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ALMOST COMPLEX CURVES OF TYPE (III) IN THE NEARLY KÄHLER 6-SPHERE

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Firstly, we prove that a complete almost complex curve of type (III) and of non-negative sectional curvature immersed in nearly Kähler 6-sphere must be flat. Secondly, we announce the results in [HU] about the first non-trivial example of almost complex 2-torus of type(III) in nearly Kähler 6-sphere.

Introduction

It is well known that 6-dimensional standard sphere S^6 has a nearly Kähler structure(see [KN], pp139-140). We denote by J the nearly Kähler structure. It is an interesting problem to classify J-invariant submanifolds of S^6. Due to a result of A. Gray[Gr], there is no 4-dimensional J-invariant submanifolds. Thus, the only possibility of J-invariant submanifold is the case of the image of Riemann surface by some immersion, which we call an *almost comples curve*(or *J-holomorphic curve*). Let $\varphi : S \longrightarrow S^6$ be an almost complex curve. If φ is linearly full and the ellipses of curvature and the second curvature are circles, then φ is said to be *superminimal.* On the other hand, Bolton-Vrancken-Woodward([BVW]) proved that if φ is linearly full in some totally geodesic S^m in S^6 then $m = 2, 5$ or 6. This, together with a general result on superminimal surfaces by Calabi([Ca]), means that almost complex curves are divided into the following four types :
(I) linearly full in S^6 and superminimal,
(II) linearly full in S^6 and non-superminimal,
(III) linearly full in some totally geodesic S^5 in S^6 (necessarily non-superminimal),

(IV) totally geodesic.
For type (I), Bryant([Br]) gave the construction using the so-called twistor method. The automorphism group of the nearly Kähler structure of S^6 is the exceptional Lie group G_2 and S^6 may be represented as a homogeneous space $G_2/SU(3)$.

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Let $\pi : Q^5 = G_2/U(2) \longrightarrow S^6 = G_2/SU(3)$ be the $P^2(\mathbf{C})$-bundle associated to the principal $SU(3)$-bundle $G_2 \longrightarrow G_2/SU(3)$. Then, any almost complex curve of type (I) can be lifted to some horizontal holomorphic curve in Q^5. Moreover, Bryant gave the representation formula of the horizontal holomorphic curves in Q^5. Recently, Hashimoto([Ha]) gave the representation formula more explicitly and calculated the Gaussian curvature of almost complex curve of type (I). For types (II) and (III), Bolton-Pedit-Woodward([BPW]) proved that φ can be lifted to $\widetilde{\varphi} : S \longrightarrow G_2/T^2$, where T^2 is the maximal torus of $SU(3)$, and φ has a Toda-framing $F : S \longrightarrow G_2$, of which the integrable condition is some periodic Toda equation. Consequently, any almost complex torus of type (II) or (III) in S^6 can be lifted to a *primitive map of finite type* into G_2/T^2(for primitive map of finite type, see [OU]). By a recent result in [OU], we find that almost complex torus of type (II) or (III) is itself of finite type. However, any non-trivial example of such torus has been unknown.

In this paper, we concentrate on almost complex curve of type (III). Firstly, we prove that any complete almost complex curve of type (III) with non-negative Gaussian curvature must be necessarily flat. Secondly, we announce some results in [HU] on the construction of the differential geometric concrete example of almost complex torus of type (III) and of *all* almost complex torus of type (III) in terms of the Prym-theta functions.

1. Preliminaries

Let $\mathcal{O}$ be the Cayley number field and we identify $\mathbf{R}^7$ with the purely imaginary part of $\mathcal{O}$. Let $\mathcal{H}$ be the quaternion number field with the basis $1, i, j, k$. Then, any element of $\mathcal{O}$ is represented as $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, where $p_1, p_2 \in \mathcal{H}$. The product of two elements $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathcal{O}$ is defined by

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_1 q_1 - \overline{q_2} p_2 \\ q_2 p_1 + p_2 \overline{q_1} \end{pmatrix},$$

where the conjugation is that of $\mathcal{H}$. The conjugation of $\mathcal{O}$ is defined by

$$\overline{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}} = \begin{pmatrix} \overline{p_1} \\ -p_2 \end{pmatrix},$$

where the conjugation of the entry in the right hand side is that of $\mathcal{H}$. For $\mathbf{x} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathcal{O}$, we have $\mathbf{x} \cdot \overline{\mathbf{x}} = \begin{pmatrix} p_1 \overline{p_1} + \overline{p_2} p_2 \\ 0 \end{pmatrix}$, which is real with respect to the conjugation of $\mathcal{O}$. If we define the norm by $\mid \mathbf{x} \mid = \sqrt{p_1 \overline{p_1} + \overline{p_2} p_2}$ then we may verify the following fundamental relations :

