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GEOMETRY OF RULED REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

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Abstract. This paper is the survey of the works with T. Adachi, H. Tanabe and M. Kimura. Ruled real hypersurfaces are typical examples of non-Hopf hypersurfaces in a nonflat complex space form. These examples are constructed by the same method in a complex projective space and a complex hyperbolic space. We shall explain geometric properties of ruled real hypersurfaces which depend on the sign of the sectional curvature of those ambient spaces.

1. Introduction

Among real hypersurfaces isometrically immersed into a nonflat complex space form $\widetilde{M}_n(c)(=\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c))\,, n \geqq 2$, we pay particular attention to ruled real hypersurfaces. Every ruled real hypersurface M in $\widetilde{M}_n(c)$ is constructed by attaching totally geodesic complex hypersurfaces $\widetilde{M}^{(s)}_{n-1}(c)$ on a smooth real curve $\gamma : I \to \widetilde{M}_n(c)$ with its arclength s defined on some open interval $I(\subset \mathbb{R})$ in such a way that the hypersurface $\widetilde{M}^{(s)}_{n-1}(c)$ is orthogonal to the real plane spanned by $\{\dot\gamma(s), J\dot\gamma(s)\}$ at each point $\gamma(s)$, where J is the Kähler structure of the ambient space $\widetilde{M}_n(c)$. Then we get a ruled real hypersurface M defined by $M := \bigcup_{s\in I}\widetilde{M}^{(s)}_{n-1}(c)$.

The class of ruled real hypersurfaces in $\widetilde{M}_n(c)$ is an abundant class which gives fruitful results in the theory of real hypersurfaces in $\widetilde{M}_n(c)$. For example, in the class of all homogeneous real hypersurfaces M^{2n-1} in $\mathbb{C}H^n(c)$ (that is, they are orbits of some subgroups of the full isometry group $\mathrm{I}(\mathbb{C}H^n(c))$ of $\mathbb{C}H^n(c)$), there exists just one example which is ruled in $\mathbb{C}H^n(c)$. Moreover, in this class M is minimal if and only if M is ruled in the ambient space $\mathbb{C}H^n(c)$ (cf. [4, 6, 5]). On

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the contrary, in the class of all homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$, there exist six examples which are minimal in $\mathbb{C}P^n(c)$. However, note that they are *not* ruled in $\mathbb{C}P^n(c)$ (cf. [19, 20]). So we can see that this fact depends on the sign of the sectional curvature of the ambient space $\widetilde{M}_n(c)$.

The main purpose of this paper is to clarify geometric properties of ruled real hypersurfaces in $\widetilde{M}_n(c)$ from this viewpoint.

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2. Ruled real hypersurfaces

Real hypersurface M in $\widetilde{M}_n(c)$ is said to be *ruled* if the holomorphic distribution $T^0M := \{X \in TM | \eta(X) = 0\}$ is integrable and each of its leaves (i.e., maximal integral manifolds) is a totally geodesic complex hypersurface $\widetilde{M}_{n-1}(c)$ of $\widetilde{M}_n(c)$, where η is the contact form on M. The construction of ruled real hypersurfaces in Introduction means that in general a ruled real hypersurface has singularities. So we must omit such points.

Remark 2.1. Every leaf $\widetilde{M}_{n-1}(c)$ is totally geodesic in a given ruled real hypersurface M because M is isometrically immersed into $\widetilde{M}_n(c)$ and $\widetilde{M}_{n-1}(c)$ is totally geodesic in the ambient space $\widetilde{M}_n(c)$.

We define two functions $\mu, \nu : M \to \mathbb{R}$ by $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$. These functions μ and ν are important quantities which measure how far the characteristic vector field ξ is from being a principal curvature vector. A characterization of ruled real hypersurfaces in terms of the functions μ, ν and the shape operator A is given as follows.

