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Duality Theorems for Separable Convex Programming without Qualifications

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Abstract In the research of mathematical programming, duality theorems are essential and important elements. Recently, Lagrange duality theorems for separable convex programming have been studied. Tseng proves that there is no duality gap in Lagrange duality for separable convex programming without any qualifications. In other words, although the infimum value of the primal problem equals to the supremum value of the Lagrange dual problem, Lagrange multiplier does not always exist. Jeyakumar and Li prove that Lagrange multiplier always exists without any qualifications for separable sublinear programming. Furthermore, Jeyakumar and Li introduce a necessary and sufficient constraint qualification for Lagrange duality theorem for separable convex programming. However, separable convex constraints do not always satisfy the constraint qualification, that is, Lagrange duality does not always hold for separable convex programming.

In this paper, we study duality theorems for separable convex programming without any qualifications. We show that a separable convex inequality system always satisfies the closed cone constraint qualification for quasiconvex programming, and investigate a Lagrange-type duality theorem for separable convex programming. In addition, we introduce a duality theorem and a necessary and sufficient optimality condition for a separable convex programming problem, whose constraints do not satisfy the Slater condition.

Keywords separable convex programming · duality theorem · constraint qualification · generator of quasiconvex functions

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1 Introduction

In the research of mathematical programming, duality theorems are essential and important elements. Various types of duality theorems have been investigated by many researchers; for example, Lagrange duality, Fenchel duality, surrogate duality, Wolfe duality, and so on. Constraint qualifications for duality theorems are also essential elements for mathematical programming. Recently, necessary and sufficient constraint qualifications for Lagrange duality have been studied; see [1–6]. To find a necessary and sufficient constraint qualification is one of the destinations of the study of mathematical programming.

In [7], Lagrange duality for separable convex programming has been studied by Tseng. It is proved that there is no duality gap without any qualifications. In other words, although the infimum value of the primal problem equals to the supremum value of the Lagrange dual problem, Lagrange multiplier does not always exist for separable convex programming. In [8], it is shown that Lagrange duality theorem for separable sublinear programming always holds without any qualifications, that is, if the objective function is sublinear and the constraint functions are separable sublinear, then there exists a Lagrange multiplier. Furthermore, in [9], a necessary and sufficient constraint qualification for Lagrange duality theorem for separable convex programming is introduced by Jeyakumar and Li. However, separable convex constraints do not always satisfy the constraint qualification, that is, exact duality does not always hold for separable convex programming.

In [10–15], Lagrange-type duality theorems for quasiconvex programming are studied by the authors. The closed cone constraint qualification for quasiconvex programming, Q-CCCQ, is introduced as a necessary and sufficient constraint qualification for Lagrange-type duality. A Lagrange-type duality theorem is also valid for convex programming. Actually, even if Lagrange exact duality does not hold for convex constraints, we may choose a suitable generator such that Lagrange-type exact duality holds; in detail, see [10–15].

In this paper, we study duality theorems for separable convex programming without any qualifications. We show that a separable convex inequality system always satisfies Q-CCCQ, and investigate a Lagrange-type duality theorem for separable convex programming. Furthermore, we introduce a duality theorem and a necessary and sufficient optimality condition for a separable convex programming problem, whose constraints do not satisfy the Slater condition.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we study a Lagrange-type duality theorem for separable convex programming without any qualifications. In Section 4, we show a duality theorem and a necessary and sufficient optimality condition for a separable convex programming problem whose constraints do not satisfy the Slater condition. We show an example which illustrate usefulness of our results.

2 Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given a set $A \subset \mathbb{R}^n$, we denote the closure, the convex hull, and the conical hull generated by A, by cl A, conv A, and cone A, respectively. We stipulate that cone $\emptyset := \{0\}$. We denote the unit sphere of $\mathbb{R}^n$ by S, that is, $S := \{x \in \mathbb{R}^n : \|x\| = 1\}$. The indicator function δ_A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & otherwise. \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, \infty]$. A function f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. The epigraph of f is defined as $\text{epi}f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$, and f is said to be convex if epi$f$ is convex. The subdifferential of f at x is defined as $\partial f(x) := \{v \in \mathbb{R}^n : \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle v, y - x\rangle\}$. The Fenchel conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as $f^*(v) := \sup_{x\in\mathbb{R}^n}\{\langle v, x\rangle - f(x)\}$. Define the $\diamond$-level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) := \{x \in \mathbb{R}^n : f(x) \diamond \beta\}$$

for each $\beta \in \mathbb{R}$. A function f is said to be quasiconvex if for each $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is convex. Any convex function is quasiconvex, but the opposite is not true. A function f is said to be quasiaffine if f and $-f$ are quasiconvex. In quasiconvex analysis, it is known that f is lower semicontinuous (lsc) quasiaffine if and only if there exists $k \in Q$ and $w \in \mathbb{R}^n$ such that $f = k \circ w$, where $Q := \{h : \mathbb{R} \to \overline{\mathbb{R}}, \text{ lsc and non-decreasing}\}$. In addition, f is lsc quasiconvex if and only if there exists $\{(k_j, w_j) : j \in J\} \subset Q \times \mathbb{R}^n$ such that $f = \sup_{j\in J} k_j \circ w_j$; see [16, 17] for more details. This result indicates that a lsc quasiconvex function f consists of a supremum of a some family of lsc quasiaffine functions. Based on this result, in [10], the authors define a notion of a generator of a quasiconvex function, that is, $G := \{(k_j, w_j) : j \in J\} \subset Q \times \mathbb{R}^n$ is said to be a generator of f if $f = \sup_{j\in J} k_j \circ w_j$. All lsc quasiconvex functions have at least one generator. The following function h^{-1} is said to be the hypo-epi-inverse of a extended real-valued function h on $\mathbb{R}$:

