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Nonlinear Error Bounds for Quasiconvex Inequality Systems

Satoshi Suzuki · Daishi Kuroiwa

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Abstract The error bound is an inequality that restricts the distance from a vector to a given set by a residual function. The error bound has so many useful applications, for example in variational analysis, in convergence analysis of algorithms, in sensitivity analysis, and so on. For convex inequality systems, Lipschitzian error bounds are studied mainly. If an inequality system is not convex, it is difficult to show the existence of a Lipschitzian global error bound in general. Hence for nonconvex inequality systems, Hölderian error bounds and nonlinear error bounds have been investigated. For quasiconvex inequality systems, there are so many examples such that systems do not have Lipschitzian and Hölderian error bounds. However, the research of nonlinear error bounds for quasiconvex inequality systems have not been investigated yet as far as we know.

In this paper, we study nonlinear error bounds for quasiconvex inequality systems. We show the existence of a global nonlinear error bound by a generator of a quasiconvex function and a constraint qualification. We show well-posedness of a quasiconvex function by the error bound.

Keywords nonlinear error bound · quasiconvex inequality system · generator of a quasiconvex function · constraint qualification · well-posedness

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S. Suzuki (corresponding author)
Department of Mathematics, Shimane University, Shimane, Japan
E-mail: suzuki@math.shimane-u.ac.jp

D. Kuroiwa
Department of Mathematics, Shimane University, Shimane, Japan
E-mail: kuroiwa@math.shimane-u.ac.jp

1 Introduction

The error bound is an inequality that restricts the distance from a vector in a test set to a given set by a residual function. In many cases, the given set is defined by an inequality system. The test set consists of vectors whose distances to the given set are of interests in applications. Some important examples of the test set are the whole space, a compact subset, and a neighborhood of a vector. If the test set is the whole space, the error bound is called global, and if the test set is a neighborhood of a vector, the error bound is called local. The error bound says that the value of the residual function at a test vector is a surrogate measure of the distance from the vector to the given set. The error bound has so many useful applications, for example in variational analysis, in convergence analysis of algorithms, in sensitivity analysis, and so on. One of the most important applications is well-posedness that plays a key role in convergence analysis of algorithms.

In the research of error bounds, various results are introduced by many researchers, see [1–15]. Lipschitzian error bounds, which have linear-type residual functions, are widely studied, see [1–4, 6–15]. Especially, for convex inequality systems, Lipschitzian error bounds have been investigated mainly. On the other hand, if inequality system is not convex, it is difficult to show the existence of a Lipschitzian global error bound in general. In this case, Hölderian error bounds, whose residual functions are written by power functions, and nonlinear error bounds, which have nonlinear-type residual functions, are studied, see [4–6, 10, 12–15]. In quasiconvex analysis, relations between well-posedness, well-behavior, and other related notions are studied precisely by Penot in [14–18]. However, the research of error bounds for quasiconvex inequality systems have not been investigated yet as far as we know. Moreover, for quasiconvex inequality systems, there are so many examples such that systems do not have Lipschitzian and Hölderian error bounds.

In this paper, we study nonlinear error bounds for quasiconvex inequality systems. We introduce a notion of generators of quasiconvex functions in [19–23]. We show the existence of a nonlinear global error bound by a generator and a constraint qualification. We show well-posedness of a quasiconvex function by the error bound.

The remainder of the paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we study nonlinear global error bounds for quasiconvex inequality systems. In Section 4, we show well-posedness of a quasiconvex function by the error bound. In addition, we show examples of nonlinear global error bounds for quasiconvex inequality systems.

2 Preliminaries

Let X be a Hilbert space, and $\langle x, y \rangle$ denote the inner product of two vectors x and y. Given a set $A \subset X$, we denote the closure, the convex hull, the boundary, and the conical hull generated by A, by $\mathrm{cl}A$, $\mathrm{co}A$, $\mathrm{bd}A$, and $\mathrm{cone}A$, respectively. By convention, we define cone $\emptyset = \{0\}$. The normal cone of A at $x \in A$ is defined as $N_A(x) := \{v \in X : \forall y \in A, \langle v, y - x\rangle \leq 0\}$. We define $d(x, A) := \inf_{y\in A} d(x, y)$. The indicator function δ_A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & otherwise. \end{cases}$$

