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STUDY OF SUBMANIFOLDS BY CURVES OF ORDER 2

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Abstract. We give a survey of our recent results [KM, MA, Su, SMA] on submanifolds from the viewpoint of curves of order 2. We characterize some of nice submanifolds by the extrinsic shape of circles.

1. Introduction

Let $f : M \to \widetilde{M}$ be an isometric immersion of a Riemannian manifold M into an ambient Riemannian manifold $\widetilde{M}$. In this paper, we keep our mind on each circle γ on the submanifold M and study the extrinsic shape $f \circ \gamma$ in the ambient space $\widetilde{M}$. From this point of view we recall the following two surfaces. Let f_1 be a totally umbilic imbedding of a 2-dimensional standard sphere $S^2(c)$ of curvature c into a Euclidean space $\mathbb{R}^5$ and $f_2 = \iota \circ f$ be an isometric parallel immersion of $S^2(c)$ into $\mathbb{R}^5$. Here f is the second standard minimal immersion of $S^2(c)$ into $S^4(3c)$ and ι is a totally umbilic imbedding of $S^4(3c)$ into $\mathbb{R}^5$. We know that for each great circle γ on $S^2(c)$, both of the curves $f_1 \circ \gamma$ and $f_2 \circ \gamma$ are circles in the ambient space $\mathbb{R}^5$. This implies that we cannot distinguish f_1 from f_2 by the extrinsic shape of *geodesics* of $S^2(c)$ in $\mathbb{R}^5$. However we emphasize that we can distinguish these two isometric immersions f_1 and f_2 by the extrinsic shape of (small) *circles* of $S^2(c)$ in $\mathbb{R}^5$. In fact, for each small circle γ on $S^2(c)$, the curve $f_1 \circ \gamma$ is also a circle in $\mathbb{R}^5$ but the curve $f_2 \circ \gamma$ is a helix of proper order 4 in the ambient space $\mathbb{R}^5$ (for details, see Proposition 1).

It is hence interesting to investigate the extrinsic shape of *circles* of the submanifold. We here recall the following well-known fact due to Nomizu and Yano[NY]. Let M^n be a Riemannian submanifold of $\widetilde{M}^{n+p}$ through an isometric immersion f. Then M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$ if and only if, for some positive constant k and for every circle $\gamma = \gamma(s)$ of curvature k on M^n, the curve $f \circ \gamma$ is a circle in $\widetilde{M}^{n+p}$.

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We relax the condition that $f\circ\gamma$ is a circle to the condition that it is a curve of order 2 in order to improve this characterization of extrinsic spheres (see Theorem 1). The notion of curves of order 2 is a natural extension of that of circles (for details, see section 2). Motivated by Theorem 1, throughout this paper we study the problem: For an isometric immersion $f : M \to \widetilde{M}$ if some circles on M are mapped to curves of order 2 in $\widetilde{M}$ through f, what can we say about the submanifold M?

Along this context we chracterize all parallel isometric immersions of complex projective spaces, quaternionic projective spaces and Cayley projective plane into a complete simply connected real space form $\widetilde{M}^m(\tilde{c})$ of constant curvature $\tilde{c}$, which is either a standard sphere $S^m(\tilde{c})$, a Euclidean space $\mathbb{R}^m$ or a real hyperbolic space $H^m(\tilde{c})$ (see Theorems 2,3 and 4). We also give a characterization of all totally geodesic Kähler isometric immersions into an arbitrary Kähler manifold (see Theorem 5). Theorems 1, 2, 3 and 4 are improvements of the results in [AMO1, AMO2, KM, Su].

2. Curves of order 2

Let M be a Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$. In this section we introduce the notion of curves of order 2. A smooth curve γ on M parametrized by its arclength s is called a *curve of order* 2 if it satisfies the following differential equation:

$$\|\nabla_{\dot\gamma}\dot\gamma\|^2\Big\{\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma+\|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma\Big\}=\langle\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle\nabla_{\dot\gamma}\dot\gamma, \tag{2.1}$$

where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M.