$$h^{-1}(a) := \sup\{b \in \mathbb{R} : h(b) \leq a\}.$$

It is known that, if $h \in Q$ has an inverse function, then the inverse and the hypo-epi-inverse of h are the same; in detail, see [17]. In this paper, we denote the hypo-epi-inverse of h by h^{-1}. Let g be an extended real-valued function on $\mathbb{R}^n$. We define the function g_w from $\mathbb{R}$ to $\overline{\mathbb{R}}$ for each $w \in \mathbb{R}^n$ as follows:

$$g_w(t) := \inf\{g(x) : \langle w, x\rangle \geq t\}.$$

Clearly, g_w is non-decreasing and $g(x) \geq g_w(\langle w, x\rangle)$ for each $x \in \mathbb{R}^n$.

In mathematical programming, constraint qualifications for duality theorems are essential elements. Especially in convex programming, necessary and sufficient constraint qualifications for Lagrange duality theorems have been investigated extensively; see [1–6]. In [7], Tseng proves that there is no duality gap in Lagrange duality for separable convex programming without any qualifications: if f and f_i, $i \in I := \{1, \ldots, m\}$, are separable convex, then

$$\inf\{f(x) : \forall i \in I, f_i(x) \leq 0\} = \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda_i f_i(x) \right\}.$$

Although there is no duality gap, a Lagrange multiplier does not always exist. In [8], Jeyakumar and Li prove that Lagrange duality theorem always holds for separable sublinear programming without qualifications: if f is sublinear and f_i is separable sublinear for each $i \in I = \{1, \ldots, m\}$, then

$$\inf\{f(x) : \forall i \in I, f_i(x) \leq 0\} = \max_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda_i f_i(x) \right\}.$$

Furthermore, in [9], Jeyakumar and Li introduce a necessary and sufficient constraint qualification for the Lagrange duality theorem for separable convex programming; see the following Theorem 2.1.

Theorem 2.1 *For each $i = 1, 2, \ldots, m$, let $g_i : \mathbb{R}^n \to \mathbb{R}$ be a separable convex function. Then, the following statements are equivalent:*

(i)

$$\operatorname{epi}\left(\inf_{\lambda \in \mathbb{R}^m_+} \left(\sum_{i=1}^m \lambda_i g_i \right)^* \right) = \bigcup_{\lambda \in \mathbb{R}^m_+} \operatorname{epi} \left(\sum_{i=1}^m \lambda_i g_i \right)^*,$$

(ii) for each real-valued convex function f on $\mathbb{R}^n$,

$$\inf_{x \in \mathbb{R}^n} \{f(x) : g_i(x) \leq 0, i = 1, \ldots, m\} = \max_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda_i g_i(x) \right\}.$$

The statement (i) of Theorem 2.1 is a necessary and sufficient constraint qualification for Lagrange duality for convex programming problems with separable convex constraints. It is known that the Slater condition implies the statement (i). In addition, by Example 3.2 in [9], the statement (i) is not always satisfied for separable convex constraints. This means that the Lagrange exact duality does not always hold for separable convex programming.

In [10, 15], the authors study the closed cone constraint qualification for quasiconvex programming, Q-CCCQ, as a necessary and sufficient constraint qualification for Lagrange-type duality for quasiconvex programming.

Definition 2.1 [15] Let I be an arbitrary index set, $\{g_i : i \in I\}$ a family of lsc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) : j \in J_i\} \subset Q \times \mathbb{R}^n$ a generator of g_i for each $i \in I$, and $T = \{t = (i, j) : i \in I, j \in J_i\}$. Assume that $A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$ is a non-empty set.

A quasiconvex inequality system $\{g_i(x) \leq 0 : i \in I\}$ is said to satisfy the closed cone constraint qualification for quasiconvex programming (Q-CCCQ) w.r.t. $\{(k_t, w_t) : t \in T\}$ if

$$\text{cone conv} \bigcup_{t \in T} \{(w_t, \delta) \in \mathbb{R}^{n+1} : k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[$$

is closed.

Let I be an arbitrary index set, $\{g_i : i \in I\}$ a family of lsc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) : j \in J_i\} \subset Q \times \mathbb{R}^n$ a generator of g_i for each $i \in I$, $g = \sup_{i \in I} g_i$, and $T = \{t = (i,j) : i \in I, j \in J_i\}$. Then, $\{(k_t, w_t) : t \in T\}$ is a generator of g. We can check easily that $G_g := \{(k, w) \in Q \times \mathbb{R}^n : k \circ w \leq g\}$ is a generator of g. A quasiconvex inequality system $\{g_i(x) \leq 0 : i \in I\}$ is said to satisfy the Q-CCCQ if $\{g(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. G_g.

In [13], we introduce a necessary and sufficient condition of the Q-CCCQ.

Theorem 2.2 *Let $\{g_i : i \in I\}$ be a family of lsc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and $g = \sup_{i \in I} g_i$. Assume that $A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$ is a non-empty set.*

Then, the following statements are equivalent:

(i) $\{g_i(x) \leq 0 : i \in I\}$ *satisfies the Q-CCCQ,*
(ii) for all $v \in \mathbb{R}^n \setminus \{0\}$ *and* $t > \delta_A^*(v)$, $g_v(t) = \inf\{g(x) : \langle v, x \rangle \geq t\} > 0$.