$$|\,\mathbf{x}\,| = 0 \iff \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.1}$$

$$|\,\mathbf{x}\cdot\mathbf{y}\,| = |\,\mathbf{x}\,||\,\mathbf{y}\,| \tag{1.2}$$

$$\overline{\mathbf{x}\cdot\mathbf{y}} = \overline{\mathbf{y}}\cdot\overline{\mathbf{x}} \qquad \text{for} \quad \mathbf{x},\mathbf{y}\in\mathcal{O} \tag{1.3}$$

Note that $\mathbf{x_0} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is purely-imaginary if and only if p_1 is purely-imaginary. Hence, the imaginary part of $\mathcal{O}$, denoted by $\mathrm{Im}(\mathcal{O})$, consists of the elements of the form $\mathbf{x} = \begin{pmatrix} a_1 i + a_2 j + a_3 k \\ a_4 + a_5 i + a_6 j + a_7 k \end{pmatrix}$ and the identification between $\mathrm{Im}(\mathcal{O})$ and $\mathbf{R}^7$ is given by

$$\mathrm{Im}(\mathcal{O}) \ni \mathbf{x} \longleftrightarrow \mathbf{x_R} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} \in \mathbf{R}^7.$$

If we denote by $<\ \ ,\ \ >$ the standard inner product on $\mathbf{R}^7$ then we see that for $\mathbf{x},\mathbf{y}\in\mathrm{Im}(\mathcal{O})$

$$<\mathbf{x_R},\mathbf{y_R}> = -\frac{1}{2}(\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x}).$$

We define a vector cross product on $\mathbf{R}^7$ by

$$\mathbf{x_R}\times\mathbf{y_R} = \frac{1}{2}(\mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x}), \tag{1.4}$$

where the equality is the identification between $\mathbf{R}^7$ and $\mathrm{Im}(\mathcal{O})$. Then we have $\mathbf{y_R}\times\mathbf{x_R} = -\mathbf{x_R}\times\mathbf{y_R}$. Let $\{e_1, e_2, \cdots, e_7\}$ be the standard orthonormal basis of $(\mathbf{R}^7, <\ \ ,\ \ >)$. The above identification means that

$$e_1 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} j \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} k \\ 0 \end{pmatrix},$$
$$e_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 \\ i \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 \\ j \end{pmatrix}, \quad e_7 = \begin{pmatrix} 0 \\ k \end{pmatrix}.$$

We then have

$$e_1\times e_2 = e_3, \quad e_1\times e_4 = e_5, \quad e_2\times e_4 = e_6, \quad e_3\times e_4 = e_7. \tag{1.5}$$

Moreover we may verify the following relation.

(1.6)
$$u \times (v \times w) + (u \times v) \times w = 2 < u, w > v - < u, v > w - < w, v > u \quad ,$$

where $u, v, w \in \mathbf{R}^7$. Conversely, the vector cross product may be defined as a $\mathbf{R}$-linear skew-symmetric homomorphism from $\mathbf{R}^7 \times \mathbf{R}^7$ to $\mathbf{R}^7$ satisfying the conditions (1.5) and (1.6). The group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\mathbf{R}^7$ preserving the vector cross product.

Definition. An ordered orthonormal basis $F_1, F_2, \cdots, F_7$ of $\mathbf{R}^7$ is said to be a G_2-frame if

(1.7)
$$F_1 \times F_2 = F_3, \quad F_1 \times F_4 = F_5, \quad F_2 \times F_4 = F_6, \quad F_3 \times F_4 = F_7.$$

Of course, the standard orthonormal basis of $\mathbf{R}^7$ is a G_2-frame. It follows from (1.6) that we have the following multiplication table :

$\times$	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	0	F_3	$-F_2$	F_5	$-F_4$	$-F_7$	F_6
F_2	$-F_3$	0	F_1	F_6	F_7	$-F_4$	$-F_5$
F_3	F_2	$-F_1$	0	F_7	$-F_6$	F_5	$-F_4$
F_4	$-F_5$	$-F_6$	$-F_7$	0	F_1	F_2	F_3
F_5	F_4	$-F_7$	F_6	$-F_1$	0	$-F_3$	F_2
F_6	F_7	F_4	$-F_5$	$-F_2$	F_3	0	$-F_1$
F_7	$-F_6$	F_5	F_4	$-F_3$	$-F_2$	F_1	0