To see that the class of curves of order 2 is very wide, we recall the notion of Frenet curves. A smooth curve $\gamma=\gamma(s)$ parametrized by its arclength s is called a *Frenet curve of proper order* d if there exist orthonormal frame fields $\{V_1=\dot\gamma, V_2,\dots,V_d\}$ along γ and positive functions $\kappa_1(s),\dots,\kappa_{d-1}(s)$ which satisfy the following system of ordinary equations

$$\nabla_{\dot\gamma}V_j(s)=-\kappa_{j-1}(s)V_{j-1}(s)+\kappa_j(s)V_{j+1}(s),\quad j=1,\dots,d, \tag{2.2}$$

where $V_0\equiv V_{d+1}\equiv 0$. Equation (2.2) is called the Frenet formula for the Frenet curve γ. The functions $\kappa_j(s)$ $(j=1,\dots,d-1)$ and the orthonormal frame $\{V_1,\dots,V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. We sometimes call κ_j the j-th curvature.

A Frenet curve is called a *Frenet curve of order* d if it is a Frenet curve of proper order $r(\leqq d)$. For a Frenet curve of order d which is of proper order $r(\leqq d)$, we use the convention in (2.2) that $\kappa_j\equiv 0$ $(r\leqq j\leqq d-1)$ and $V_j\equiv 0$ $(r+1\leqq j\leqq d)$. We call a Frenet curve a *helix* when all its curvatures are constant. A helix of order 1 is nothing but a geodesic. A helix of order 2, namely a curve which satisfies the following differential equations, is called a *circle of curvature* k:

$$\nabla_{\dot\gamma}V_1(s) = kV_2(s), \nabla_{\dot\gamma}V_2(s) = -kV_1(s) \text{ and } V_1(s) = \dot\gamma(s). \tag{2.3}$$

We regard a geodesic as a circle of null curvature.

Lemma 1. (1) *A Frenet curve γ of order 2 is a curve of order 2.*

(2) *If a curve γ of order 2 satisfies $\|\nabla_{\dot\gamma}\dot\gamma(s)\| > 0$, for all s, then it is a Frenet curve of proper order 2, whose curvature and Frenet frame are*

$$\kappa(s) = \|\nabla_{\dot\gamma}\dot\gamma(s)\| \quad \textit{and} \quad \{\dot\gamma, V_2 = \nabla_{\dot\gamma}\dot\gamma/\|\nabla_{\dot\gamma}\dot\gamma\|\}, \ \textit{respectively.}$$

Proof. (1) When γ is a geodesic, it is clear that γ satisfies (2.1). When γ is a Frenet curve of proper order 2, since it satisfies

$$\nabla_{\dot\gamma}\dot\gamma(s) = \kappa(s)V_2(s), \ \nabla_{\dot\gamma}V_2(s) = -\kappa(s)\dot\gamma(s),$$

we find

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(s) = -\dot\kappa(s)\dot\gamma(s) - \kappa^2(s)\dot\gamma(s).$$

This guarantees that γ satisfies (2.1).

(2) If we put $\kappa(s) = \|\nabla_{\dot\gamma}\dot\gamma(s)\|$, we have $\kappa\kappa' = \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle$. Therefore by (2.1) the vector $V_2 = (1/\kappa)\nabla_{\dot\gamma}\dot\gamma$ satisfies

$$\nabla_{\dot\gamma}V_2 = \frac{1}{\kappa^3}\left(\kappa^2\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma - \kappa\kappa'\nabla_{\dot\gamma}\dot\gamma\right) = -\kappa\dot\gamma.$$

We get the conclusion. □

Following this lemma, for a curve γ of order 2 we shall call the nonnegative function $\|\nabla_{\dot\gamma}\dot\gamma\|$ its curvature. We see in particular all geodesics and circles satisfy the equation (2.1), and there are many examples of curves of order 2. But, in general, a curve of order 2 is not a Frenet curve of order 2. In fact, we admit the case that a curve γ of order 2 has an inflection point $\gamma(s_0)$, that is a point which satisfies $(\nabla_{\dot\gamma}\dot\gamma)(s_0) = 0$, so that every plane curve is a curve of order 2. Here a curve is said to be a *plane curve* if it is locally contained on some real 2-dimensional totally geodesic submanifold.