Proposition 2.2 ([17]). *Let M be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then the following three conditions are mutually equivalent.*

(1) *M is a ruled real hypersurface.*
(2) *The shape operator A of M satisfies $\langle AX, Y\rangle = 0$ for any tangent vectors $X, Y \in T_xM$ orthogonal to ξ_x at each point $x \in M$.*
(3) *The set $M_1 = \{x \in M \mid \nu(x) \neq 0\}$ is an open dense subset of M and there exists a unit vector field U on M_1 such that it is orthogonal to ξ and satisfies that*

$$A\xi = \mu\xi + \nu U, \quad AU = \nu\xi \quad \textit{and} \quad AX = 0 \tag{2.1}$$

for an arbitrary tangent vector X orthogonal to both ξ and U.

The reason why we consider the set M_1 is that we define the vector field U clearly by $U = (A\xi - \mu\xi)/\nu$. By this definition, the vector field U cannot be extended to a smooth vector field on M in general. But in some cases, considering $-U$ instead of U and $-\nu$ instead of ν on some connected components of M_1 , we can define a smooth vector field U and a smooth function ν on M satisfying the equalities in (2.1). Hence in this paper, considering a ruled real hypersurface M, we treat M as a differentiable manifold such that $M = M_1 \cup \{x \in M | \nu(x) = 0\}$.

3. A characterization of the homogeneous ruled real hypersurfaces in $\mathbb{C}H^n(c)$

We first show the following:

Proposition 3.1. *In $\mathbb{C}P^n(c), n \geqq 2$ every ruled real hypersurface is not complete. But, in $\mathbb{C}H^n(c), n \geqq 2$ there do exist complete ruled real hypersurfaces.*

Proof. Let M be a ruled real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$, $n \geqq 2$. By direct computation (for details, see [8]) we have

$$\phi U \nu = \nu^2 + \frac{c}{4}. \tag{3.1}$$

Here, ϕ is the structure tensor on M defined by $\phi X := JX - \eta(X)\xi$ for all $X \in TM$. Solving the differential equation (3.1), the function ν is of the following form on each integral curve ρ with $\dot\rho(0) = \phi U_{\rho(0)}$ through the point $\rho(0)$ (cf. [1]). When $c > 0$,

$$\nu(\rho(s)) = \pm\Big(\frac{\sqrt{c}}{2}\Big)\tan\Big(\frac{\sqrt{c}}{2}s + a\Big) \tag{3.2}$$

with some constant a and when $c < 0$,

$$\nu(\rho(s)) = \begin{cases} \pm\Big(\frac{\sqrt{|c|}}{2}\Big)\tanh\Big(\frac{\sqrt{|c|}}{2}s + a\Big), \\ \pm\Big(\frac{\sqrt{|c|}}{2}\Big), \\ \pm\Big(\frac{\sqrt{|c|}}{2}\Big)\coth\Big(\frac{\sqrt{|c|}}{2}s + a\Big) \end{cases} \tag{3.3}$$

with some constant a, according as the initial condition $|\nu(\rho(0))|$ is less than $\sqrt{|c|}/2$, equal to $\sqrt{|c|}/2$ or greater than $\sqrt{|c|}/2$. Since we have a choice of directions for the vector field U, we put double signs in (3.2) and (3.3). Moreover, we can show that every integral curve $\rho = \rho(s)$ of ϕU is a geodesic on each ruled real hypersurface M in $\widetilde{M}_n(c)$, which means that every ruled real hypersurface of $\mathbb{C}P^n(c)$ is *not* complete. When $c < 0$, we emphasize that there exist two minimal ruled real hypersurfaces which are complete. One is homogeneous and the other is not homogeneous in the ambient space $\mathbb{C}H^n(c)$ (for details, see the next section). $\square$

Motivated by Proposition 3.1, we are interested in the homogeneous ruled real hypersurface M in $\mathbb{C}H^n(c)$. This homogeneous real hypersurface M is an orbit of the subgroup (of the full isometry group $\mathrm{I}(\mathbb{C}H^n(c))$ which is the direct of $\mathrm{I}(\mathbb{C}H^{n-1}(c))$ and a one-parameter subgroup $\{\varphi_s\}$ whose orbit is a horocycle in $\mathbb{R}H^2(c/4)$, i.e., a circle of positive curvature $\sqrt{|c|}/2$ on a totally real totally geodesic $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in $\mathbb{C}H^n(c)$. Here, $\mathrm{I}(\mathbb{C}H^{n-1}(c))$ is the full isometry group of a totally geodesic complex hypersurface $\mathbb{C}H^{n-1}(c)$ of $\mathbb{C}H^n(c)$). Note that the homogeneous ruled real hypersurface M is obtained by attaching holomorphic hyperplanes $\mathbb{C}H^{n-1}(c)$ on the horocycle (for details, see [10, 6, 12]).