The standard nearly Kähler structure J on the 6-dimensional unit sphere S^6 is given by

$$Ju = x \times u, \qquad u \in T_x S^6, \qquad x \in S^6. \tag{1.8}$$

In fact, it follows from (1.6) that $J^2 u = x \times (x \times u) = -u$. Moreover, if we put $G(X, Y) = (\widetilde{\nabla}_X J)(Y)$, then we obtain

$$G(X, Y) = X \times Y - < x \times X, Y > x,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on S^6. Since $< u \times v, w > = < u, v \times w >$, we see that $G(X, Y) = -G(Y, X)$, which implies that J is a nearly Kähler structure.

Let $\varphi : S \longrightarrow S^6$ be an almost complex curve. Then, the second fundamental form α and the shape operator A_ξ in the direction $\xi \in T^\perp S$ satisfy the following equations :

$$\begin{cases} \alpha(X, JY) = J\alpha(X, Y), \\ A_{J\xi} = JA_\xi = -A_\xi J, \\ \nabla^\perp_X J\xi = G(X, \xi) + J\nabla^\perp_X \xi, \\ (\nabla\alpha)(X, Y, JZ) = J(\nabla\alpha)(X, Y, Z) + G(X, \alpha(Y, Z)), \end{cases} \tag{1.9}$$

where $\nabla^{\perp}$ is the normal connection of the normal bundle $T^{\perp}S$ of the immersion φ. Assume that S does not contain any totally geodesic points. Let p be any point of S and let V be an arbitrary unit tangent vector field on a neighborhood W of p. Define a non-zero function $\mu = \| \alpha(V,V) \|$, which does not depend on the choice of V because $\{V, JV\}$ forms a basis of T_pW and if $X = aV + bJV$ is an arbitrary unit tangent vector then it follows from (1.9) that

$$\begin{aligned}\| \alpha(X,X) \|^2 &= \| (a^2-b^2)\alpha(V,V) + 2abJ\alpha(V,V) \|^2 \\ &= (a^2-b^2)^2 \| \alpha(V,V) \|^2 + 4a^2b^2 \| J\alpha(V,V) \|^2 \\ &= \| \alpha(V,V) \|^2 .\end{aligned}$$

Let $U = JV$. Then we have a G_2-frame defined by

(1.10)
$$\begin{cases} F_1 = \varphi, \quad F_2 = \varphi_*(V), \quad F_3 = \varphi_*(U) = J\varphi_*(V), \quad F_4 = \alpha(V,V)/\mu, \\ F_5 = J\alpha(V,V)/\mu, \quad F_6 = F_2 \times \alpha(V,V)/\mu, \quad F_7 = F_3 \times \alpha(V,V)/\mu. \end{cases}$$

Lemma 1.1 (cf. [DV]). *Set*

$$(\nabla\alpha)(V,V,V) = \mu(a_1F_4 + a_2F_5 + a_3F_6 + a_4F_7).$$

Then

$$(\nabla\alpha)(V,V,U) = \mu(-a_2F_4 + a_1F_5 + (a_4+1)F_6 - a_3F_7).$$

Moreover, the following statements are true :
(1) φ is of type (I) if and only if $a_3 = 0$ and $a_4 = -\frac{1}{2}$,
(2) φ is of type (III) if and only if $a_3^2 + a_4^2 + a_4 = 0$.

Proof. The first claim follows from the fourth equation of (1.9). Set $z = V - iU$, which is a $(1,0)$-vector of S. Since $\alpha(z,z)$ is a $(1,0)$-vector by the first equation of (1.9), we have $< \alpha(z,z), \alpha(z,z) >= 0$. This means that the ellipse of curvatures is a circle. The ellipse of second curvatures is a circle if $< (\nabla\alpha)(z,z,z), (\nabla\alpha)(z,z,z) >= 0$. Hence, φ is superminimal if and only if $< (\nabla\alpha)(z,z,z), (\nabla\alpha)(z,z,z) >= 0$. We now calculate

$$\begin{aligned}(\nabla\alpha)(z,z,z) &= 2\{(\nabla\alpha)(V,V,V) - iJ(\nabla\alpha)(V,V,V)\} \\ &\quad - 2i\{(\nabla\alpha)(V,V,U) - iJ(\nabla\alpha)(V,V,U)\} - 2i\mu\{F_6 - iF_7\}.\end{aligned} \tag{1.11}$$

Note that the third term of the right hand side of (1.11) is $(0,1)$-vector by the multiplication table. Therefore, we obtain