At the mention of inflection points, we here introduce the notion of a Frenet curve of order 2 in a wide sense. A smooth curve γ parametrized by its arclength is called a Frenet curve of order 2 in a wide sense if there exist a smooth unit vector field V along γ which is orthogonal to $\dot\gamma$ and a smooth function κ satisfying that

$$\nabla_{\dot\gamma}\dot\gamma(s) = \kappa(s)V(s), \ \nabla_{\dot\gamma}V(s) = -\kappa(s)\dot\gamma(s). \tag{2.4}$$

Here we do not suppose κ to be positive. We shall also call this function κ the curvature of γ. When γ is a Frenet curve of order 2 in a wide sense which is not a geodesic, the pair (κ, V) is determined up to their signatures, that is either (κ, V) or $(-\kappa, -V)$ satisfies (2.4). Every smooth plane curve parametrized by its arclength is a Frenet curve of order 2 in a wide sense. In a Euclidean space, every Frenet curve of order 2 in a wide sense is a plane curve. The proof of Lemma 1 tells us that every Frenet curve of order 2 in a wide sense is a curve of order 2, but not vice versa. The following example tells us that even in a Euclidean space there exists a curve of order 2 which is not a plane curve.

Example 1. Let γ be a smooth curve in a Euclidean space $\mathbb{R}^3$ defined by

$$\gamma(t) = \begin{cases} (t, e^{-1/t^2}, 0), & t < 0, \\ (0, 0, 0), & t = 0, \\ (t, 0, e^{-1/t^2}), & t > 0. \end{cases}$$

When we reparametrize t to the arclength parameter s, the curve $\gamma(s)$ satisfies Equation (2.1). This example shows that there is a curve of order 2 in $\mathbb{R}^3$ which is not a plane curve, namely, it is not contained in a plane $\mathbb{R}^2$. Hence this curve is not a Frenet curve of order 2 in a wide sense. Note that $\ddot{\gamma}(0)(= (\nabla_{\dot{\gamma}}\dot{\gamma})(0)) = 0$ and that we can not smoothly extend the vector field $\nabla_{\dot{\gamma}}\dot{\gamma}(s)/\|\nabla_{\dot{\gamma}}\dot{\gamma}(s)\|$ $(-\epsilon < s < 0,\ 0 < s < \epsilon)$ along γ to the origin.

At an inflection point we should take care in handling curves of order 2. For example, the differential equation (2.1) may have a bifurcation point.

Example 2. Let ρ be a smooth curve in a Euclidean space $\mathbb{R}^3$ defined by

$$\rho(t) = \begin{cases} (t, e^{-1/t^2}, 0), & t < 0, \\ (t, 0, 0), & t \geq 0. \end{cases}$$

When we reparametrize t to the arclength parameter s, the curve $\rho(s)$ also satisfies Equation (2.1). Comparing this with the curve $\gamma(s)$ in Example 1, we find a solution of (2.1) branches at the origin. We remark that this curve ρ is a plane curve, so that it is a Frenet curve of order 2 in a wide sense.

On the contrary, we have the following result on Frenet curves of order 2 in a wide sense in a complete Riemannian manifold M. Given a smooth function $\kappa(s), -\infty < s < \infty$ and a pair $X, Y \in T_xM$ of orthonormal vectors at an arbitrary point $x \in M$, we have a unique Frenet curve γ of order 2 in a wide sense with curvature κ and with initial condition $\gamma(0) = x, \dot{\gamma}(0) = X, \nabla_{\dot{\gamma}}\dot{\gamma}(0) = \kappa(0)Y$.

3. Extrinsic spheres

We shall start our study on submanifolds by extending the result of Nomizu and Yano which is stated in the introduction.