Our aim here is to characterize the homogeneous ruled real hypersurface M in $\mathbb{C}H^n(c)$ by observing some geodesics and all integral curves of the characteristic vector field ξ on M. For this purpose, by observing the extrinsic shape of some geodesics we prepare the following lemma which is a characterization of all ruled real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$.

Lemma 3.2. *Let M be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$, $n \geqq 2$ through an isometric immersion. Then M is ruled in the ambient space $\widetilde{M}_n(c)$ if and only if at an arbitrary point $p \in M$ there exist such orthonormal vectors $v_1, v_2, \dots, v_{2n-2} (\in T_pM)$ orthogonal to the characteristic vector ξ_p that every geodesic $\gamma_{ij,p}$ on M through the point $\gamma_{ij,p}(0) = p$ in the direction of $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ is an extrinsic geodesic, namely $\gamma_{ij,p}$ is also a geodesic in $\widetilde{M}_n(c)$.*

Remark 3.3. In the statement of Lemma 3.2, when $c > 0$, at every fixed point $p \in M$ if we delete the condition that all geodesics through the point $\gamma_{ij,p}(0) = p$ in the direction of $v_i + v_j$ $(1 \leqq i < j \leqq 2n-2)$ (, i.e., in this case the initial vector $\dot{\gamma}_{ij,p}(0)$ is expressed as $\dot{\gamma}_{ij,p}(0) = (v_i + v_j)/\sqrt{2}\,$) are extrinsic geodesics, this lemma is no longer true (see Remark 3 in [18]). Indeed, let M be a tube of radius $\pi/(2\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^{(n-1)/2}(c)$, where $n (\geqq 3)$ is odd. Then M has three distinct constant principal curvatures 0 with multiplicity 1, $\sqrt{c}/2$ with multiplicity $n-1$ and $-\sqrt{c}/2$ with multiplicity $n-1$, where $A\xi = 0$. We take orthonormal bases $e_1, e_2, \dots, e_{n-1}$ and $f_1, f_2, \dots, f_{n-1}$ of principal curvatures $\sqrt{c}/2$ and $-\sqrt{c}/2$, respectively. We here set an orthonormal basis $\{v_1, v_2, \dots, v_{2n-2}\}$ of T^0M defined by $v_i := (e_i + f_i)/\sqrt{2}$ $(1 \leqq i \leqq n-1)$ and $v_{n-1+i} := (e_i - f_i)/\sqrt{2}$ $(1 \leqq i \leqq n-1)$. Then by the construction of the orthonormal basis $\{v_1, v_2, \dots, v_{2n-2}\}$ of T^0M, the Gauss formula: $\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle \mathcal{N}$ and the property of our real hypersurface M : $\langle (\nabla_X A)X, X \rangle = 0$ for all $X \in TM$ we can see that our homogeneous real hypersurface M of type (A_2) satisfies that every geodesic $\gamma_{ii,p}(0) = p$ in the direction of v_i $(i = 1, 2, \dots, 2n-2)$ is an extrinsic geodesic.

When $c < 0$, in the statement of Lemma 3.2 we do *not* know the answer to the question "Can we delete the condition that all geodesics through the point $\gamma_{ij,p}(0) = p$ in the direction of $v_i + v_j$ $(1 \leqq i < j \leqq 2n-2)$ are extrinsic geodesics?".

At the end of Remark 3.3 we review the notion of real hypersurfaces of type (A_2) in a complex projective space. A real hypersurface M of $\mathbb{C}P^n(c), n \geqq 3$ is of type (A_2) if M is a tube of radius r $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^\ell(c)$ with $1 \leqq \ell \leqq n-2$ in $\mathbb{C}P^n(c)$.