In [15], we show the following Lagrange-type duality theorem with its necessary and sufficient constraint qualification, Q-CCCQ.

Theorem 2.3 *Let $\{g_i : i \in I\}$ be a family of lsc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and $g = \sup_{i \in I} g_i$. Assume that $A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$ is a non-empty set.*

Then, the following statements are equivalent:

(i) $\{g_i(x) \leq 0 : i \in I\}$ *satisfies the Q-CCCQ,*
(ii) for each real-valued convex function f on $\mathbb{R}^n$, there exist a finite subset $S_0 = \{w_1, \ldots, w_m\} \subset S$ such that $g_{w_j}^{-1}(0) \in \mathbb{R}$ for each $j \in \{1, \ldots, m\}$, and

$$\inf_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^m} \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^m \lambda_j (w_j(x) - g_{w_j}^{-1}(0)) \right\}.$$

In Theorem 2.3, the statement (ii) is Lagrange duality for the convex programming problem with the objective function f and affine constraint functions $\{w - g_w^{-1}(0) : w \in S\}$. We call the statement (ii) Lagrange-type duality.

3 Lagrange-Type Duality Theorem without Qualifications

In this paper, we consider the following separable convex programming problem:

(P) min $f(x)$, s.t. $x \in A := \{x \in \mathbb{R}^n : g(x) \leq 0\}$,

where f is a real-valued convex function on $\mathbb{R}^n$, g_i is a real-valued convex function on $\mathbb{R}$ for each $i \in \{1, \ldots, n\}$, and $g(x) = \sum_{i=1}^n g_i(x_i)$. If a problem has multiple separable convex constraints, that is, $A = \{x \in \mathbb{R}^n : \forall t \in T,\ h_t(x) \leq 0\}$, then we can obtain the single constraint $\{x \in \mathbb{R}^n : \sup_{t\in T} h_t \leq 0\}$. Additionally, if T is finite and h_t is separable convex for each $t \in T$, then $\sup_{t\in T} h_t$ is also separable and convex. Hence, the problem (P) contains the problems in [7–9].

In the rest of the paper, we often use the following assumption:

(A) the set $A = L(g, \leq, 0)$ is nonempty whereas $L(g, <, 0)$ is empty.

We show the following lemma.

Lemma 3.1 *Let g_i be a real-valued convex function on $\mathbb{R}$ for each $i \in \{1, \ldots, n\}$, $g(x) = \sum_{i=1}^n g_i(x_i)$, and $m_i := \inf g_i(\mathbb{R})$. Assume that (A) is satisfied.*

Then, the following statements hold:

(i) for each $i \in \{1, \ldots, n\}$, $m_i \in \mathbb{R}$,

(ii) $A = \prod_{i=1}^n L(g_i, \leq, m_i)$,

(iii) $\sum_{i=1}^n m_i = 0$.

Proof (i) Clearly, for each $i \in \{1, \ldots, n\}$, $m_i < \infty$. If $m_i = -\infty$, then there exists $a_0 \in \mathbb{R}$ such that $g_i(a_0) < -\sum_{j\neq i} g_j(1) - 1$. Hence,

$$g(1, \ldots, 1, a_0, 1, \ldots, 1) = \sum_{j\neq i} g_j(1) + g_i(a_0) < -1.$$

This is a contradiction.

(ii) Let $x \in A$. Since $L(g, <, 0)$ is empty, $g(x) = \sum_{i=1}^n g_i(x) = 0$. If there exists $i_0 \in \{1, \ldots, n\}$ such that $g_{i_0}(x_{i_0}) > m_{i_0}$, then there exists $y_{i_0} \in \mathbb{R}$ such that $g_{i_0}(x_{i_0}) > g_{i_0}(y_{i_0})$. Hence,

$$g(x_1, \ldots, x_{i_0-1}, y_{i_0}, x_{i_0+1}, \ldots, x_n) < g(x) = 0.$$

This is a contradiction. Hence, $x \in \prod_{i=1}^n L(f_i, \leq, m_i)$.

Let $x \notin A$. Then, $g(x) > 0$. Since A is non-empty, there exists $y \in A$ such that

$$g(x) = \sum_{i=1}^n g_i(x_i) > 0 = g(y) = \sum_{i=1}^n g_i(y_i).$$

Clearly, there exists $i_0 \in \{1, \dots, n\}$ such that $g_{i_0}(x_{i_0}) > g_{i_0}(y_{i_0}) \geq m_{i_0}$. This shows that $x \notin \prod_{i=1}^n L(g_i, \leq, \inf_{a\in\mathbb{R}} g_i(a))$.

(iii) Since A is non-empty, there exists $x \in A$. By the condition (ii) and (A),

$$0 = g(x) = \sum_{i=1}^n g_i(x_i) \leq \sum_{i=1}^n m_i \leq \sum_{i=1}^n g_i(x_i) = 0.$$

This completes the proof. □

We introduce the following lemma without proofs.

Lemma 3.2 *Let h be a real-valued convex function on $\mathbb{R}$. Then, the following conditions hold:*

(i) there exists a, $b \in \overline{\mathbb{R}}$ such that $L(h, \leq, \inf_{t\in\mathbb{R}} h(t)) = \{t \in \mathbb{R} : a \leq t \leq b\}$ and $a \leq b$,
(ii) if $b \in \mathbb{R}$, then for each $\varepsilon > 0$ and $t \geq b + \varepsilon$, $h(t) \geq h(b + \varepsilon) > \inf_{t\in\mathbb{R}} h(t)$,
(iii) if $a \in \mathbb{R}$, then for each $\varepsilon > 0$ and $t \leq a - \varepsilon$, $h(t) \geq h(a - \varepsilon) > \inf_{t\in\mathbb{R}} h(t)$.