(1.12)
$$< (\nabla\alpha)(z,z,z), (\nabla\alpha)(z,z,z) > = -16\mu^2 i(2a_3 - (2a_4+1)i) \quad ,$$

which is zero if and only if $a_3 = 0$ and $a_4 = -\dfrac{1}{2}$, proving the assertion (1). Finally we show (2). Since the normal space is spanned by

$$\{\alpha(V,V), \alpha(V,U), (\nabla\alpha)(V,V,V), (\nabla\alpha)(V,V,U)\},$$

we see that φ is of type (III) if and only if the components of $(\nabla\alpha)(V,V,V)$ and $(\nabla\alpha)(V,V,U)$ in the direction of the second normal space are propotional to each other, which holds if and only if $a_3F_6 + a_4F_7$ is propotional to $(a_4+1)F_6 - a_3F_7$. Therefore, φ is of type (III) if and only if $a_3^2 + a_4^2 + a_4 = 0$. q.e.d.

2. Minimal surfaces in S^n

In this section, we prove a certain extension of the result in [HS]. As a corollary, we obtain some result on complete almost complex curve of type (III) in S^6. The method is similar to that of Calabi in [Ca], but differes from it in delicate sense.

Let M be an oriented minimal surface in S^n. Let $\{e_1, e_2\}$ be local fields of orthonormal basis. Set $z = \frac{1}{\sqrt{2}}(e_1 - ie_2)$ and $\overline{z} = \frac{1}{\sqrt{2}}(e_1 + ie_2)$. We define the m-th second fundamental form σ^m using

$$\begin{cases} \sigma^2(z,z) = \sigma(z,z), \\ \sigma^m(\overbrace{z,\cdots,z}^{m-\text{times}}) = (\nabla_z\sigma^{m-1})(\overbrace{z,\cdots,z}^{m-1-\text{times}}) \\ \qquad = \nabla_z^{\perp}\sigma^{m-1}(\overbrace{z,\cdots,z}^{m-1-\text{times}}) - (m-1)\sigma^{m-1}(\nabla_z z, \overbrace{z,\cdots,z}^{m-2-\text{times}}). \end{cases}$$

Set $L_m = <\sigma^m(\overbrace{z,\cdots,z}^{m-\text{times}}), \sigma(z,z)>$. We then have

Lemma 2.1. *Assume that $L_j \equiv 0$ for $2 \leqq j \leqq m$ for some $m \geqq 2$. Then, the following are true :*

$$(1) \qquad (\nabla_z\sigma^{m+1})(\overbrace{z,\cdots,z}^{m+1-\text{times}}) = \sum_{j=2}^{m} f_j\sigma^j(\overbrace{z,\cdots,z}^{i-\text{times}}),$$

where $f_2, \cdots, f_m$ are locally defined C^∞-functions.
(2) If m is even then we have $L_{m+1} \equiv 0$, $\nabla_{\overline{z}}L_{m+2} \equiv 0$ and

$$(2.1) \qquad \frac{1}{2}\Delta \mid L_{m+2} \mid^2 = \mid \nabla_z L_{m+2} \mid^2 + (m+4)K \mid L_{m+2} \mid^2,$$

where K is the Gaussian curvature of M.

Proof. (1) For $m = 2$ we have

$$\begin{aligned}(\nabla_{\overline{z}}\sigma^3)(z,z,z) &= 2K\sigma(z,z) - R^{\perp}(z,\overline{z})\sigma(z,z) \\ &= 2K\sigma(z,z) - <\sigma(z,z), \overline{\sigma(z,z)}> \sigma(z,z)\ ,\end{aligned}$$

where $R^{\perp}$ is the curvature of the normal bundle and we have used the assumption $L_2 \equiv 0$. Hence, the case of $m = 2$ is verified. Suppose that the equation holds for $m = j$. For $m = j + 1$, we have

$$
\begin{aligned}
&(\nabla_{\overline{z}}\sigma^{j+2})(\overbrace{z,\cdots,z}^{j+2-\text{times}})\\
&=(\nabla_z\nabla_{\overline{z}}\sigma^{j+1})(\overbrace{z,\cdots,z}^{j+1-\text{times}}) + (j+1)K\sigma^{j+1}(\overbrace{z,\cdots,z}^{j+1-\text{times}}) - R^{\perp}(z,\overline{z})\sigma^{j+1}(\overbrace{z,\cdots,z}^{j+1-\text{times}})\\
&=\sum_{i=2}^{j+1} f_i\sigma^i(\overbrace{z,\cdots,z}^{i-\text{times}}) + (j+1)K\sigma^{j+1}(\overbrace{z,\cdots,z}^{j+1-\text{times}})\\
&\qquad - <\sigma^{j+1}(\overbrace{z,\cdots,z}^{j+1-\text{times}}), \overline{\sigma(z,z)}> \sigma(z,z),
\end{aligned}
$$