We are now in a position to show the following:

Theorem 3.4 ([12]). *A real hypersurface M isometrically immersed into $\mathbb{C}H^n(c)$, $n \geqq 2$ is locally congruent to the homogeneous ruled real hypersurface if and only if it satisfies the following three conditions.*

i) *At an arbitrary point $p \in M$, there exist such orthonormal vectors $v_1, v_2, \dots,$ $v_{2n-2} (\in T_pM)$ orthogonal to the characteristic vector ξ_p that every geodesic*

$\gamma_{ij,p}$ on M through the point $\gamma_{ij,p}(0) = p$ in the direction of $v_i + v_j$ ($1 \leqq i \leqq j \leqq 2n-2$) is an extrinsic geodesic, namely $\gamma_{ij,p}$ is also a geodesic in $\mathbb{C}H^n(c)$.

ii) *At an arbitrary point $p \in M$, the integral curve γ_p of the characteristic vector field ξ on M through $\gamma_p(0) = p$ lies locally on a totally real totally geodesic 2-dimensional real hyperbolic space $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$.*

iii) *The curvature function $\kappa_p = \|\widetilde{\nabla}_{\dot\gamma_p}\dot\gamma_p\|$ of the curve γ_p in* ii) *does not depend on the choice of γ_p, where $\widetilde{\nabla}$ is the Riemannian connection of the ambient space $\mathbb{C}H^n(c)$. This means that for any curves γ_p, γ_q in* ii) *their curvature functions $\kappa_p(s)$ and $\kappa_y(s)$ satisfy the following equality with some constant s_0 : $\kappa_p(s) = \kappa_q(s+s_0)$ for $-\infty < \forall s < \infty$, where p, q are any points of M.*

Due to the works of J. Berndt and others ([4, 5, 6]), we find that the class of all homogeneous real hypersurfaces M in $\mathbb{C}H^n(c)$ has just one example which is minimal in this space. Furthermore, in this ambient space, a homogeneous real hypersurface M is minimal if and only if it is ruled. This fact implies that it is interesting to study this minimal homogeneous real hypersurface from the viewpoint of the geometry of ruled real hypersurfaces. At the end of this section we characterize this minimal homogeneous real hypersurface in $\mathbb{C}H^n(c)$ in the class of all ruled real hypersurfaces M by investing the first curvature of all integral curves of the characteristic vector field ξ and the vector field U. Note that there exist minimal non-homogeneous ruled real hypersyrfaces in $\mathbb{C}H^n(c)$ (see [10, 1]).

We here show the following:

Theorem 3.5 ([15]). *Let M^{2n-1} be a ruled real hypersurface of $\mathbb{C}H^n(c), n \geqq 2$. Then M is homogeneous in this ambient space, i.e., M is an orbit of a subgroup of the full isometry group* $\mathrm{I}(\mathbb{C}H^n(c))$ *of $\mathbb{C}H^n(c)$ if and only if M satisfies the following two conditions*:

(1) *Every integral curve γ_U of the vector field U in* (2.1) *is mapped to a curve of positive first curvature $\kappa_U := \|\widetilde{\nabla}_U U\|$ with $\kappa_U \leqq \sqrt{|c|}$ in $\mathbb{C}H^n(c)$;*

(2) *Every integral curve γ_ξ of the characteristic vector field ξ in* (2.1) *is mapped to a curve of positive first curvature $\kappa_\xi := \|\widetilde{\nabla}_\xi \xi\|$ with $\kappa_\xi \leqq \sqrt{|c|}/2$ in $\mathbb{C}H^n(c)$.*

Here, $\widetilde{\nabla}$ is the Riemannian connection of the ambient space $\mathbb{C}H^n(c)$.

We remark that all curves $\gamma_U = \gamma_U(s)$ in Condition (1) in Theorem 3.5 are mapped to circles of the same curvature $\sqrt{|c|}$ on a holomorphic line $\mathbb{C}H^1(c)$ of $\mathbb{C}H^n(c)$ (, i.e., they are mapped to horocycles in $\mathbb{C}H^1(c)$) and all curves $\gamma_\xi = \gamma_\xi(s)$ in Condition (2) in Theorem 3.5 are mapped to circles of the same curvature $\sqrt{|c|}/2$ on a totally real totally geodesic surface $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ (, i.e., they are mapped to horocycles in $\mathbb{R}H^2(c/4)$) in the ambient space $\mathbb{C}H^n(c)$.