Let f be a function from X to $\overline{\mathbb{R}} := [-\infty, \infty]$. We denote the domain of f by dom$f$, that is, dom$f := \{x \in X : f(x) < \infty\}$. The epigraph of f is defined as epi$f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$, and f is said to be convex if epi$f$ is convex. Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) := \{x \in X : f(x) \diamond \beta\}$$

for each $\beta \in \mathbb{R}$. A function f is said to be quasiconvex if for each $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true. A function f is said to be quasiaffine if f and $-f$ are quasiconvex. It is important to notice that f is lower semi-continuous (lsc) quasiaffine if and only if there exists $k \in Q$ and $w \in X$ such that $f = k \circ w$, where $Q := \{h : \mathbb{R} \to \overline{\mathbb{R}} : h \text{ is lsc and non-decreasing}\}$. Furthermore, f is lsc quasiconvex if and only if there exists $\{(k_j, w_j) : j \in J\} \subset Q \times X$ such that $f = \sup_{j\in J} k_j \circ w_j$, see [24,25] for more details. This result indicates that a lsc quasiconvex function consists of the supremum of a some family of lsc quasiaffine functions. A set $G = \{(k_j, w_j) : j \in J\} \subset Q \times X$ is said to be a generator of f if $f = \sup_{j\in J} k_j \circ w_j$. All lsc quasiconvex functions have at least one generator. Next, we introduce a generalized notion of the inverse function. The following function h^{-1} is said to be the hypo-epi-inverse of $h \in Q$:

$$h^{-1}(a) := \sup\{b \in \mathbb{R} : h(b) \leq a\}$$

for each $a \in \overline{\mathbb{R}}$. It is known that if $h \in Q$ has the inverse function, then the hypo-epi-inverse of h equals to the inverse, see [25]. In the present paper, we denote the hypo-epi-inverse of h by h^{-1}. Furthermore, we define $[\alpha]_+ := \max\{0, \alpha\}$ for each $\alpha \in \mathbb{R}$. It is clear that if $\alpha \geq \beta$, then $[\alpha]_+ \geq [\beta]_+$.

In this paper, we study the existence of the following non-decreasing function h on $\mathbb{R}_+ := [0, \infty)$ satisfying $h(0) = 0$, and for each $x \in T$,

$$d(x, A) \leq h([g(x)]_+), \tag{1}$$

where T is a subset of a Hilbert space X. Inequality (1) is called an error bound and $h([g(\cdot)]_+)$ is called a residual function. If h is a linear function, inequality (1) is called a Lipschitzian error bound. Especially, for convex inequality systems, Lipschitzian error bounds have been investigated mainly. On the other hand, for nonconvex inequality systems, Hölderian error bounds, $h(t) = t^\gamma$ for some $\gamma > 0$, and nonlinear error bounds, h is a nonlinear function, are studied. The set T consists of vectors whose distances to A are of interests in applications. If T is the whole space, inequality (1) is called a global error bound, and if T is a neighborhood of a vector, inequality (1) is called a local error bound. Inequality (1) says that the value of the residual function at $x \in T$ is a surrogate measure of the distance from x to A.

3 Nonlinear Global Error Bounds for Quasiconvex Inequality Systems

Throughout this paper, let X be a Hilbert space, g a lsc quasiconvex function from X to $\overline{\mathbb{R}}$, and $A = \{x \in X : g(x) \leq 0\}$ a nonempty set.

In this section, we study nonlinear global error bounds for quasiconvex inequality systems. At first, we introduce a suitable generator for nonlinear global error bounds. For each $w \in X$, let g_w be the following function from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$:

$$g_w(a) := \inf\{g(x) : \langle w, x\rangle \geq a\}.$$

It is clear that g_w is non-decreasing. Additionally, if g is lsc quasiconvex, then

$$g = \sup_{\|w\|=1} g_w \circ w = \sup_{\|w\|=1} (\mathrm{cl}\ g_w) \circ w,$$

where cl g_w is the lsc hull of g_w. We can prove the above equation by the separation theorem. Actually, it is clear that $g \geq \sup_{\|w\|=1} g_w \circ w \geq \sup_{\|w\|=1}(\mathrm{cl}\ g_w) \circ w$. By using the separation theorem for $x \in X$ and $L(g, \leq, g(x) - \varepsilon)$ for each $\varepsilon > 0$, we can show that $g = \sup_{\|w\|=1}(\mathrm{cl}\ g_w) \circ w$, in detail, see [21, 23, 25]. This means that $G = \{(\mathrm{cl}\ g_w, w) : \|w\| = 1\} \subset Q \times X$ is a generator of g. By using this generator $G = \{(\mathrm{cl}\ g_w, w) : \|w\| = 1\}$, we study nonlinear global error bounds for quasiconvex inequality systems.