where we have used the assumption $L_{j+1} \equiv 0$ and the assumption of the induction. Thus, the desired equation is proved. Next, we show (2). If $L_j \equiv 0$ for $2 \leqq j \leqq m$ then

$$
<\sigma^{\alpha}(\overbrace{z,\cdots,z}^{\alpha-\text{times}}), \sigma^{\beta}(\overbrace{z,\cdots,z}^{\beta-\text{times}})> \equiv 0 \qquad \text{for} \quad \alpha+\beta \leqq m+2 \quad \text{with} \quad \alpha,\beta \geqq 2.
$$

Therefore, we obtain

$$
L_{m+1} = (-1)^{\frac{m}{2}-1} < \sigma^{\frac{m}{2}+2}(\overbrace{z,\cdots,z}^{\frac{m}{2}+2-\text{times}}), \sigma^{\frac{m}{2}+1}(\overbrace{z,\cdots,z}^{\frac{m}{2}+1-\text{times}}) > .
$$

On the other hand, for $2\alpha = m + 2$ we have

$$
<\sigma^{\alpha+1}(\overbrace{z,\cdots,z}^{\alpha+1-\text{times}}), \sigma^{\alpha}(\overbrace{z,\cdots,z}^{\alpha-\text{times}})> = - <\sigma^{\alpha}(\overbrace{z,\cdots,z}^{\alpha-\text{times}}), \sigma^{\alpha+1}(\overbrace{z,\cdots,z}^{\alpha+1-\text{times}})>,
$$

whence $L_{m+1} \equiv 0$. Next we have

(2.2)

$$
\nabla_{\overline{z}}L_{m+2} = \nabla_{\overline{z}} <\sigma^{m+2}(\overbrace{z,\cdots,z}^{m+2-\text{times}}), \sigma(z,z)> = <\nabla_{\overline{z}}\sigma^{m+2}(\overbrace{z,\cdots,z}^{m+2-\text{times}}), \sigma(z,z)>,
$$

where we have used the minimality of M and the Codazzi equation, that is, $(\nabla_{\overline{z}}\sigma^2)(z,z) = (\nabla_z\sigma^2)(z,\overline{z}) = 0$. It follows from the equation of Lemma 2.1-(1) and (2.2) that $\nabla_{\overline{z}}L_{m+2}$ is a linear combinations of $L_2,\cdots,L_{m+1}$. Therefore, we have $\nabla_{\overline{z}}L_{m+2} \equiv 0$. Using $\nabla_{\overline{z}}L_{m+2} \equiv 0$, we may compute $\nabla_{\overline{z}}\nabla_z \mid L_{m+2} \mid^2$ as follows :

$$
\begin{aligned}
\nabla_{\overline{z}}\nabla_z \mid L_{m+2} \mid^2 &= \mid \nabla_z L_{m+2} \mid^2 + \nabla_{\overline{z}}\nabla_z L_{m+2} \cdot \overline{L_{m+2}}\ ,\\
\nabla_{\overline{z}}\nabla_z L_{m+2} &= (m+4)KL_{m+2}\ ,
\end{aligned}
$$

where the second equation follows from the equation of Lemma 2.1-(1) and the Ricci equation. q.e.d.

We now have

Theorem 2.2. *Let $\varphi : M \longrightarrow S^n$ be an oriented complete minimal surface with nonnegative Gaussian curvature K. If all the $\mid \sigma^m \mid^2$ are bounded on M, then either φ is superminimal or $K \equiv 0$ and congruent to $T^2 \longrightarrow S^n$ constructed and classified by Kenmotsu ([Ke]).*

Proof. If $K \geqq 0$ then the Gaussian equation implies that $\mid L_2 \mid^2$ is bounded on M. It follows from Lemma 2.1-(2) that $\mid L_2 \mid^2$ is a bounded subharmonic function, hence $\mid L_2 \mid^2$ is constant by Huber's theorem ([Hu]). If K is positive at some point of M, then $L_2 \equiv 0$. It then follows from Lemma 2.1-(2), our assumption and Huber's theorem that $L_m \equiv 0$ for any $m \geqq 2$. This is, in fact, equivalent to saying that φ is superminimal(see [BVW]). q.e.d.