A convex inequality system $\{g(x) \leq 0\}$ is said to satisfy the Slater condition if $L(g, <, 0)$ is non-empty. It is well known that the Slater condition implies the existence of Lagrange multiplier for convex programming. Furthermore, Farkas Minkowski (FM) is known as a necessary and sufficient constraint qualification for Lagrange duality for convex programming, and FM implies Q-CCCQ; in detail, see [1–6,10–15,18–20]. Hence, if the Slater condition is satisfied, then $\{g(x) \leq 0\}$ satisfies the Q-CCCQ.

In the following theorem, we show that a separable convex inequality system always satisfies the Q-CCCQ.

Theorem 3.1 *Let g_i be a convex function from $\mathbb{R}$ to $\mathbb{R}$ for each $i \in \{1, \dots, n\}$, $g(x) = \sum_{i=1}^n g_i(x_i)$, $m_i = \inf g_i(\mathbb{R})$, and $A = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ is non-empty. Then, $\{g(x) \leq 0\}$ satisfies the Q-CCCQ.*

Proof As mentioned above, if $L(g, <, 0)$ is non-empty, then $\{g(x) \leq 0\}$ satisfies the Q-CCCQ. Assume that $L(g, <, 0)$ is empty and let $v \in \mathbb{R}^n \setminus \{0\}$ and $t > \delta_A^*(v)$. By Lemma 3.1 and Lemma 3.2, there exists a_i, $b_i \in \overline{\mathbb{R}}$ such that

$$A = \{x \in \mathbb{R}^n : \forall i \in \{1, \dots, n\}, a_i \leq x_i \leq b_i\}.$$

Let $V_+ := \{i \in \{1, \dots, n\} : v_i > 0\}$, $V_- := \{i \in \{1, \dots, n\} : v_i < 0\}$, and $V_0 := \{i \in \{1, \dots, n\} : v_i = 0\}$. Since A is non-empty, $b_i > -\infty$ and $a_i < \infty$ for each $i \in \{1, \dots, n\}$. In addition, for each $i \in V_+$, $b_i \in \mathbb{R}$. Actually, if $b_i = \infty$, then we can check that $\delta_A^*(v) = \infty$. This is a contradiction. Similarly, we can prove that for each $i \in V_-$, $a_i \in \mathbb{R}$. Let $d_i := \frac{t - \delta_A^*(v)}{2n|v_i|}$ for each $i \in V_+ \cup V_-$. Next we prove the following inclusion:

$$L(v, \geq, t) \subset \left\{ \bigcup_{i\in V_+} \{x : x_i \geq b_i + d_i\} \right\} \bigcup \left\{ \bigcup_{i\in V_-} \{x : x_i \leq a_i - d_i\} \right\}.$$

Let $x \in L(v, \geq, t)$ and assume that for each $i \in V_+$ and $j \in V_-$, $x_i < b_i + d_i$ and $x_j > a_j - d_i$. Then,

$$\begin{aligned}\langle v, x\rangle &= \sum_{i\in V_+} v_i x_i + \sum_{i\in V_-} v_i x_i \\ &< \sum_{i\in V_+} v_i\,(b_i + d_i) + \sum_{i\in V_-} v_i\,(a_i - d_i) \\ &= \sum_{i\in V_+} v_i b_i + \sum_{i\in V_-} v_i a_i + \sum_{i\in V_+\cup V_-} \frac{t-\delta_A^*(v)}{2n} \\ &\leq \sum_{i\in V_+} v_i b_i + \sum_{i\in V_-} v_i a_i + \frac{t-\delta_A^*(v)}{2}.\end{aligned}$$

Since $A = \{x \in \mathbb{R}^n : \forall i\{1,\dots,n\}, a_i \leq x_i \leq b_i\}$ is non-empty and b_i, $a_j \in \mathbb{R}$ for each $i \in V_+$ and $j \in V_-$, there exists $c_k \in \mathbb{R}$ for each $k \in V_0$ such that $\bar{y} = ((b_i)_{i\in V_+}, (a_j)_{j\in V_-}, (c_k)_{k\in V_0}) \in A$. Hence,

$$\langle v, x\rangle < \sum_{i\in V_+} v_i b_i + \sum_{i\in V_-} v_i a_i + \frac{t-\delta_A^*(v)}{2} \leq \delta_A^*(v) + \frac{t-\delta_A^*(v)}{2} < t.$$

This is a contradiction. By Lemma 3.1 and Lemma 3.2, for each $x \in L(v, \geq, t)$,

$$g(x) \geq m := \min\left\{\min_{i\in V_+} y_i, \min_{i\in V_-} z_i\right\} > \sum_{i=1}^n m_i = 0,$$

where $y_i := g_i\,(b_i + d_i) + \sum_{j\neq i} m_j$, and $z_i := g_i\,(a_i - d_i) + \sum_{j\neq i} m_j$. Hence, $g_v(t) = \inf\{g(x) : \langle v, x\rangle \geq t\} \geq m > 0$. By Theorem 2.2, $\{g(x) \leq 0\}$ satisfies the Q-CCCQ. □

Theorem 2.3 and Theorem 3.1 imply the following Lagrange-type duality theorem.