For each $w \in X$ with $\|w\| = 1$ and $g_w^{-1}(0) \in \mathbb{R}$, let h_w be the following function from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$:

$$h_w(t) = g_w^{-1}(t) - g_w^{-1}(0).$$

Lemma 3.1 *Let $w_0 \in X$ with $\|w_0\| = 1$ and $g_{w_0}^{-1}(0) \in \mathbb{R}$. Then, the following statements hold:*

(i) h_{w_0} *is non-decreasing,*
(ii) $h_{w_0}(0) = 0$,
(iii) $\inf_{t>0} h_{w_0}(t) = 0$,
(iv) for each $t \in \overline{\mathbb{R}}$, $h_{w_0} \circ g_{w_0}(t) \geq t - g_{w_0}^{-1}(0)$.

Proof We can easily check that (i) and (ii) hold by the definition of h_{w_0}.

(iii) By the statements (i) and (ii), $\inf_{t>0} h_{w_0}(t) \geq h_{w_0}(0) = 0$. Assume that $\inf_{t>0} h_{w_0}(t) > 0$. Then, there exists $a, b \in \mathbb{R}$ such that $\inf_{t>0} h_{w_0}(t) > a > b > 0$. Since $a > b > 0 = g_{w_0}^{-1}(0) - g_{w_0}^{-1}(0)$, $g_{w_0}(a + g_{w_0}^{-1}(0)) \geq g_{w_0}(b + g_{w_0}^{-1}(0)) > 0$. Hence, there exists $t_0 \in \mathbb{R}$ such that $g_{w_0}(b + g_{w_0}^{-1}(0)) > t_0 > 0$. This implies that $a + g_{w_0}^{-1}(0) > b + g_{w_0}^{-1}(0) \geq g_{w_0}^{-1}(t_0) = \sup\{t \in \mathbb{R} \mid g_w(t) \leq t_0\}$, that is, $a > g_{w_0}^{-1}(t_0) - g_{w_0}^{-1}(0) = h_{w_0}(t_0) \geq \inf_{t>0} h_{w_0}(t)$. This is a contradiction.

(iv) For each $t \in \overline{\mathbb{R}}$,

$$\begin{aligned} h_{w_0} \circ g_{w_0}(t) &= g_{w_0}^{-1}(g_{w_0}(t)) - g_{w_0}^{-1}(0) \\ &= \sup\{a \in \mathbb{R} : g_{w_0}(a) \leq g_{w_0}(t)\} - g_{w_0}^{-1}(0) \\ &\geq t - g_{w_0}^{-1}(0). \end{aligned}$$

This completes the proof. □

Let h be the following function on $\mathbb{R}_+$:

$$h(t) = \sup_{\substack{\|w\|=1 \\ g_w^{-1}(0) \in \mathbb{R}}} h_w(t).$$

If $A \neq X$, there exists $w_0 \in X$ such that $\|w_0\| = 1$ and $g_{w_0}^{-1}(0) \in \mathbb{R}$. Actually, if $g_w^{-1}(0) = \infty$ for each $w \in X$ with $\|w\| = 1$, $g_w(t) \leq 0$ for each $t \in \mathbb{R}$. Then, $A = X$ since $0 \geq \sup_{\|w\|=1} g_w \circ w(x) = g(x)$ for each $x \in X$. This is a contradiction. Hence we can prove that h is a non-decreasing, $h(0) = 0$, and $\{h(t) : t \in \mathbb{R}_+\} \subset [0, \infty]$ by Lemma 3.1.

In the following theorem, we show that if a lsc quasiconvex inequality system satisfies a certain condition, then it has a nonlinear global error bound.

Theorem 3.1 *Assume that $A \neq X$, and for each $x \in \mathrm{bd}A$,*

$$N_A(x) \subset \mathrm{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x \rangle\}.$$

Then, g has a nonlinear global error bound, that is, for each $x \in \mathrm{dom}g$,

$$d(x, A) \leq h([g(x)]_+).$$

Proof Let $x \in \mathrm{dom}g$. If $x \in A$, then it is clear that the inequality holds since $g(x) \leq 0$ and $h(0) = 0$.

Assume that $x \notin A$, then $g(x) > 0$. Since A is closed convex, there exists the metric projection of x on A, $P_A(x) \in \mathrm{bd}A$. Then, $d(x, A) = \|x - P_A(x)\|$ and $0 \neq x - P_A(x) \in N_A(P_A(x))$. By the assumption, there exists $w_0 \in X$ and $\lambda > 0$ such that $x - P_A(x) = \lambda w_0$, $\|w_0\| = 1$, and $g_{w_0}^{-1}(0) = \langle w_0, P_A(x) \rangle$. Since $\|w_0\| = 1$, $\lambda = \|x - P_A(x)\|$. Hence,

$$\begin{aligned}
\|x - P_A(x)\|^2 &= \langle x - P_A(x), x - P_A(x) \rangle \\
&= \langle \lambda w_0, x - P_A(x) \rangle \\
&= \lambda(\langle w_0, x \rangle - \langle w_0, P_A(x) \rangle) \\
&= \|x - P_A(x)\|(\langle w_0, x \rangle - g_{w_0}^{-1}(0)).
\end{aligned}$$