Remark. The case of $n = 4$ for Theorem 2.2 is due to [HS]. In that case, $K \geqq 0$ implies that $\mid \sigma^2 \mid^2$ is bounded.

Corollary. *Let $\varphi : S \longrightarrow S^6$ be a complete almost complex curve with nonnegative Gaussian curvature K. If φ is of type (III), then $K \equiv 0$.*

Proof. If φ is of type (III), then $a_3^2 + a_4^2 + a_4 = 0$ by Lemma 1.1. On the other hand, $L_2 = L_3 \equiv 0$ and

$$< (\nabla\alpha)(z, z, z), (\nabla\alpha)(z, z, z) > = -2\mu^2 i(2a_3 - (2a_4 + 1)i),$$

which implies that $\mid L_4 \mid^2$ is bounded on S. It follows from Lemma 2.1-(2) that $\mid L_4 \mid^2$ is bounded subharmonic function on S, whence $\mid L_4 \mid^2$ is constant. If $L_4 \equiv 0$ then φ is superminimal. However, this is impossible by a result of Calabi([Ca]) and Bolton-Vrancken-Woodward([BVW]). Hence, $K \equiv 0$ by Lemma 2.1-(2). q.e.d.

3. Kähler angle of horizontal surface in S^5 and examples of almost complex curves of type (III)

In this section, we announce some results in [HU].

Let M be a Riemann surface. Let $\varphi : M \longrightarrow S^5 \subset S^6$ be a conformal immersion with $e^\omega =< \dfrac{\partial\varphi}{\partial z}, \dfrac{\partial\varphi}{\partial \bar{z}} >$, where z is a local complex coordinate system of M. By some correspondense we may identify $\varphi : M \longrightarrow S^5$ with $\varphi_{\mathbf{C}} : M \longrightarrow S^5_{\mathbf{C}} \subset \mathbf{C}^3$. Denote by θ the Kähler angle of φ, i.e.,

$$< J\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} > = 2e^\omega \cos\theta \ ,$$

where $z = x + iy$. We then have the following.

Lemma 3.1([HU]). *If $\varphi_{\mathbf{C}}$ is horizontal with respect to the Hopf fibration $S^5_{\mathbf{C}} \longrightarrow P^2(\mathbf{C})$ then*

$$\cos\theta = \mathrm{Re}\{i \ \det(\varphi_{\mathbf{C}} \quad e^{-\frac{\omega}{2}}\frac{\partial\varphi_{\mathbf{C}}}{\partial z} \quad e^{-\frac{\omega}{2}}\frac{\partial\varphi_{\mathbf{C}}}{\partial \bar{z}})\}. \tag{3.1}$$

where $i = \sqrt{-1}$.

Using this Lemma, we obtain the general method of constructing almst complex curve of type (III) in S^6.

Proposition 3.2([HU]). *Let* $s_0 : M \longrightarrow \mathbf{C}^3$ *be a smooth map and* $\omega : M \longrightarrow \mathbf{R}$ *a smooth function. Set* $s_1 = e^{-\frac{\omega}{2}}\dfrac{\partial s_0}{\partial z}, s_2 = e^{-\frac{\omega}{2}}\dfrac{\partial s_0}{\partial \bar{z}}$. *If* $S = (s_0 \quad s_1 \quad s_2)$ *has values in* $U(3)$ *and satisfies* $\det S = -i$, *then* $\varphi : M \longrightarrow S^5 \subset S^6$ *corresponding to* $\varphi_{\mathbf{C}} : M \longrightarrow S^5_{\mathbf{C}} \subset \mathbf{C}^3$ *defined by* $\varphi_{\mathbf{C}} = s_0$ *is a conformal immersion and an almost complex curve with respect to* J. *The converse is also true.*

In fact, $S : M \longrightarrow U(3)$ is a Toda framing.

Proposition 3.3 ([HU]).