4. Classification theorems of minimal ruled real hypersurfaces in $\widetilde{M}_n(c)$

We first classify minimal ruled real hypersurfaces in $\mathbb{C}P^n(c)$.

Theorem 4.1 ([1]). *There exists the unique minimal ruled real hypersurface in $\mathbb{C}P^n(c)$, $n \geqq 2$ up to the action of the full isometry group $\mathrm{I}(\mathbb{C}P^n(c))$. It is constructed by attaching holomorphic hyperplanes $\mathbb{C}P^{n-1}(c)$ on a geodesic on a totally real totally geodesic $\mathbb{R}P^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}P^n(c)$.*

We shall classify minimal ruled real hypersurfaces in $\mathbb{C}H^n(c)$.

Theorem 4.2 ([1]). *There exist three minimal ruled real hypersurfaces in $\mathbb{C}H^n(c)$, $n \geqq 2$ up to the action of the full isometry group $\mathrm{I}(\mathbb{C}H^n(c))$. They are constructed by attaching holomorphic hyperplanes $\mathbb{C}H^{n-1}(c)$ on a circle of curvature $k(\geqq 0)$ on a totally real totally geodesic $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}H^n(c)$. Minimal ruled real hypersurfaces of parabolic type (, i.e., $k = \sqrt{|c|}/2$) and axial type (, i.e., $0 \leqq k < \sqrt{|c|}/2$) are complete, but the minimal ruled real hypersurface of elliptic type (, i.e., $k > \sqrt{|c|}/2$) is not complete.*

For the sake of your convenience we here give images of ruled real hypersurfaces of axial, parabolic and elliptic types in a ball model $D^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\}$ of a complex hyperbolic space $\mathbb{C}H^n$. In figures, dotted lines imply leaves of ruled real hypersurfaces and lines inside of the balls imply totally real circles on a totally real totally geodesic $\mathbb{R}H^2$, which are denoted by γ_k in the above. If we regard these figures as models of $\mathbb{R}H^2$ containing the totally real circles, the dotted lines show integral curves of ϕU. We note that all the totally real circles have the same pairs of points at infinity as we can see in Fig. 1. In Fig. 2, two lines show horocyclic totally real circles having the same points at infinity. One may easily guess that the only parabolic one is homogeneous.

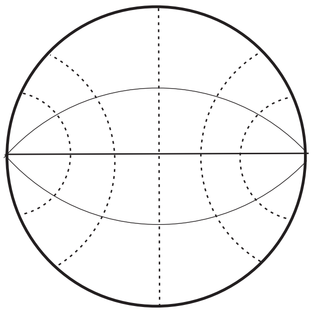

Figure 1.
axial type

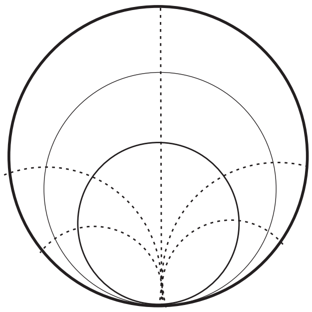

Figure 2.
parabolic type

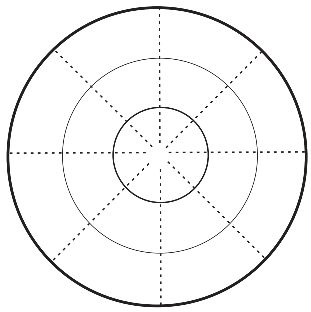

Figure 3.
elliptic type

Remark 4.3. In the minimal ruled real hypersurface M of axial type, if we delete the subset $\{x \in M | \nu(x) = 0\}$ of M, then M is *not* complete.