Theorem 1. *Let M^n be a Riemannian submanifold of $\widetilde{M}^{n+p}$ through an isometric immersion f. Then the following conditions are equivalent.*

(1) *M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$.*
(2) *There exists some positive constant k satisfying that for every circle γ of curvature k on M^n the curve $f \circ \gamma$ is a curve of order 2 in $\widetilde{M}^{n+p}$.*

For an isometric immersion $f : M \to \widetilde{M}$ we denote by σ the second fundamental form of f and by A_ξ the shape operator in the direction of ξ. In this study of submanifolds the formulae of Gauss and Weingarten are basic relations. If we denote by $\widetilde{\nabla}$ the Riemannian connection of $\widetilde{M}$ and by D the covariant differentiation in the normal bundle, these formulae are

$$\widetilde{\nabla}_X Z = \nabla_X Z + \sigma(X, Z), \quad \widetilde{\nabla}_X \xi = D_X \xi - A_\xi X.$$

We define the covariant differentiation $\bar{\nabla}$ of the second fundamental form σ with respect to the connection in (tangent bundle)+(normal bundle) as follows:

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

When $\bar{\nabla}\sigma = 0$, we call an isometric immersion f parallel. A submanifold is called an *extrinsic sphere* if it is totally umbilic with parallel second fundamental form.

To prove Theorem 1 we give the following a priori lemma.

Lemma 2. *Let $f : M \to \widetilde{M}$ be an isometric immersion. If the extrinsic shape $f \circ \gamma$ of a circle γ of curvature k on M is a curve of order 2 in $\widetilde{M}$, then the following equalities hold*:

$$\kappa^2 \Big\{ - A_{\sigma(\dot\gamma, \dot\gamma)} \dot\gamma + \|\sigma(\dot\gamma, \dot\gamma)\|^2 \dot\gamma \Big\} = \kappa\kappa' \nabla_{\dot\gamma} \dot\gamma, \tag{3.1}$$

$$\kappa^2 \Big\{ 3\sigma(\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) + (\bar{\nabla}_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma) \Big\} = \kappa\kappa' \sigma(\dot\gamma, \dot\gamma), \tag{3.2}$$

where $\kappa = \sqrt{k^2 + \|\sigma(\dot\gamma, \dot\gamma)\|^2}$ is the curvature of $f \circ \gamma$ and $\widetilde{\nabla}_{\dot\gamma}$ denotes the covariant differentiation along $f \circ \gamma$ with respect to $\widetilde{\nabla}$.

In particular, for an orthonormal pair $X, Y \in T_x M$ at an arbitrary point $x \in M$ if the extrinsic shape $f \circ \gamma$ of a circle γ of curvature $k(> 0)$ with initial condition that $\gamma(0) = x, \dot\gamma(0) = X$ and $\nabla_{\dot\gamma}\dot\gamma(0) = kY$ is a curve of order 2, then we have

$$\Big(3k^2 + \|\sigma(X, X)\|^2\Big) \langle \sigma(X, X), \sigma(X, Y) \rangle + k \langle \sigma(X, X), (\bar{\nabla}_X \sigma)(X, X) \rangle = 0. \tag{3.3}$$

Proof. In the following, we also denote $f \circ \gamma$ by γ. By hypothesis the curve $f \circ \gamma$ satisfies the following differential equation which corresponds to Equation (2.1):

$$\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|^2 \Big\{ \widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma + \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|^2 \dot\gamma \Big\} = \langle \widetilde{\nabla}_{\dot\gamma}\dot\gamma, \widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma \rangle \widetilde{\nabla}_{\dot\gamma}\dot\gamma. \tag{3.4}$$

On the other hand, since γ is a circle of curvature k, which satisfies (2.3), it follows from the formulae of Gauss and Weingarten that

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \sigma(\dot\gamma,\dot\gamma) = kV_2 + \sigma(\dot\gamma,\dot\gamma), \tag{3.5}$$

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -k^2\dot\gamma - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + 3k\sigma(\dot\gamma, V_2) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma). \tag{3.6}$$