$$S^{-1}\frac{\partial S}{\partial z} = \begin{pmatrix} 0 & 0 & -e^{\frac{\omega}{2}} \\ e^{\frac{\omega}{2}} & \frac{\omega_z}{2} & 0 \\ 0 & ie^{-\omega} & -\frac{\omega_z}{2} \end{pmatrix} =: U,$$

and

$$S^{-1}\frac{\partial S}{\partial \bar{z}} = \begin{pmatrix} 0 & -e^{\frac{\omega}{2}} & 0 \\ 0 & -\frac{\omega_{\bar{z}}}{2} & ie^{-\omega} \\ e^{\frac{\omega}{2}} & 0 & \frac{\omega_{\bar{z}}}{2} \end{pmatrix} =: V = \overline{U},$$

where $\omega_z = \dfrac{\partial \omega}{\partial z}$ *and* $\omega_{\bar{z}} = \overline{\omega_z}$. *The integrability condition of this system is given by so-called Tzitzéica equation :*

$$\omega_{z\bar{z}} = e^{-2\omega} - e^{\omega}. \tag{3.2}$$

[**The special solution of** (3.2)] We assume that ω depends on only a variable x, hence we write as $\omega = \omega(x)$. The solution for this case is treated by Castro-Urbano ([CU]). However, their choices of the coordinate $z = x + \sqrt{-1}y$ and a parameter of the solution $\omega(x)$ do not fit into our framework. We modify the calculation a little bit. The details about it will be described in [HU].

Remark. Before the work of Castro-Urbano appeared, Ejiri([Ej]) had already treated the equation (3.2) and observed the existence of periodic solutions of (3.2) (cf. (7.5) in [Ej]). Ejiri proved the existence of countably many totally real immersions of $S^1 \times S^{n-1}$ into $P^n(\mathbf{C})$ with arbitrary prescribed non-negative constant mean curvature H (Corollary 8 in [Ej]). If we set $n = 2$ and $H = 0$ in his result, we obtain countably many totally real minimal immersions of tori into $P^2(\mathbf{C})$.

Consider the elliptic integral of first kind :

$$x = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - p^2 \sin^2 \theta}} =: F(\varphi) \quad , \qquad (- \leqq p \leqq 1).$$

Then we define the Jacobi elliptic functions $sn(x,p), cn(x,p)$ and $dn(x,p)$ as follows :

(3.3)

$$sn(x,p) := \sin \varphi, \quad cn(x,p) := \cos \varphi, \quad dn(x,p) := \sqrt{1 - p^2 \sin^2 \varphi}.$$

We then easily see that $F(-\varphi) = -F(\varphi), F(\varphi + \pi) = F(\varphi) + F(\pi)$. Setting $\varphi = -\frac{\pi}{2}$, it follows that $F(\frac{\pi}{2}) = \frac{1}{2}F(\pi)$, which is called the complete integral of first kind and denoted by $K(p)$ (or simply K if the value of p is fixed.) By the same reason as that of $K = K(p)$, we simply write $sn(x), cn(x), dn(x)$ in place of writing $sn(x,p), cn(x,p), dn(x,p)$. Moreover, we see that

$$\begin{cases} sn(x + 2K) = sn(x + F(\pi)) = \sin(\varphi + \pi) = -\sin(\varphi) = -sn(x) \\ cn(x + 2K) = -cn(x) \quad , \qquad dn(x + 2K) = dn(x) \quad . \end{cases}$$

Now, if we put $Y = e^{\omega(x)}$ then (3.2) becomes to, after integration one time,

$$(Y')^2 + 8Y^3 - 8aY^2 + 4 = 0, \tag{3.4}$$

where $Y' = \frac{dY}{dx}$ and a is a constant of the integration. We give an initial condition for $\omega(x)$ by

$$e^{\omega(0)} = \frac{\alpha}{2}, \quad \frac{d\omega}{dx}(0) = 0.$$

Since a Weierstrass $\mathfrak{p}$-function satisfies (3.4) and a Weirestrass $\mathfrak{p}$-function may be described in terms of the Jacobi sn-function, our choice of the initial condition of $\omega(x)$ means that we may put

$$Y = \frac{\alpha}{2}\left(1 - q^2 sn^2(rx, p)\right) \quad ,$$

for some real numbers p, q, r. In fact, this Y satisfies (3.4) if and only if

$$\begin{cases} a = \frac{1}{2}\alpha + 2\alpha^{-2} \quad , \qquad q^2 = \frac{\alpha^3 - 2 - 2\sqrt{\alpha^3 + 1}}{\alpha^3}, \\ r^2 = \frac{\alpha^3 - 2 + 2\sqrt{\alpha^3 + 1}}{\alpha^2} \quad , \qquad p^2 = \frac{\alpha q^2}{r^2}. \end{cases} \tag{3.5}$$

[**A solution of** (3.1)] We fix the choice of real numbers p, q, r which satisfy (3.5). Then we have $\alpha \geqq 2$. Consider an curve parametrized by x on S^2 :