Theorem 3.2 *Let g_i be a convex function from $\mathbb{R}$ to $\mathbb{R}$ for each $i \in \{1,\dots,n\}$, $g(x) = \sum_{i=1}^n g_i(x_i)$, and $A = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ is non-empty.*

Then, for each real-valued convex function f on $\mathbb{R}^n$, there exists a finite subset $S_0 = \{w_1,\dots,w_m\} \subset S$ such that $g_{w_j}^{-1}(0) \in \mathbb{R}$ for each $j \in \{1,\dots,m\}$, and

$$\inf_{x\in A} f(x) = \max_{\lambda\in\mathbb{R}_+^m} \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^m \lambda_j (w_j(x) - g_{w_j}^{-1}(0)) \right\}.$$

4 A Duality Theorem for Separable Convex Programming without the Slater Condition

If a convex inequality system satisfies the Slater condition, then we can obtain Lagrange duality theorem in order to solve the problem. However, the Slater condition is often not satisfied for many problems arising in applications. In this section, we study a duality theorem for a separable convex programming problem, whose constraints do not satisfy the Slater condition.

In the following theorem, we characterize the hypo-epi-inverse of g_w for separable convex function g.

Theorem 4.1 *Let* $I := \{1, \ldots, n\}$, g_i *be a real-valued convex function on* $\mathbb{R}$ *for each* $i \in I$, $g(x) = \sum_{i=1}^{n} g_i(x_i)$, $A = \{x \in \mathbb{R}^n : g(x) \leq 0\}$, $m_i = \inf g_i(\mathbb{R})$, $L(g_i, \leq, m_i) = \{t \in \mathbb{R} : a_i \leq t \leq b_i\}$, $w \in \mathbb{R}^n \setminus \{0\}$, $W_+ := \{i \in I : w_i > 0\}$, $W_- := \{i \in I : w_i < 0\}$, *and* $W_0 := \{i \in I : w_i = 0\}$. *Assume that (A) is satisfied. Then,*

$$g_w^{-1}(0) = \sum_{i \in W_+} w_i b_i + \sum_{i \in W_-} w_i a_i.$$

Proof Since A is non-empty, there exists $\bar{x} \in A$. For each $i \in \{1, \ldots, n\}$, $a_i \leq \bar{x}_i \leq b_i$ by Lemma 3.1 (ii), and hence $b_i > -\infty$ and $a_i < \infty$. Let $r := \sum_{i \in W_+} w_i b_i + \sum_{i \in W_-} w_i a_i$ and $d_i := \frac{\varepsilon}{2nw_i}$ for each $i \in I$.

Case 1. "there exists $i_0 \in W_+$ such that $b_{i_0} = \infty$" or "there exists $i_0 \in W_-$ such that $a_{i_0} = -\infty$".
Clearly, $r = \infty$. Assume that there exists $i_0 \in W_+$ such that $b_{i_0} = \infty$. Let $t \in \mathbb{R}$ and

$$\bar{x}^t = \left(\bar{x}_1, \ldots, \bar{x}_{i_0 - 1}, \max \left\{ a_i, \frac{t - \sum_{j \neq i_0} w_j \bar{x}_j}{w_{i_0}} \right\}, \bar{x}_{i_0+1}, \ldots, \bar{x}_n \right);$$

then, $\langle w, \bar{x}^t \rangle \geq t$ and $g(\bar{x}^t) = 0$. Hence, $g_w(t) \leq 0$ for each $t \in \mathbb{R}$. This shows that $g_w^{-1}(0) = \infty$. The proof of the case "there exists $i_0 \in W_-$ such that $a_{i_0} = -\infty$" is similar and omitted.

Case 2. "for each $i \in W_+$, $b_i \in \mathbb{R}$" and "for each $i \in W_-$, $a_i \in \mathbb{R}$".
Now we prove that

$$r = \max\{t \in \mathbb{R} : g_w(t) \leq 0\} = g_w^{-1}(0).$$

By Lemma 3.1,

$$g_w(r) = \inf \{g(x) : \langle w, x \rangle \geq r\} \leq \sum_{i \in W_+} g_i(b_i) + \sum_{i \in W_-} g_i(a_i) + \sum_{i \in W_0} g_i(\bar{x}_i) = 0.$$

Let $t > r$ and $\varepsilon = t - r > 0$. Then, for each $x \in \mathbb{R}^n$ with $\langle w, x \rangle \geq t$, "there exists $i_0 \in W_+$ such that $w_i x_i \geq w_i b_i + \frac{\varepsilon}{2n}$" or "there exists $i_0 \in W_-$ such that $w_i x_i \geq w_i a_i + \frac{\varepsilon}{2n}$". Hence,

$$\begin{aligned}
&g_w(t) \\
&= \inf\{g(x) : \langle w, x \rangle \geq t\} \\
&\geq \min \left\{ \min_{i \in W_+} \left\{ g_i(b_i + d_i) + \sum_{j \neq i} m_j \right\}, \min_{i \in W_-} \left\{ g_i(a_i + d_i) + \sum_{j \neq i} m_j \right\} \right\} \\
&> 0.
\end{aligned}$$

This shows that $g_w^{-1}(0) = r$. □

In the following theorem, we show a duality theorem for a separable convex programming problem, whose constraints do not satisfy the Slater condition.

