) - g(s)| < \varepsilon_0$ for all $t \geq 0$ and $s \geq 0$ with $|t - s| < \delta_0$ and suppose that $\tilde{h}$ belongs to $\mathscr{F}_{[\mathrm{WIP}]}$. Then, under the assumption* (4.2)*, the equilibrium of* (4.1) *is asymptotically stable if and only if $M^{-1}K$ is a positive definite matrix and*

$$\int_0^\infty \frac{\int_0^t e^{\widetilde{H}(s)}ds}{e^{\widetilde{H}(t)}}\, dt = \infty. \tag{4.3}$$

Proof Let λ_1, λ_2, ..., λ_n be eigenvalues of $M^{-1}K$. Since $M^{-1}K$ is a symmetric $n \times n$ real matrix, all eigenvalues λ_i $(i = 1, 2, \ldots, n)$ are real numbers. Let $\mathbf{v}_i$ be an eigenvector for $M^{-1}K$ corresponding to the eigenvalue λ_i and let B be the $n \times n$ matrix such that

$$B = (\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \cdots \;\; \mathbf{v}_n)^{\mathrm{T}}.$$

Then, by the same manner as in the proof of Theorem 1.1, we can confirm that $BM^{-1}K = DB$, where

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.$$

Multiplying Eq. (4.1) by the matrix BM^{-1} from the left, we get

$$BE\mathbf{x}'' + BM^{-1}C(t)\mathbf{x}' + BM^{-1}K\mathbf{x} = \mathbf{0},$$

where E is the $n \times n$ identity matrix. From (4.2) it follows that

$$BM^{-1}C(t) = f(t)B + g(t)BM^{-1}K \quad \text{for } t \geq 0.$$

Since $BM^{-1}K = DB$, we see that Eq. (4.1) is equivalent to

$$(B\mathbf{x})'' + \big(f(t)E + g(t)D\big)(B\mathbf{x})' + D(B\mathbf{x}) = \mathbf{0} \tag{4.4}$$

provided that (4.2) holds. Let $y_i = \mathbf{v}_i^{\mathrm{T}}\mathbf{x}$ $(i = 1, 2, \ldots, n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} = B\mathbf{x}$. Then, we obtain

$$\mathbf{y}'' + \big(f(t)E + g(t)D\big)\mathbf{y}' + D\mathbf{y} = \mathbf{0}; \tag{4.5}$$

namely, the isolated system of second-order differential equations

$$\begin{cases} y_1'' + \big(f(t) + \lambda_1 g(t)\big)y_1' + \lambda_1 y_1 = 0, \\ y_2'' + \big(f(t) + \lambda_2 g(t)\big)y_2' + \lambda_2 y_2 = 0, \\ \qquad \cdots\cdots\cdots\cdots\cdots \\ y_n'' + \big(f(t) + \lambda_n g(t)\big)y_n' + \lambda_n y_n = 0. \end{cases}$$

Sufficiency. Since $M^{-1}K$ is a positive definite matrix, all eigenvalues λ_i $(i = 1, 2, \ldots, n)$ are positive. For $i = 1, 2, \ldots, n$, let $\varepsilon_i = (1 + \lambda_i)\varepsilon_0$. Then, by the assumption of f and g, we have

$$\big| f(t) + \lambda_i g(t) - \big(f(s) + \lambda_i g(s)\big)\big| \le |f(t) - f(s)| + \lambda_i\, |g(t) - g(s)| < \varepsilon_i$$

for all $t \ge 0$ and $s \ge 0$ with $|t - s| < \delta_0$.

If $\lambda_i \ge 1$ for some i, then

$$\frac{1}{\lambda_i}\big(f(t) + \lambda_i g(t)\big) \le \tilde{h}(t) \le f(t) + \lambda_i g(t)$$

for $t \ge 0$, because $f(t) \ge 0$ and $g(t) \ge 0$ for $t \ge 0$. Since $\tilde{h}$ belongs to $\mathscr{F}_{[\mathrm{WIP}]}$, we see that

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} \big(f(t) + \lambda_i g(t)\big)\, dt \ge \sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} \tilde{h}(t) dt = \infty.$$

Hence, $f + \lambda_i g$ also belongs to $\mathscr{F}_{[\mathrm{WIP}]}$. Let

$$F(t) = \int_0^t f(s) ds \quad \text{and} \quad G(t) = \int_0^t g(s) ds.$$

Then, we have

$$\begin{aligned} \widetilde{H}(t) - \widetilde{H}(s) &= \int_s^t \tilde{h}(s) ds \ge \frac{1}{\lambda_i} \int_s^t \big(f(\tau) + \lambda_i g(\tau)\big)\, d\tau \\ &= \frac{1}{\lambda_i}\Big(F(t) - F(s) + \lambda_i\big(G(t) - G(s)\big)\Big), \end{aligned}$$

and therefore,

$$\begin{aligned} \int_0^\infty \frac{\int_0^t e^{F(s)+\lambda_i G(s)} ds}{e^{F(t)+\lambda_i G(t)}}\, dt &= \int_0^\infty \int_0^t e^{-(F(t)-F(s)+\lambda_i (G(t)-G(s)))} ds\, dt \\ &\ge \int_0^\infty \int_0^t e^{-\lambda_i(\widetilde{H}(t)-\widetilde{H}(s))} ds\, dt = \int_0^\infty \frac{\int_0^t e^{\lambda_i \widetilde{H}(s)} ds}{e^{\lambda_i \widetilde{H}(t)}}\, dt. \end{aligned}$$

From (4.3) and Proposition 1.2, we see that

$$\int_0^\infty \frac{\int_0^t e^{\lambda_i \widetilde{H}(s)} ds}{e^{\lambda_i \widetilde{H}(t)}}\, dt = \infty.$$

Hence, by means of Theorem 2.1, we conclude that every solution y_i of the single oscillator

$$y_i'' + \big(f(t) + \lambda_i g(t)\big) y_i' + \lambda_i y_i = 0, \qquad i = 1, 2, \ldots, n \tag{4.6}$$

and its derivative y_i' approach zero as t tends to ∞.