By Lemma 3.1,

$$\begin{aligned}
d(x, A) &= \|x - P_A(x)\| \\
&= \langle w_0, x \rangle - g_{w_0}^{-1}(0) \\
&\leq h_{w_0} \circ g_{w_0}(\langle w_0, x \rangle) \\
&\leq h_{w_0} \circ g(x) \\
&\leq h \circ g(x) \\
&= h([g(x)]_+).
\end{aligned}$$

This completes the proof. □

Remark 3.1 The following assumption

$$N_A(x) \subset \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x\rangle\}$$

is called the basic constraint qualification for quasiconvex programming (Q-BCQ). Q-BCQ is a necessary and sufficient constraint qualification for an optimality condition via quasiconvex programming. If A is compact, or f is convex and $L(f, <, 0)$ is nonempty, then the assumption holds. Hence, Q-BCQ is not so strong for quasiconvex inequality systems, in detail, see [20, 21] and Section 4.

4 Discussion

In this section, we discuss applications and usefulness of our results. We study well-posedness of a quasiconvex function as an application of a nonlinear global error bound. We show examples of nonlinear global error bounds which are not consequences of previous results. In addition, we observe the basic constraint qualification for quasiconvex programming, Q-BCQ, as an assumption of our results.

At first, we show the following result concerned with well-posedness.

Corollary 4.1 *Assume that* $A \neq X$,

$$N_A(x) \subset \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x\rangle\}$$

for each $x \in \text{bd}A$, *and* $\inf_{t>0} h(t) = 0$. *Let* $\{x_k\} \subset X$ *such that* $g(x_k)$ *converges to* 0. *Then,* $\{d(x_k, A)\}$ *converges to* 0.

Proof Since $g(x_k)$ converges to 0, we assume that $\{x_k\} \subset \text{dom}g$ without loss of generality. Now we prove that $h([g(x_k)]_+)$ converges to 0. Since $\inf_{t>0} h(t) = 0$, for each $\varepsilon > 0$, there exists $t_0 > 0$ such that $h(t_0) < \varepsilon$. Since $g(x_k)$ converges to 0, there exists $K \in \mathbb{N}$ such that for each $k \geq K$, $g(x_k) < t_0$. Since h is non-decreasing, for each $k \geq K$,

$$h([g(x_k)]_+) \leq \max\{h(0), h(g(x_k))\} \leq h(t_0) < \varepsilon.$$

Hence $h([g(x_k)]_+)$ converges to 0. By Theorem 3.1, for each $k \in \mathbb{N}$,

$$0 \leq d(x_k, A) \leq h([g(x_k)]_+),$$

This shows that $d(x_k, A)$ converges to 0. □

Remark 4.1 A sequence $\{x_k\} \subset X$ satisfying $g(x_k)$ converges to $\inf_{x\in X} g(x)$ is called a minimizing sequence. A function g is well-posed if for each minimizing sequence $\{x_k\} \subset X$, $\{d(x_k, S)\}$ converges to 0, where $S = \{x \in X : g(x) = \inf_{x\in X} g(y)\}$. Hence, if $\emptyset \neq S \neq X$, $\inf_{x\in X} g(y) = 0$,

$$N_S(x) \subset \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x\rangle\}$$

for each $x \in \text{bd}S$, and $\inf_{t>0} h(t) = 0$, then g is well-posed by Corollary 4.1.

Next, we show examples of nonlinear global error bounds for quasiconvex inequality systems.

Example 4.1 Let g be the following function on $\mathbb{R}^2$:

$$g(x) := \begin{cases} \log(\|x\|) & x \neq 0, \\ -\infty & x = 0. \end{cases}$$

Then, g is lsc quasiconvex, and $A = \{x \in \mathbb{R}^2 : g(x) \leq 0\} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. It is clear that the inequality system does not have Lipschitzian and Hölderian global error bounds. We study a nonlinear global error bounds by our results.

Let $w \in \mathbb{R}^2$ with $\|w\| = 1$. Then,

$$g_w(t) = \begin{cases} \log t & t > 0, \\ -\infty & t \leq 0, \end{cases}$$

and $g_w^{-1}(t) = e^t$.