Theorem 4.2 *Let $I = \{1, \ldots, n\}$, g_i be a convex function from $\mathbb{R}$ to $\mathbb{R}$ for each $i \in I$, $g(x) = \sum_{i=1}^n g_i(x_i)$, $L(g_i, \leq, \inf_{t\in\mathbb{R}} g_i(t)) = \{t \in \mathbb{R} : a_i \leq t \leq b_i\}$ for each $i \in I$, $I^+ := \{i \in I : b_i \in \mathbb{R}\}$, $I^- := \{i \in I : a_i \in \mathbb{R}\}$, $\bar{b} \in \mathbb{R}^n$ satisfying $\bar{b}_i = b_i$ for each $i \in I^+$, and $\bar{a} \in \mathbb{R}^n$ satisfying $\bar{a}_i = a_i$ for each $i \in I^-$. Assume that (A) is satisfied.*

Then, for each real-valued convex function f on $\mathbb{R}^n$, there exist v, $w \in \mathbb{R}^n_+$ such that $v_i = 0$ for each $i \notin I^+$, $w_i = 0$ for each $i \notin I^-$, and

$$\inf_{x\in A} f(x) = \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \langle v, x - \bar{b}\rangle - \langle w, x - \bar{a}\rangle \right\}.$$

Proof Let f be a real-valued convex function on $\mathbb{R}^n$. By Theorem 3.2, there exist a finite subset $S_0 = \{w_1, \ldots, w_m\} \subset S$ and $\bar{\lambda} \in \mathbb{R}^m_+$ such that $g_{w_j}^{-1}(0) \in \mathbb{R}$ for each $j \in \{1, \ldots, m\}$, and

$$\inf_{x\in A} f(x) = \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^m \bar{\lambda}_j (w_j(x) - g_{w_j}^{-1}(0)) \right\}.$$

Let $j \in \{1, \ldots, m\}$, $W_+^j = \{i : (w_j)_i > 0\}$ and $W_-^j = \{i : (w_j)_i < 0\}$. By Theorem 4.1, $g_{w_j}^{-1}(0) = \sum_{i\in W_+^j} (w_j)_i b_i + \sum_{i\in W_-^j} (w_j)_i a_i$. Since $g_{w_j}^{-1}(0) \in \mathbb{R}$, $W_+^j \subset I^+$ and $W_-^j \subset I^-$. Let $\bar{v}$, $\bar{w} \in \mathbb{R}^n$ with $\bar{v}_t = \sum_{j=1}^m \sum_{t\in W_+^j} \bar{\lambda}_j (w_j)_t$ for each $t \in \bigcup_{j=1}^m W_+^j$, $\bar{v}_t = 0$ for each $t \notin \bigcup_{j=1}^m W_+^j$, $\bar{w}_t = \sum_{j=1}^m \sum_{t\in W_-^j} \bar{\lambda}_j |(w_j)_t|$ for each $i \in \bigcup_{j=1}^m W_-^j$ and $\bar{w}_t = 0$ for each $i \notin \bigcup_{j=1}^m W_-^j$. Then,

$$\begin{aligned}
&\inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^m \bar{\lambda}_j (w_j(x) - g_{w_j}^{-1}(0)) \right\} \\
&= \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^m \bar{\lambda}_j \left(\sum_{i\in W_+^j} (w_j)_i (x_i - b_i) + \sum_{i\in W_-^j} (w_j)_i (x_i - a_i) \right) \right\} \\
&= \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^m \sum_{i\in W_+^j} \bar{\lambda}_j (w_j)_i (x_i - b_i) + \sum_{j=1}^m \sum_{i\in W_-^j} \bar{\lambda}_j |(w_j)_i| (-x_i + a_i) \right\} \\
&= \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \langle \bar{v}, x - \bar{b}\rangle - \langle \bar{w}, x - \bar{a}\rangle \right\},
\end{aligned}$$

Let $i \in I$. If $i \in \bigcup_{j=1}^m W_+^j$, then $b_i \in \mathbb{R}$. Hence, if $b_i \notin \mathbb{R}$, then $i \notin \bigcup_{j=1}^m W_+^j$, that is, $\bar{v}_i = 0$. Similarly, we can prove that $a_i \notin \mathbb{R}$ implies $\bar{w}_i = 0$. This completes the proof. □

By Theorem 4.2, we show the following necessary and sufficient optimality condition.

Theorem 4.3 *Let f be a real-valued convex function f on $\mathbb{R}^n$, g_i a real-valued convex function on $\mathbb{R}$ for each $i \in \{1, \dots, n\}$, $g(x) = \sum_{i=1}^{n} g_i(x_i)$, and $L(g_i, \leq, \inf_{t \in \mathbb{R}} g_i(t)) = \{t \in \mathbb{R} : a_i \leq t \leq b_i\}$ for each $i \in I$. Assume that (A) is satisfied.*

Then, $\bar{x} \in A$ is a global minimizer of f in A if and only if there exist v, $w \in \mathbb{R}^n_+$ such that $b_i > \bar{x}_i$ implies $v_i = 0$, $a_i < \bar{x}_i$ implies $w_i = 0$, and

$$w - v \in \partial f(\bar{x}).$$

Proof Let $\bar{b}$, $\bar{a} \in \mathbb{R}^n$ such that $\bar{b}_i = b_i$ for each $i \in I^+$, and $\bar{a}_i = a_i$ for each $i \in I^-$. Assume that $\bar{x} \in A$ is a global minimizer of f in A. By Theorem 4.2, there exist v, $w \in \mathbb{R}^n_+$ such that $v_i = 0$ for each $i \notin I^+$, $w_i = 0$ for each $i \notin I^-$, and

$$\inf_{x \in A} f(x) = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \left\langle v, x - \bar{b} \right\rangle - \langle w, x - \bar{a} \rangle \right\}.$$