If $0 < \lambda_i < 1$ for some i, then

$$f(t) + \lambda_i g(t) \le \tilde{h}(t) \le \frac{1}{\lambda_i}\big(f(t) + \lambda_i g(t)\big)$$

for $t \ge 0$, because $f(t) \ge 0$ and $g(t) \ge 0$ for $t \ge 0$. Hence, $\tilde{h} \in \mathscr{F}_{[\mathrm{WIP}]}$ implies $\lambda_i(f+g) \in \mathscr{F}_{[\mathrm{WIP}]}$ and $f+\lambda_i g \in \mathscr{F}_{[\mathrm{WIP}]}$. Also, we see that

$$\begin{aligned}\widetilde{H}(t)-\widetilde{H}(s) &= \int_s^t \tilde{h}(s)ds \ge \int_s^t \big(f(\tau)+\lambda_i g(\tau)\big)\,d\tau \\ &= F(t)-F(s)+\lambda_i\big(G(t)-G(s)\big).\end{aligned}$$

From this estimation and (4.3), we obtain

$$\begin{aligned}\int_0^\infty \frac{\int_0^t e^{F(s)+\lambda_i G(s)}ds}{e^{F(t)+\lambda_i G(t)}}\,dt &= \int_0^\infty\!\!\int_0^t e^{-(F(t)-F(s)+\lambda_i(G(t)-G(s)))}ds\,dt \\ &\ge \int_0^\infty\!\!\int_0^t e^{-(\widetilde{H}(t)-\widetilde{H}(s))}ds\,dt = \int_0^\infty \frac{\int_0^t e^{\widetilde{H}(s)}ds}{e^{\widetilde{H}(t)}}\,dt = \infty.\end{aligned}$$

Hence, by virtue of Theorem 2.1, we conclude that every solution y_i of (4.6) and its derivative y_i' approach zero as t tends to ∞.

Thus, every solution $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ of (4.5) satisfies

$$\lim_{t\to\infty} \mathbf{y}(t) = \lim_{t\to\infty} \mathbf{y}'(t) = 0.$$

Taking into account that $\mathbf{y} = B\mathbf{x}$ and $\det B \neq 0$, we see that every solution $\mathbf{x}$ of (4.4) satisfies condition (1.2). We therefore conclude that the equilibrium of (4.1) is asymptotically stable provided that (4.2) holds.

Necessity. There are two cases to be considered: (i) $M^{-1}K$ is not a positive definite matrix; (ii) $M^{-1}K$ is a positive definite matrix and condition (1.3) does not hold. In both cases, we can proceed our argument by the same way as the proof of the necessity of Theorem 1.1.

Case (i). There exists a nonpositive eigenvalue λ_j of $M^{-1}K$ for some integer j with $1 \le j \le n$. Proposition 2.2 and Remark 2.3 assert that the equilibrium of the equation

$$y_j'' + \big(f(t)+\lambda_j g(t)\big)y_j' + \lambda_j y_j = 0$$

is not asymptotically stable. Since $\mathbf{y} = B\mathbf{x}$ and $\det B \neq 0$, the equilibrium of (4.4) is also not asymptotically stable. Hence, the equilibrium of (4.1) is not asymptotically stable provided that (4.2) holds.

Case (ii). All eigenvalues of $M^{-1}K$ are positive. Hence, from Theorem 2.1 it turns out that the equilibrium of (4.6) is not asymptotically stable for all i. This means that the equilibrium of (4.5) is not asymptotically stable. Since $\det B \neq 0$ and $BM^{-1}K = DB$, we conclude that the equilibrium of (4.1) is not asymptotically stable provided that (4.2) holds. □

Remark 4.1 For $i = 1, 2, \ldots, n$, let $h_i = f + \lambda_i g$. In Theorem 4.1, we may assume that there exist an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that $|t-s| < \delta_0$ implies $|h_i(t) - h_i(s)| < \varepsilon_0$ for all i, instead of the assumption of f and g.

Remark 4.2 Since $f(t) \geq 0$ and $g(t) \geq 0$ for $t \geq 0$, if $f \in \mathscr{F}_{[\mathrm{WIP}]}$ or $g \in \mathscr{F}_{[\mathrm{WIP}]}$, then $\tilde{h} = f + g \in \mathscr{F}_{[\mathrm{WIP}]}$. However, the converse is not true. For example, let

$$f(t) = \begin{cases} 1/(1+t) & \text{if } 2(n-1) \leq t < 2n-1, \\ 0 & \text{if } 2n-1 \leq t < 2n \end{cases}$$

and

$$g(t) = \begin{cases} 0 & \text{if } 2(n-1) \leq t < 2n-1, \\ 1/(1+t) & \text{if } 2n-1 \leq t < 2n \end{cases}$$

for $n \in \mathbb{N}$. Then, $\tilde{h}(t) = 1/(1+t)$ for $t \geq 0$. Hence, $\tilde{h}$ belongs to $\mathscr{F}_{[\mathrm{WIP}]}$ though both f and g do not belong to $\mathscr{F}_{[\mathrm{WIP}]}$.

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