Now we show that

$$N_A(x) \subset \text{cone}\{w \in \mathbb{R}^2 : \|w\| = 1, g_w^{-1}(0) = \langle w, x \rangle\}$$

for each $x \in \text{bd}A$. Let $x_0 \in \text{bd}A$ and $v \in N_A(x_0)$. Then $\|x_0\| = 1$ and $v \in \text{cone}\{x_0\}$. Furthermore,

$$g_{x_0}^{-1}(0) = e^0 = 1 = \langle x_0, x_0 \rangle .$$

This shows that $x_0 \in \{w \in \mathbb{R}^2 : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0 \rangle\}$. Hence,

$$v \in \text{cone}\{x_0\} \subset \text{cone}\{w \in \mathbb{R}^2 : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0 \rangle\}.$$

By Theorem 3.1, g has a nonlinear global error bound.

For each $w \in \mathbb{R}^2$ with $\|w\| = 1$,

$$h_w(t) = g_w^{-1}(t) - g_w^{-1}(0) = e^t - 1.$$

Hence,

$$h(t) = \sup_{\substack{\|w\|=1 \\ g_w^{-1}(0) \in \mathbb{R}}} h_w(t) = e^t - 1.$$

Let $x \in \mathbb{R}^2$. If $x \notin A$, then

$$h([g(x)]_+) = h(g(x)) = e^{\log(\|x\|)} - 1 = \|x\| - 1 = d(x, A).$$

Additionally, if $x \in A$, then

$$h([g(x)]_+) = h(0) = 0 = d(x, A).$$

This shows that g has a nonlinear global error bound.

In addition, since $\inf_{t>0} h(t) = 0$, for each $\{x_k\} \subset X$ satisfying $g(x_k)$ converges to 0, $\{d(x_k, A)\}$ converges to 0 by Corollary 4.1.

Example 4.2 Let g be the following function on $\mathbb{R}^2$:

$$g(x_1,x_2) := \begin{cases} -\frac{1}{2} & (x_1,x_2)=(0,0), \\ \frac{1}{2} & (x_1,x_2) \in (\mathbb{R}\setminus\{0\})\times\{0\}, \\ x_2+\frac{1}{2} & (x_1,x_2)\in\mathbb{R}\times(0,\infty), \\ \frac{|x_1|}{\sqrt{x_1^2+x_2^2}}-\frac{1}{2} & (x_1,x_2)\in\mathbb{R}\times[-1,0), \\ -x_2-\frac{1}{2} & (x_1,x_2)\in\mathbb{R}\times(-\infty,-1). \end{cases}$$

Then, g is lsc quasiconvex, and

$$A=\{x\in\mathbb{R}^2 : g(x)\le 0\} = \mathrm{co}\left\{(0,0), \left(\frac{1}{\sqrt{3}},-1\right)\left(-\frac{1}{\sqrt{3}},-1\right)\right\}.$$

We can check that the inequality system does not have Lipschitzian and Hölderian global error bounds.

Since A is compact,

$$N_A(x)\subset \mathrm{cone}\{w\in X : \|w\|=1, g_w^{-1}(0)=\langle w,x_0\rangle\}$$

for each $x\in A$, in detail, see [21] and the latter half of this section. Hence by Theorem 3.1, g has a nonlinear global error bound.

We can calculate $h_{(\pm1,0)}$ and $h_{(0,\pm1)}$ as follows:

$$h_{(\pm1,0)}(t) = \begin{cases} -\infty & t\in\left(-\infty,-\frac{1}{2}\right), \\ \frac{2t+1}{\sqrt{4-(2t+1)^2}}-\frac{1}{\sqrt{3}} & t\in\left[-\frac{1}{2},\frac{1}{2}\right], \\ \infty & t\in\left(\frac{1}{2},\infty\right), \end{cases}$$

$$h_{(0,\pm1)}(t) = \begin{cases} -\infty & t\in\left(-\infty,-\frac{1}{2}\right), \\ 0 & t\in\left[-\frac{1}{2},\frac{1}{2}\right], \\ t-\frac{1}{2} & t\in\left(\frac{1}{2},\infty\right). \end{cases}$$

Let $\hat{h}=\max\{h_{(0,1)},h_{(0,-1)},h_{(1,0)},h_{(-1,0)}\}$, then $h\ge\hat{h}$. In this example, we can show the existence of nonlinear global error bounds by $\hat{h}$. Actually,

$$\hat{h}(t) = \begin{cases} \frac{2t+1}{\sqrt{4-(2t+1)^2}}-\frac{1}{\sqrt{3}} & t\in\left[0,\frac{1}{2}\right], \\ \infty & t\in\left(\frac{1}{2},\infty\right), \end{cases}$$