5. Sectional curvatures of ruled real hypersurfaces in $\widetilde{M}_n(c)$

In classical differential geometry, it is well-known that every ruled surface in 3-dimensional Euclidean space $\mathbb{R}^3$ has nonpositive Gaussian curvature. Hence we have an interest in the sectional curvature of every ruled real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Since every leaf $\widetilde{M}_{n-1}(c)$ is totally geodesic in M (see Remark 2.1(2)), in the case of $n \geqq 3$ we find easily the fact that the sectional curvature K of every ruled real hypersurface M in $\widetilde{M}_n(c)$ satisfies sharp inequalities $|c/4| \leqq |K(X,Y)| \leqq |c|$ for an arbitrary pair of orthonormal vectors X and Y *orthogonal* to the characteristic vector ξ_p at each point p of M. Here, note that in the case of $n = 2$ we find also the fact that the sectional curvature K of every ruled real hypersurface M in $\widetilde{M}_n(c)$ satisfies $K(X,Y) = c$ for each pair $\{X,Y\}$ of orthonormal vectors X and Y that are orthogonal to ξ.

Inspired by this fact we study ruled real hypersurfaces $M^{2n-1}, n \geqq 2$ having the sectional curvature K with $|c/4| \leqq |K| \leqq |c|$, that is, the sectional curvature K satisfies sharp inequalities either $c/4 \leqq K(X,Y) \leqq c$ $(c > 0)$ or $c \leqq K(X,Y) \leqq c/4$ $(c < 0)$ for an arbitrary pair of orthonormal vectors X and Y that are *not necessarily* tangent to the leaf at each point p of M.

First of all we give the following inequalities for the sectional curvature K of a ruled real hypersurface M (cf. [21]).

Lemma 5.1. *Let M be a ruled real hypersurface in $\widetilde{M}_n(c), n \geqq 2$. Then the sectional curvature K of M satisfies the following* :

1. *If $c > 0$, we have $(c/4) - \nu(p)^2 \leqq K \leqq c$ at every point $p \in M$* ;
2. *If $c < 0$, we have $(c/4) - \nu(p)^2 \leqq K \leqq c/4$ at every point $p \in M$ with $\nu(p)^2 \geqq 3|c|/4$ and $c \leqq K \leqq c/4$ at the point $p \in M$ with $\nu(p)^2 < 3|c|/4$.*

These estimates are sharp in the sense that at each point of M we can take a pair of orthonormal tangent vectors X and Y satisfying $K(X,Y) = k$ for given k with $K_{\min} \leqq k \leqq K_{\max}$.

In [10], Lohnherr-Reckziegel studied ruled real hypersurfaces by parameterizing them by maps of the form $f : \mathrm{I} \times \widetilde{M}_{n-1}(c) \to \widetilde{M}_n(c)$ and show properties on these maps. The following lemma is due to them.

Lemma 5.2 ([10]). *For every $s \in I$ the function ν satisfies the following*:

1. *When $c > 0$, on each leaf $\widetilde{M}^{(s)}_{n-1}(c)$ we have $0 \leqq \nu < \infty$* ;
2. *When $c < 0$, on each leaf $\widetilde{M}^{(s)}_{n-1}(c)$ we have either $0 \leqq \nu < \sqrt{|c|}/2$, $\nu \equiv \sqrt{|c|}/2$ or $\sqrt{|c|}/2 < \nu < \infty$.*

Here we recall the definition of Frenet curves in a Riemannian manifold $\widetilde{M}$. A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist a field of orthonormal frames $\{V_1 = \dot{\gamma}, V_2, \ldots, V_d\}$ along γ and positive smooth functions $\kappa_1(s), \ldots, \kappa_{d-1}(s)$ satisfying the following system of ordinary differential equations