In particular, we have $\kappa^2 = \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|^2 = k^2 + \|\sigma(\dot\gamma,\dot\gamma)\|^2$. From these three equalities (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}\kappa^2\Big\{ - k^2\dot\gamma - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + 3k\sigma(\dot\gamma, V_2) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma) + \kappa^2\dot\gamma\Big\}& \\ = \kappa\kappa'\{kV_2 + \sigma(\dot\gamma,\dot\gamma)\}&.\end{aligned}$$

Considering the tangential and normal components of this equality, we get

$$\kappa^2\Big\{ - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + \|\sigma(\dot\gamma,\dot\gamma)\|^2\dot\gamma\Big\} = k\kappa\kappa' V_2, \tag{3.1'}$$

$$\kappa^2\Big\{3k\sigma(\dot\gamma, V_2) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)\Big\} = \kappa\kappa'\sigma(\dot\gamma,\dot\gamma), \tag{3.2'}$$

which are the desired equalities (3.1) and (3.2).

We now take the inner product of the both sides of (3.1') with V_2 and that of the both sides of (3.2') with $\sigma(\dot\gamma,\dot\gamma)$. When $k > 0$, as we can see $\kappa > 0$, we have

$$\begin{gathered} - \kappa\langle\sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, V_2)\rangle = k\kappa', \\ \kappa\Big\{3k\langle\sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, V_2)\rangle + \langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma), \sigma(\dot\gamma,\dot\gamma)\rangle\Big\} = \kappa'\|\sigma(\dot\gamma,\dot\gamma)\|^2,\end{gathered}$$

which lead us to

$$\Big\{3k^2 + \|\sigma(\dot\gamma,\dot\gamma)\|^2\Big\}\langle\sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, V_2)\rangle + k\langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma), \sigma(\dot\gamma,\dot\gamma)\rangle = 0. \tag{3.7}$$

Evaluating (3.7) at $s = 0$, we obtain the equality (3.3). $\square$

We are now in a position to prove Theorem 1. By our hypothesis we have

$$\Big(3k^2 + \|\sigma(X,X)\|^2\Big)\langle\sigma(X,X), \sigma(X,Y)\rangle + k\langle\sigma(X,X), (\bar\nabla_X\sigma)(X,X)\rangle = 0.$$

for an arbitrary orthonormal pair $X, Y \in TM$ of vectors. As the pair $X, -Y$ is also orthonomal, we see

$$-\Big(3k^2 + \|\sigma(X,X)\|^2\Big)\langle\sigma(X,X), \sigma(X,Y)\rangle + k\langle\sigma(X,X), (\bar\nabla_X\sigma)(X,X)\rangle = 0.$$

These two equalities show that $\langle \sigma(X,X), \sigma(X,Y)\rangle = 0$ for each orthonormal pair of vectors $X, Y \in TM$.

For each circle γ on M, taking the inner product of both sides in Equation (3.1) with $\nabla_{\dot\gamma}\dot\gamma$, we have

$$k^2\kappa\kappa' = -\kappa^2\langle \sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\rangle = 0,$$

because $\dot\gamma$ and $\nabla_{\dot\gamma}\dot\gamma$ are orthogonal. Since the curvature κ of the curve $f\circ\gamma$ of order 2 is positive, we find it is constant, hence $f\circ\gamma$ is a circle in the ambient space $\widetilde{M}^{n+p}$. Therefore by virtue of the result of Nomizu and Yano we get the conclusion of Theorem 1.

We shall close this section by recalling the notion of isotropic immersions for the use in the later sections. An isometric immersion $f : M \to \widetilde{M}$ is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2 (= \lambda(x))$ does not depend on the choice of $X(\neq 0) \in T_xM$. If the immersion is isotropic at every point, then the immersion is said to be isotropic. When the function $\lambda = \lambda(x)$ is constant on M, we call M a constant (λ-)isotropic submanifold. Note that a totally umbilic immersion is isotropic, but not *vice versa*. The following is well-known ([O]).