and

$$\hat{h}([g(x_1,x_2)]_+) = \begin{cases} \infty & (x_1,x_2)\in\mathbb{R}\times((-\infty,-1)\cup(0,\infty)), \\ \infty & (x_1,x_2)\in(\mathbb{R}\setminus\{0\})\times\{0\}, \\ 0 & (x_1,x_2)\in A, \\ -\frac{|x_1|}{x_2}-\frac{1}{\sqrt{3}} & (x_1,x_2)\in A^c\cap(\mathbb{R}\times[-1,0)), \end{cases}$$

where A^c is the complement of A. Let $x=(x_1,x_2)\in\mathbb{R}^2$. If $x\in A$, it is clear that

$$\hat{h}([g(x)]_+)=0=d(x,A).$$

If $x \notin A$ and $x_2 = -1$, then

$$\hat{h}([g(x)]_+) = |x_1| - \frac{1}{\sqrt{3}} = d(x, A). \tag{2}$$

If $x \notin A$ and $0 > x_2 > -1$, then

$$\begin{aligned}
\hat{h}([g(x)]_+) &= -\frac{|x_1|}{x_2} - \frac{1}{\sqrt{3}} \\
&= -\frac{\left|\frac{x_1}{x_2}\right|}{-1} - \frac{1}{\sqrt{3}} \\
&= \hat{h}\left(\left[g\left(\frac{x_1}{x_2}, -1\right)\right]_+\right) \\
&= d\left(\left(\frac{x_1}{x_2}, -1\right), A\right) \\
&> d(x, A).
\end{aligned}$$

Otherwise,

$$\hat{h}([g(x)]_+) = \infty > d(x, A).$$

Hence, g has a nonlinear global error bound with a residual function $\hat{h}(g[\cdot]_+)$. It is not necessary to calculate h_w for each $w \in \mathbb{R}^2$ with $\|w\| = 1$ in this example.

Furthermore, since $\inf_{t>0} \hat{h}(t) = 0$, for each $\{x_k\} \subset X$ satisfying $g(x_k)$ converges to 0, $\{d(x_k, A)\}$ converges to 0 by Corollary 4.1.

Example 4.3 Let g be the following function on $\mathbb{R}$:

$$g(x) := \begin{cases} 0 & x \in [-1, 1], \\ (x+1)^2 & x \in [-2, -1], \\ 1 & x \in [-3, -2] \cup (1, 2], \\ x - 1 & x \in [2, 3], \\ -x - 2 & x \in [-4, -3], \\ 2 & x \in (-\infty, -4] \cup [3, \infty). \end{cases}$$

It is clear that the inequality system does not have Lipschitzian and Hölderian global error bounds. Additionally, we can check easily that

$$N_A(x) \subset \operatorname{cone}\{w \in \mathbb{R} : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0 \rangle\}$$

for each $x \in A$. Hence by Theorem 3.1, g has a nonlinear global error bound.

We can calculate that

$$g_1(t) = \begin{cases} 0 & t \in (\infty, 1], \\ 1 & t \in (1, 2], \\ t - 1 & t \in [2, 3], \\ 2 & t \in [3, \infty), \end{cases}$$

and

$$g_{-1}(t) = \begin{cases} 0 & t \in (\infty, 1], \\ (t-1)^2 & t \in [1, 2], \\ 1 & t \in [2, 3] \\ t-2 & t \in [3, 4], \\ 2 & t \in [4, \infty). \end{cases}$$

Hence

$$h_1(t) = \begin{cases} -\infty & t \in (\infty, 0), \\ 0 & t \in [0, 1), \\ t & t \in [1, 2), \\ \infty & t \in [2, \infty), \end{cases}$$

$$h_{-1}(t) = \begin{cases} -\infty & t \in (\infty, 0), \\ \sqrt{t} & t \in [0, 1), \\ t+1 & t \in [1, 2) \\ \infty & t \in [2, \infty), \end{cases}$$

and

$$h(t) := \max\{h_1(t), h_{-1}(t)\} = \begin{cases} -\infty & t \in (\infty, 0), \\ \sqrt{t} & t \in [0, 1), \\ t+1 & t \in [1, 2) \\ \infty & t \in [2, \infty), \end{cases}$$

Then

$$h([g(x)]_+) = \begin{cases} 0 & x \in [-1, 1], \\ |x+1| & x \in (-4, -3] \cup (-2, -1], \\ 2 & x \in [-3, -2] \cup (1, 2], \\ x & x \in [2, 3), \\ \infty & x \in (-\infty, -4] \cup [3, \infty), \end{cases}$$

that is, g has a nonlinear global error bound with a residual function $h(g[\cdot]_+)$.
In addition, since $\inf_{t>0} h(t) = 0$, g is well-posed by Corollary 4.1.