Since $a_i \leq \bar{x}_i \leq b_i$ for each $i \in I$,

$$\begin{aligned} f(\bar{x}) &= \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \left\langle v, x - \bar{b} \right\rangle - \langle w, x - \bar{a} \rangle \right\} \\ &\leq f(\bar{x}) + \left\langle v, \bar{x} - \bar{b} \right\rangle - \langle w, \bar{x} - \bar{a} \rangle \\ &\leq f(\bar{x}). \end{aligned}$$

This shows that $\left\langle v, \bar{x} - \bar{b} \right\rangle = \langle w, \bar{x} - \bar{a} \rangle = 0$ and $\bar{x}$ is a global minimizer of $f + \left\langle v, \cdot - \bar{b} \right\rangle - \langle w, \cdot - \bar{a} \rangle$ in $\mathbb{R}^n$. Therefore,

$$0 \in \partial(f + \left\langle v, \cdot - \bar{b} \right\rangle - \langle w, \cdot - \bar{a} \rangle)(\bar{x}) = \partial f(\bar{x}) + v - w,$$

that is, $w - v \in \partial f(\bar{x})$. Since $b_i \notin \mathbb{R}$ implies $v_i = 0$,

$$\left\langle v, \bar{x} - \bar{b} \right\rangle = \sum_{i \in I^+} v_i(\bar{x}_i - \bar{b}_i).$$

For each $i \in I^+$, $v_i(\bar{x}_i - \bar{b}_i) = 0$ since $\bar{x}_i \leq b_i = \bar{b}_i$ and $\left\langle v, \bar{x} - \bar{b} \right\rangle = 0$. Now we assume that $b_i > \bar{x}_i$. If $b_i \notin \mathbb{R}$, then $v_i = 0$. If $b_i \in \mathbb{R}$, then $\bar{b}_i = b_i > \bar{x}_i$. This implies that $v_i = 0$. Similarly, we can prove that $a_i < \bar{x}_i$ implies $w_i = 0$.

Conversely, assume that there exist v, $w \in \mathbb{R}^n_+$ such that $b_i > \bar{x}_i$ implies $v_i = 0$, $a_i < \bar{x}_i$ implies $w_i = 0$, and $w - v \in \partial f(\bar{x})$. Then, we can check easily that $\bar{x}$ is a global minimizer of $f + \left\langle v, \cdot - \bar{b} \right\rangle - \langle w, \cdot - \bar{a} \rangle$ in $\mathbb{R}^n$. Furthermore, $\left\langle v, \bar{x} - \bar{b} \right\rangle = \langle w, \bar{x} - \bar{a} \rangle = 0$. Actually, if $b_i \notin \mathbb{R}$, then $b_i = \infty > \bar{x}_i$. This implies $v_i = 0$. If $b_i \in \mathbb{R}$, then $\bar{b}_i = b_i \geq \bar{x}_i$. This shows that $v_i(\bar{b}_i - \bar{x}_i) = 0$ since "$b_i > \bar{x}_i$ implies $v_i = 0$". Similarly, we can prove that $w_i(\bar{x}_i - \bar{a}_i) = 0$ for each $i \in \{1, \dots, n\}$. Hence $\left\langle v, \bar{x} - \bar{b} \right\rangle = \langle w, \bar{x} - \bar{a} \rangle = 0$.

Therefore,

$$\begin{aligned}\inf_{x\in A} f(x) &\geq \inf_{x\in\mathbb{R}^n} \left\{f(x) + \left\langle v, x-\bar{b}\right\rangle - \langle w, x-\bar{a}\rangle\right\} \\ &= f(\bar{x}) + \left\langle v, \bar{x}-\bar{b}\right\rangle - \langle w, \bar{x}-\bar{a}\rangle \\ &= f(\bar{x}) \\ &\geq \inf_{x\in A} f(x)\end{aligned}$$

since $\left\langle v, x-\bar{b}\right\rangle - \langle w, x-\bar{a}\rangle \leq 0$ for each $x \in A$. This completes the proof. □

We can also prove Theorem 4.3 using the optimality condition in terms of the subdifferential and the normal cone to the feasible set A. In this paper, we show Theorem 4.3 as an application of Theorem 4.2.

If the Slater condition is satisfied, then we can solve the separable convex programming problem by Lagrange duality. Even if the Slater condition is not satisfied, we can solve the separable convex programming problem by the necessary and sufficient optimality condition in Theorem 4.3. Finally, we show the following example, which illustrate usefulness of our results.

Example 4.1 Let g_1 be the following function on $\mathbb{R}$:

$$g_1(t) := \begin{cases} (t+1)^2, \; t \leq -1, \\ 0, \qquad\quad -1 \leq t \leq 1, \\ (t-1)^2, \; t \geq 1. \end{cases}$$

Let g be the following function on $\mathbb{R}^2$: for each $x = (x_1, x_2) \in \mathbb{R}^2$,

$$g(x) = g_1(x_1) + g_1(x_2).$$

Clearly, g is separable convex, $A = \{x \in \mathbb{R}^2 : g(x) \leq 0\} = [-1,1] \times [-1,1]$, $L(g_1, \leq, 0) = [a_1, b_1] = [-1,1]$, and $L(g, <, 0)$ is empty. Additionally, since

$$\operatorname{epi}\left(\inf_{\lambda\in\mathbb{R}_+} (\lambda g)^*\right) = \{(v_1, v_2, \alpha) \in \mathbb{R}^3 : |v_1| + |v_2| \leq \alpha\},$$

and

$$\bigcup_{\lambda\in\mathbb{R}_+} \operatorname{epi}(\lambda g)^* = \{(v_1, v_2, \alpha) \in \mathbb{R}^3 : |v_1| + |v_2| < \alpha\} \cup \{(0,0,0)\},$$

the statement (i) of Theorem 2.1 is not satisfied.