(3.6)
$$^t\Big(\sqrt{\frac{r_2}{r_1+r_2}}dn(rx,p), \sqrt{\frac{r_1}{r_1+r_2}}cn(rx,p), \sqrt{\frac{r_2p^2+r_1}{r_1+r_2}}sn(rx,p)\Big)$$

where this is, in fact, a curve on S^2 precisely when

$$\text{(3.7)} \qquad \begin{cases} r_1 = \dfrac{\sqrt{\alpha^3+1}+1}{\alpha} \quad , \qquad r_2 = \dfrac{\sqrt{\alpha^3+1}-1}{\alpha}, \\ r_3 = \dfrac{2}{\alpha}. \end{cases}$$

We deine $\hat{s}_0$ as a S^1-orbit of this curve as follows :

(3.8)
$$\hat{s}_0 =$$
$$^t\Big(\sqrt{\frac{r_2}{r_1+r_2}}e^{ir_1y}dn(rx,p), \sqrt{\frac{r_1}{r_1+r_2}}e^{-ir_2y}cn(rx,p), \sqrt{\frac{r_2p^2+r_1}{r_1+r_2}}e^{-ir_3y}sn(rx,p)\Big).$$

Define $\hat{S}$ by $\hat{S} = (\hat{s}_0 \quad e^{-\frac{\omega(x)}{2}}\dfrac{\partial \hat{s}_0}{\partial z} \quad e^{-\frac{\omega(x)}{2}}\dfrac{\partial \hat{s}_0}{\partial \overline{z}})$. Then, a direst computation shows that $\det(\hat{S}) = 1$. If we define s_0 by

$$s_0 = \tau \hat{s}_0, \qquad \text{for} \quad \tau = e^{\frac{\pi}{2}i + \frac{2n}{3}\pi i}, \quad (n = 0, 1, 2),$$

then it follows from Proposition 3.2 that $\varphi^0 : \mathbf{R}^2 \longrightarrow S^6$ corresponding to $\varphi^0_{\mathbf{C}}$ defined by $\varphi^0_{\mathbf{C}} = s_0$ defines an almost complex curve of type (III). The $\varphi^0_{\mathbf{C}}$ is written as follows :

$$\begin{aligned} \varphi^0_{\mathbf{C}} = \Big(&\sqrt{\frac{r_2}{r_1+r_2}}(-\sin(r_1y) + i\cos(r_1y))dn(rx,p), \\ &\sqrt{\frac{r_1}{r_1+r_2}}(\sin(r_2y) + i\cos(r_2y))cn(rx,p), \\ &\sqrt{\frac{r_2p^2+r_1}{r_1+r_2}}(\sin(r_3y) + i\cos(r_3y))sn(rx,p)\Big). \end{aligned}$$

[**Double-periodicity of** φ^0] Since the Jacobi elliptic functions $sn(rx), cn(rx)$

and $dn(rx)$ are invariant under the translation $x \longrightarrow x + \frac{4K}{r}$, φ^0 is doubly-periodic if and only if the ratio $\frac{2\pi}{r_1} : \frac{2\pi}{r_2} : \frac{2\pi}{r_3}$ is that of rational numbers. In particular, if $\alpha = \sqrt[3]{(\frac{m}{n})^2 - 1}$ with $n, m \in \mathbf{N}, (n, m) = 1$ and $\alpha \geqq 2$, then φ^0 is doubly-periodic and the lattice Γ of the 2-torus is given by

$$\Gamma = \mathrm{Span}_{\mathbf{R}}\{\frac{4K}{r}e_1 + n\sqrt[3]{(\frac{m}{n})^2 - 1}\pi e_2\} \quad \text{when } n+m \quad \text{is even},$$

$$\Gamma = \mathrm{Span}_{\mathbf{R}}\{\frac{4K}{r}e_1 + 2n\sqrt[3]{(\frac{m}{n})^2 - 1}\pi e_2\} \quad \text{when } n+m \quad \text{is odd},$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbf{R}^2$.

4. Further results

In fact, all the solutions of (3.1) for $\mathbf{R}^2$ as domain may be described by Prym-theta function, i.e., a function on Prym variety defined using the theta-function. Here, the Prym variety is a Jacobian torus of a compact Riemann surface modulo some involution. This fact is already proved by Sharipov([Sh], [ChSh]). He classified all minimal 2-tori in $S^5 \subset \mathbf{C}^3$ which are complex normal. Here the term gcomplex normalh means that the immersion is horizontal with respect to the Hopf fibration $S^5_{\mathbf{C}} \longrightarrow P^2(\mathbf{C})$. Hence, using our Proposition 3.2 we may describe all the almost complex 2-tori of type (III) in S^5 in terms of the Prym-theta function (see [HU]).

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