Lemma A. *Let f be an isometric immersion of M into $(\widetilde{M}, \langle\ ,\ \rangle)$. Then f is isotropic at $x \in M$ if and only if the second fundamental form σ of f satisfies $\langle\sigma(X,X), \sigma(X,Y)\rangle = 0$ for an arbitrary orthogonal pair $X, Y \in T_xM$.*

For the extrinsic shape of a helix through a constant isotropic immersion we have by Gauss formula the following elementary lemma.

Lemma 3. *Let $f : M \to \widetilde{M}$ be a constant λ-isotropic immersion. Then for each helix γ on M the first curvature $\tilde\kappa_1$ of the curve $f\circ\gamma$ on $\widetilde{M}$ is constant. We have $\tilde\kappa_1 = \sqrt{\kappa_1^2 + \lambda^2}$ if we denote the first curvature of γ by κ_1.*

4. Parallel imbeddings of compact symmetric spaces of rank one

In this section we study each parallel and non-totally geodesic immersion f of a compact symmetric space M of rank one into a complete simply conneted real space form $\widetilde{M}^m(\tilde c)$ of constant curvature $\tilde c$. This parallel immersion f has many geometric properties. For example, for each geodesic γ on the submanifold M the curve $f\circ\gamma$ is a circle of positive curvature in the ambient space $\widetilde{M}^m(\tilde c)$, and the curvature of $f\circ\gamma$ does not depend on the choice of γ.

We shall study these parallel immersions by investigating the extrinsic shape of *circles* on the submanifold M in $\widetilde{M}^m(\tilde c)$. In her paper[Su] the first author studies the case of a real projective space. Note that every circle on a real projective space $\mathbb{R}P^n$ is contained in some totally geodesic $\mathbb{R}P^2$ of $\mathbb{R}P^n$. So it is enough to study the case $n = 2$. The following clarifies the extrinsic shape of circles of positive curvature on $\mathbb{R}P^n$ in $\widetilde{M}^m(\tilde c)$ under the parallel imbedding.

Proposition 1. *Let* $f = f_2 \circ f_1 : \mathbb{R}P^2(c/3) \xrightarrow{f_1} S^4(c) \xrightarrow{f_2} \widetilde{M}^{2+p}(\tilde{c})$ *be an isometric parallel imbedding of* $\mathbb{R}P^2(c/3)$ *into a real space form* $\widetilde{M}^{2+p}(\tilde{c})$ $(c \geqq \tilde{c})$. *Here* f_1 *is the first standard minimal imbedding of* $\mathbb{R}P^2(c/3)$ *into* $S^4(c)$ *and* f_2 *is a totally umbilic imbedding of* $S^4(c)$ *into* $\widetilde{M}^{2+p}(\tilde{c})$. *Then*

(1) *When* $c = \tilde{c}$,

(1_a) f *maps each circle of curvature* $\sqrt{c/6}$ *to a helix of proper order* 3 *whose curvatures are* $\kappa_1 = \sqrt{c/2}, \kappa_2 = \sqrt{c}$.

(1_b) f *maps each circle of curvature* $k \neq \sqrt{c/6}$ *to a helix of proper order* 4 *whose curvatures are*

$$\kappa_1 = \sqrt{\frac{3k^2+c}{3}},\ \kappa_2 = \frac{3k\sqrt{c}}{\sqrt{3k^2+c}},\ \kappa_3 = \frac{|6k^2-c|}{\sqrt{3(3k^2+c)}}.$$

(2) *When* $c > \tilde{c}$, f *maps each circle of curvature* k *to a helix of proper order* 4 *whose curvatures are*

$$\kappa_1 = \sqrt{\frac{3k^2+4c-3\tilde{c}}{3}},\ \kappa_2 = \frac{3k\sqrt{c}}{\sqrt{3k^2+4c-3\tilde{c}}},$$
$$\kappa_3 = \frac{\sqrt{4(3k^2+c)^2-3\tilde{c}(12k^2+c)}}{\sqrt{3(3k^2+4c-3\tilde{c})}}.$$