On the other hand, $\{g(x) \leq 0\}$ satisfies the Q-CCCQ. We show that

$$\begin{aligned} K &:= \operatorname{cone}\operatorname{conv} \bigcup_{(k,w)\in G_g} \{(w,\delta) \in \mathbb{R}^3 : k^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[\\ &= \{(v_1, v_2, \alpha) \in \mathbb{R}^3 : |v_1| + |v_2| \leq \alpha\}\end{aligned}$$

Clearly, $(w, k^{-1}(0)) \in \operatorname{epi}\delta_A^* = \{(v_1, v_2, \alpha) \in \mathbb{R}^3 : |v_1| + |v_2| \leq \alpha\}$ for each $(k, w) \in G_g$; in detail, see [10,13,15]. Since $\operatorname{epi}\delta_A^*$ is a closed convex cone,

$K \subset \mathrm{epi}\delta_A^*$. Conversely, let $(v_1, v_2, \alpha) \in \mathbb{R}^3$ with $|v_1| + |v_2| \leq \alpha$. By Theorem 4.1, $g_{(v_1,v_2)}^{-1}(0) = |v_1| + |v_2|$ since $A = [-1,1] \times [-1,1]$. Let $\mathrm{cl}g_{(v_1,v_2)}$ is the lower semicontinuous hull of $g_{(v_1,v_2)}$. Then, $(\mathrm{cl}g_{(v_1,v_2)}, (v_1, v_2)) \in G_g$ and $\mathrm{cl}g_{(v_1,v_2)}^{-1}(0) = g_{(v_1,v_2)}^{-1}(0)$; in detail, see [13,15]. Hence,

$$(v_1, v_2, \alpha) = \left(v_1, v_2, \mathrm{cl}g_{(v_1,v_2)}^{-1}(0)\right) + \left(0, 0, \alpha - \mathrm{cl}g_{(v_1,v_2)}^{-1}(0)\right) \in K.$$

This shows that K is closed, that is, $\{g(x) \leq 0\}$ satisfies the Q-CCCQ.

Let f be the following real-valued convex function on $\mathbb{R}^2$:

$$f(x) := (x_1 - 2)^2 + (x_2 - 2)^2.$$

Finally, we solve the following separable convex programming problem by Theorem 4.2 and Theorem 4.3:

(P) min $f(x)$, s.t. $x \in A = \{x \in \mathbb{R}^2 : g(x) \leq 0\}$.

Let $\bar{b} := (1,1)$, and $\bar{a} := (-1,-1)$. By Theorem 4.2,

$$\inf_{x \in A} f(x) = \max_{v,w \in \mathbb{R}^2_+} \inf_{x \in \mathbb{R}^2} \left\{ f(x) + \left\langle v, x - \bar{b} \right\rangle - \langle w, x - \bar{a} \rangle \right\}.$$

Actually, by completing the square,

$$\begin{aligned}
&\max_{v,w \in \mathbb{R}^2_+} \inf_{x \in \mathbb{R}^2} \left\{ f(x) + \left\langle v, x - \bar{b} \right\rangle - \langle w, x - \bar{a} \rangle \right\} \\
&= \max_{v,w \in \mathbb{R}^2_+} \left\{ -\left(\frac{v_1 - w_1 - 4}{2}\right)^2 - \left(\frac{v_2 - w_2 - 4}{2}\right)^2 + 8 - (v_1 + v_2 + w_1 + w_2) \right\} \\
&= \max_{v,w \in \mathbb{R}^2_+} \left\{ -\frac{1}{4}(v_1 - (2 + w_1))^2 - \frac{1}{4}(v_2 - (2 + w_2))^2 - 2(w_1 + w_2) + 2 \right\} \\
&= 2 \\
&= \inf_{x \in A} f(x).
\end{aligned}$$

By Theorem 4.3, $\bar{x} \in A$ is a global minimizer of f in A if and only if there exist v, $w \in \mathbb{R}^n_+$ such that $1 > \bar{x}_i$ implies $v_i = 0$, $-1 < \bar{x}_i$ implies $w_i = 0$, and $w - v \in \partial f(\bar{x})$. For each $x \in A$, we can calculate that

$$\partial f(x) \subset \{z \in \mathbb{R}^2 : z_1 < 0, z_2 < 0\}.$$

If $\bar{x}_1 < 1$ or $\bar{x}_2 < 1$, then $v_1 v_2 = 0$. This shows that $w_1 - v_1 \geq 0$ or $w_2 - v_2 \geq 0$ since $w \in \mathbb{R}^2_+$. Hence, if $\bar{x}$ is a global minimizer, then $\bar{x} = (1,1)$. Actually,

$$\inf_{x \in A} f(x) = 2 = f(1,1).$$

5 Conclusions

In this paper, we study duality theorems for separable convex programming without any qualifications. At first, in Section 3, we prove that a separable convex inequality system always satisfies Q-CCCQ by using a necessary and sufficient condition for Q-CCCQ. We show a Lagrange-type duality theorem for separable convex programming without any qualifications. If a convex inequality system satisfies the Slater condition, then we can solve a problem by using Lagrange duality. In order to solve a separable convex programming problem, whose constraints do not satisfy the Slater condition, we introduce another duality theorem in terms of Lagrange-type duality in Section 4. In addition, we introduce a necessary and sufficient optimality condition for separable convex programming. We show Example 4.1 which illustrate usefulness of our results.

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