$$\widetilde{\nabla}_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s) V_{j-1}(s) + \kappa_j(s) V_{j+1}(s), \quad j = 1, \ldots, d, \tag{5.1}$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\widetilde{\nabla}_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection $\widetilde{\nabla}$ of $\widetilde{M}$. The functions $\kappa_j(s)$ $(j = 1, \dots, d-1)$ and a field of orthonormal frames $\{V_1, \dots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. Roughly speaking, every Frenet curve can be regarded as a smooth regular real curve admitting no inflection points. We call a curve a *helix* when all of its curvatures are constant functions. A curve γ is called a *helix of order* d if it is a helix of proper order $r(\leqq d)$. For a helix of order d, which is of proper order $r(\leqq d)$, we use the convention in (5.1) that $\kappa_j = 0$ $(r \leqq j \leqq d-1)$ and $V_j = 0$ $(r+1 \leqq j \leqq d)$. Needless to say, a helix of order 1 is nothing but a geodesic and a helix of order 2 is a circle. For a Frenet curve γ of proper order d in a nonflat complex space form $\widetilde{M}_n(c)$, we define its *complex torsions* by $\tau_{ij}(s) = \langle V_i(s), JV_j(s)\rangle$ $(1 \leqq i < j \leqq d)$. In the study of Frenet curves in $\widetilde{M}_n(c)$ their complex torsions play an important role (see [13]).

Now, we consider a ruled real hypersurface associated with a Frenet curve γ and investigate the value of the function ν along γ.

Lemma 5.3 ([16])**.** *Let $\gamma = \gamma(s)$ be a Frenet curve of proper order d with curvatures $\kappa_j(s)$ $(1 \leqq j \leqq d-1)$ and complex torsions $\tau_{ij}(s)$ $(1 \leqq i < j \leqq d)$ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$ and M be a ruled real hypersurface associated with the curve γ. Then the functions $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$ satisfy the following.*

$$\mu(\gamma(s)) = -\kappa_1(s)\tau_{12}(s), \tag{5.2}$$

$$\nu(\gamma(s))^2 = \kappa_1(s)^2(1 - \tau_{12}(s)^2). \tag{5.3}$$

As a direct consequence of Lemmas 5.1, 5.2 and 5.3, we have the following.

Theorem 5.4 ([16])**.** *Every ruled real hypersurface M in $\mathbb{C}P^n(c), n \geqq 2$ has the sectional curvature K with sharp inequalities $-\infty < K \leqq c$, so that there does not exist a ruled real hypersurface having the sectional curvature K with $c/4 \leqq K \leqq c$ in this ambient space.*

Theorem 5.5 ([16])**.** *Let $\gamma : I \to \mathbb{C}H^n(c), n \geqq 2$ be a Frenet curve of proper order d defined on an open interval $I(\subset \mathbb{R})$ in $\mathbb{C}H^n(c)$ and M be a ruled real hypersurface associated with the curve γ in $\mathbb{C}H^n(c)$. Then M has the sectional curvature K with sharp inequalities $c \leqq K \leqq c/4$ if and only if the first curvature κ_1 and a complex torsion τ_{12} of the Frenet curve γ satisfy $\kappa_1(s)^2(1 - \tau_{12}(s)^2) \leqq |c|/4$ for any $s \in I$.*

Remark 5.6. Both of minimal ruled hypersurfaces M of axial type and parabolic type satisfy the inequality in Theorem 5.5, so that the sectional curvatures K of M satisfy $c \leqq K \leqq c/4$. However, note that the minimal ruled real hypersurface of elliptic type does *not* satisfy the inequality in Theorem 5.5. This ruled real hypersurface has the sectional curvature K with $-\infty < K \leqq c/4$.

6. Characterization of all ruled real hypersurfaces in $\widetilde{M}_n(c)$

It is known that every real hypersurface in a nonflat complex space form does not have parallel shape operator (for details, see [17]). So, weakening this parallelism of the shape operator, we shall characterize all ruled real hypersurfaces in $\widetilde{M}_n(c)$. Note that the following theorem is established from the viewpoint of geometric properties (of all ruled real hypersurfaces) which do *not* depend on the sign of the sectional curvature of a nonflat complex space form.

Theorem 6.1 ([9]). *A real hypersurface M isometrically immersed into a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$ is ruled in this ambient space if and only if the holomorphic distribution T^0M on M is integrable and the shape operator A of M is η-parallel, i.e., $\langle(\nabla_X A)Y, Z\rangle = 0$ for all vectors X, Y and $Z \in T^0M$ holds on M, where $\langle\ ,\ \rangle$ and ∇ are the Riemannian metric and the Riemannian connection of M, respectively.*

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