Proof. Generally, the second fundamental form σ of a λ-isotropic immersion $f : M^n(c_1) \to \widetilde{M}^{n+p}(c_2)$ satisfies the following:

$$\begin{aligned}\langle\sigma(X,Y),\sigma(Z,W)\rangle = \frac{c_1-c_2}{3}&\big(2\langle X,Y\rangle\langle Z,W\rangle - \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\big)\\ &+\frac{\lambda^2}{3}\big(\langle X,Y\rangle\langle Z,W\rangle + \langle X,W\rangle\langle Y,Z\rangle + \langle X,Z\rangle\langle Y,W\rangle\big)\end{aligned}$$

for any vector fields X, Y, Z, W on the submanifold $M^n(c_1)$.

Since our isometric imbedding f is a $\sqrt{(4c-3\tilde{c})/3}$-isotropic imbedding, it satisfies that

$$\begin{aligned}\langle\sigma(X,Y),\sigma(Z,W)\rangle = \frac{2c-3\tilde{c}}{3}&\langle X,Y\rangle\langle Z,W\rangle\\ &+\frac{c}{3}\big(\langle X,W\rangle\langle Y,Z\rangle + \langle X,Z\rangle\langle Y,W\rangle\big).\end{aligned} \tag{4.1}$$

We denote by ∇ the covariant differentiation of $\mathbb{R}P^2(c/3)$ and by $\widetilde{\nabla}$ that of $\widetilde{M}^{2+p}(\tilde{c})$. For a circle γ of curvature k on $\mathbb{R}P^2(c/3)$ satisfying $\nabla_{\dot\gamma}\dot\gamma = kY$ and $\nabla_{\dot\gamma}Y = -k\dot\gamma$, we see from the Gauss formula and (4.1) that $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa_1 V_2$, where

$$\kappa_1 = \frac{\sqrt{3k^2+4c-3\tilde{c}}}{\sqrt{3}},\quad V_2 = \frac{\sqrt{3}}{\sqrt{3k^2+4c-3\tilde{c}}}(kY+\sigma(\dot\gamma,\dot\gamma)).$$

Since our imbedding f is parallel, we get

$$\widetilde{\nabla}_{\dot\gamma} V_2 = -\kappa_1 \dot\gamma + \frac{3\sqrt{3}k}{\sqrt{3k^2+4c-3\tilde c}}\sigma(\dot\gamma, Y).$$

Therefore, from (4.1) we have $\widetilde{\nabla}_{\dot\gamma} V_2 = -\kappa_1\dot\gamma + \kappa_2 V_3$ by putting

$$\kappa_2 = \frac{3k\sqrt{c}}{\sqrt{3k^2+4c-3\tilde c}}, \quad V_3 = \sqrt{\frac{3}{c}}\,\sigma(\dot\gamma, Y).$$

Continuing calculation and using (4.1), we obtain $\widetilde{\nabla}_{\dot\gamma} V_3 = -\kappa_2 V_2 + \kappa_3 V_4$ and $\widetilde{\nabla}_{\dot\gamma} V_4 = -\kappa_3 V_3$ with

$$\kappa_3 = \frac{\sqrt{4(3k^2+c)^2 - 3\tilde c(12k^2+c)}}{\sqrt{3(3k^2+4c-3\tilde c)}}$$

and

$$\begin{aligned} V_4 = &\frac{(6k^2-4c+3\tilde c)cY - 3k(3k^2+c-3\tilde c)\sigma(\dot\gamma,\dot\gamma)}{\sqrt{c(3k^2+4c-3\tilde c)\{4(3k^2+c)^2-3\tilde c(12k^2+c)\}}} \\ &+ \frac{3k\sqrt{3k^2+4c-3\tilde c}\,\sigma(Y,Y)}{\sqrt{c\{4(3k^2+c)^2-3\tilde c(12k^2+c)\}}}. \end{aligned}$$