The nonlinear global error bound in Example 4.3 is not a consequence of previous results for nonlinear error bounds, especially, the following Theorem 4.1. Actually, if equation (3) holds for each $x \notin A$, then $\beta \equiv 0$. Since β is not a nondecreasing function and $\int_0^{d(x,A)} \beta(t)dt = 0$ for each $x \in \mathbb{R}$, we cannot show the existence of nonlinear global error bound in Example 4.3 by Theorem 4.1.

Theorem 4.1 *[5] Let (X, d) be a complete metric space, g a lsc function from X to $\mathbb{R} \cup \{\infty\}$, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$, and β a continuous and nondecreasing function from $(0, \infty)$ to $(0, \infty)$. Assume that $a < b$, $L(g, \leq, a)$ is nonempty, and for each $x \in X$ with $a < g(x) < b$,*

$$|\nabla g|(x) \geq \beta(d(x, L(g, \leq, a))), \tag{3}$$

where

$$|\nabla g|(x) = \begin{cases} 0 & x : \text{local min. of } f, \\ \limsup_{y \to x} \dfrac{f(x) - f(y)}{d(x, y)} & \text{otherwise.} \end{cases}$$

Then, for each $x \in X$ with $a < g(x) < b$,

$$f(x) - a \geq \int_0^{d(x,L(g,\leq,a))} \beta(t)dt.$$

At the last of the paper, we introduce the following constraint qualification for quasiconvex inequality systems.

Definition 4.1 [20,21] Let $\{g_i : i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) : j \in J_i\} \subset Q \times X$ a generator of g_i for each $i \in I$, $T = \{t = (i,j) : i \in I, j \in J_i\}$, and $A = \{x \in X : \forall i \in I, g_i(x) \leq 0\}$. Assume that A is non-empty.

A lsc quasiconvex inequality system $\{g_i(x) \leq 0 : i \in I\}$ is said to satisfy the basic constraint qualification for quasiconvex programming (Q-BCQ) with respect to $\{(k_t, w_t) : t \in T\}$ at $x \in A$ if

$$N_A(x) = \text{cone co} \bigcup_{t \in T(x)} \{w_t\},$$

where $T(x) = \{t \in T : \langle w_t, x \rangle = k_t^{-1}(0)\}$.

The Q-BCQ is a necessary and sufficient constraint qualification for an optimality condition and Lagrange-type min-max duality via quasiconvex programming. Furthermore, $\{g_i(x) \leq 0 : i \in I\}$ satisfies Q-BCQ with respect to $\{(k_t, w_t) : t \in T\}$ at $x \in A$ if and only if

$$N_A(x) \subset \text{cone co} \bigcup_{t \in T(x)} \{w_t\},$$

in detail, see [20–22]. The following assumption in Theorem 3.1,

$$N_A(x) \subset \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0 \rangle\},$$

is equivalent to Q-BCQ w.r.t. $G = \{(\text{cl } g_w, w) : \|w\| = 1\}$, see the following proposition.

Proposition 4.1 *Let $x_0 \in A$. Then, the following statements hold:*

(i) $\{w \in X : g_w^{-1}(0) = \langle w, x_0 \rangle\}$ *is a convex cone,*
(ii) $\{w \in X : g_w^{-1}(0) = \langle w, x_0 \rangle\} = \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0 \rangle\}$,
(iii) $\{g(x) \leq 0\}$ *satisfies Q-BCQ w.r.t.* $G = \{(\text{cl } g_w, w) : \|w\| = 1\}$ *at* x_0 *if and only if*

$$N_A(x) \subset \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0 \rangle\}.$$

Proof (i) Let $v \in \{w \in X : g_w^{-1}(0) = \langle w, x_0 \rangle\}$ and $\lambda \geq 0$. If $v = 0$ or $\lambda = 0$, then $\lambda v = 0$. Since A is nonempty,

$$g_{\lambda v}^{-1}(0) = \sup\{t \in \mathbb{R} : \inf\{g(x) : \langle 0, x \rangle \geq t\} \leq 0\} = 0 = \langle \lambda v, x_0 \rangle .$$

Hence $\lambda v \in \{w \in X : g_w^{-1}(0) = \langle w, x_0 \rangle\}$.

Assume that $v \neq 0$ and $\lambda \neq 0$. Then

$$\begin{aligned} g_{\lambda v}^{-1}(0) &= \sup\{t \in \mathbb{R} : \inf\{g(x) : \langle \lambda v, x\rangle \geq t\} \leq 0\} \\ &= \lambda \sup\{t \in \mathbb{R} : \inf\{g(x) : \langle v, x\rangle \geq t\} \leq 0\} \\ &= \lambda g_v^{-1}(0) \\ &= \langle \lambda v, x_0\rangle . \end{aligned}$$

This shows that $\lambda v \in \{w \in X : g_w^{-1}(0) = \langle w, x_0\rangle\}$. Hence, $\{w \in X : g_w^{-1}(0) = \langle w, x_0\rangle\}$ is a cone.

Let $v_1,\ v_2 \in \{w \in X : g_w^{-1}(0) = \langle w, x_0\rangle\}$. Since $g_{v_1+v_2} \circ (v_1 + v_2)(x_0) \leq g(x_0) \leq 0$, $g_{v_1+v_2}^{-1}(0) \geq (v_1+v_2)(x_0) = g_{v_1}^{-1}(0)+g_{v_2}^{-1}(0)$. Assume that $g_{v_1+v_2}^{-1}(0) > (v_1 + v_2)(x_0) = g_{v_1}^{-1}(0) + g_{v_2}^{-1}(0)$. Then there exist $t_1 > g_{v_1}^{-1}(0)$ and $t_2 > g_{v_2}^{-1}(0)$ such that

$$g_{v_1+v_2}^{-1}(0) > t_1 + t_2 > g_{v_1}^{-1}(0) + g_{v_2}^{-1}(0).$$

Since $g_{v_1+v_2}$ is non-decreasing, $g_{v_1+v_2}(t_1 + t_2) \leq 0$. For each $k \in \mathbb{N}$, there exists $x_k \in X$ such that $\langle v_1 + v_2, x_k\rangle \geq t_1 + t_2$ and $g(x_k) < \frac{1}{k}$. This shows that $\langle v_1, x_k\rangle \geq t_1$ or $\langle v_2, x_k\rangle \geq t_2$. Without loss of generality, we assume that $\{k \in \mathbb{N} : \langle v_1, x_k\rangle \geq t_1\}$ is infinite set. Then, $t_1 \leq g_{v_1}^{-1}(0)$ since $g_{v_1}(t_1) = \inf\{g(x) : \langle v_1, x\rangle \geq t_1\} \leq 0$. This is a contradiction. Hence $g_{v_1+v_2}^{-1}(0) = (v_1 + v_2)(x_0) = g_{v_1}^{-1}(0) + g_{v_2}^{-1}(0)$, that is, $v_1 + v_2 \in \{w \in X : g_w^{-1}(0) = \langle w, x_0\rangle\}$. This shows that $\{w \in X : g_w^{-1}(0) = \langle w, x_0\rangle\}$ is a convex cone.

(ii) By the statement (i),

$$\text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0\rangle\} \subset \{w \in X : g_w^{-1}(0) = \langle w, x_0\rangle\}.$$

Let $v \in \{w \in X : g_w^{-1}(0) = \langle w, x_0\rangle\}$. If $v = 0$, then $v \in \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0\rangle\}$. Assume that $v \neq 0$. We can see that

$$g_{\frac{v}{\|v\|}}^{-1}(0) = \frac{1}{\|v\|} g_v^{-1}(0) = \frac{1}{\|v\|} \langle v, x_0\rangle = \left\langle \frac{v}{\|v\|}, x_0 \right\rangle .$$

Hence

$$v = \|v\| \frac{v}{\|v\|} \in \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0\rangle\},$$

that is, $v \in \text{cone}\{w \in X : \|w\| = 1, g_w^{-1}(0) = \langle w, x_0\rangle\}$.

(iii) By the statement (iv) of Lemma 1 in [21], cl $g_w^{-1}(0) = g_w^{-1}(0)$. Additionally, by the definition of Q-BCQ and the statements (i) and (ii) in this theorem, the statement (iii) holds. □

In [19–23, 26, 27], we study constraint qualifications for some dualities via quasiconvex programming. In these results, we show that if g is a real-valued convex function, then Slater condition implies Q-BCQ w.r.t. G at for each $x \in \text{bd}A$. In addition, if A is compact, then Q-BCQ w.r.t. G holds. Hence, Q-BCQ is not so strong in quasiconvex analysis.

5 Conclusion

In this paper, we study global nonlinear error bounds for quasiconvex inequality systems. By a generator of a quasiconvex function, we introduce a nonlinear residual function. In Theorem 3.1, we prove that a lsc quasiconvex inequality system satisfying Q-BCQ has a global nonlinear error bound. In Corollary 4.1, we show well-posedness of a quasiconvex function by the error bound. In addition, we show some examples which demonstrate usefulness of our results. In these examples, we can easily compute residual functions and show standard property '$\inf_{t>0} h(t) = 0$'. However, in other cases, it might not be easy to compute these residual functions and residual functions does not always satisfy such a standard property. These are future research.

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