Here we note that

$$\begin{cases} 4(3k^2+c)^2 - 3\tilde c(12k^2+c) > 0, & \text{if } c > \tilde c, \\ 4(3k^2+c)^2 - 3\tilde c(12k^2+c) = (6k^2-c)^2, & \text{if } c = \tilde c, \end{cases}$$

we hence get the conclusion. □

Proposition 1 shows that under a parallel immersion the extrinsic shape of each circle of positive curvature on $\mathbb{R}P^n$ is never a curve of order 2 in the ambient space $\widetilde{M}^m(\tilde c)$. On the contrary, *some* circles on a complex projective space or a quaternionic projective space are mapped to circles, so that they are curves of order 2 in the ambient space under a parallel immersion. In order to explain this geometric property in detail we review the notion of Kähler circles in a Kähler manifold and that of quaternionic circles in a quaternionic Kähler manifold.

Let γ be a Frenet curve of order 2 in a wide sense which is not a geodesic and satisfies $\nabla_{\dot\gamma}\dot\gamma(s) = \kappa(s)V(s)$, $\nabla_{\dot\gamma}V(s) = -\kappa(s)\dot\gamma(s)$ with a smooth vector field V along γ and a function κ. We put $\tau_\gamma = |\langle\dot\gamma, JV\rangle|$, which is well-defined. Since we have

$$\frac{d}{ds}\langle\dot\gamma, JV\rangle = \langle\nabla_{\dot\gamma}\dot\gamma, JV\rangle + \langle\dot\gamma, J\nabla_{\dot\gamma}V\rangle = \kappa\langle V, JV\rangle - \kappa\langle\dot\gamma, J\dot\gamma\rangle = 0,$$

we see τ_γ does not depend on parameter s. We call γ *Kähler* when $\tau_\gamma = 1$. In other words, we call γ Kähler if $\dot\gamma$ and V span a holomorphic plane (i.e. either $V = J\dot\gamma$ or $V = -J\dot\gamma$). When γ is a Kähler circle of curvature k, the equation (2.3) reduces to

$$\nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma \quad \text{or} \quad \nabla_{\dot\gamma}\dot\gamma = -kJ\dot\gamma. \tag{4.2}$$

The extrinsic shape of Kähler circles of a complex projective space under a parallel immersion f into a real space form $\widetilde{M}^m(\tilde c)$ is known ([CM]). By using fact that the immersion f is parallel and isotropic, we find the following.

Proposition 2. *Let $f_1 : \mathbb{C}P^n(2nc/(n+1)) \to S^{n(n+2)-1}(c)$ be the first standard minimal imbedding and $f_2 : S^{n(n+2)-1}(c) \to \widetilde{M}^m(\tilde c)$ a totally umbilic imbedding into a real space form, where $c \geqq \tilde c$. Then the isometric parallel imbedding $f = f_2 \circ f_1 : \mathbb{C}P^n(2nc/(n+1)) \to \widetilde{M}^m(\tilde c)$ maps every Kähler circle of curvature k on $\mathbb{C}P^n(2nc/(n+1))$ to a circle of curvature $\sqrt{\{(n+1)(k^2 - \tilde c) + 2nc\}/(n+1)}$ in the ambient space $\widetilde{M}^m(\tilde c)$, so that it is a curve of order* 2.

Let M be a quaternionic Kähler manifold with local basis $\{I, J, K\}$ of quaternionic structure and γ be a Frenet curve of order 2 in a wide sense which is not a geodesic on M. Then I, J and K satisfy

$$\begin{cases} \nabla_{\dot\gamma} I = \qquad qJ - rK \\ \nabla_{\dot\gamma} J = -qI \quad + pK \\ \nabla_{\dot\gamma} K = rI - pJ, \end{cases} \tag{4.3}$$

for some functions p, q, r along γ ([I]). We see from (4.3) that $\tau_\gamma^2 := \langle\dot\gamma, IV\rangle^2 + \langle\dot\gamma, JV\rangle^2 + \langle\dot\gamma, KV\rangle^2$ is constant along γ. We call γ *quaternionic* if V is a $\mathbb{R}$-linear combination of $I\dot\gamma, J\dot\gamma$ and $K\dot\gamma$ at each point of γ. In other words, γ is quaternionic if and only if $\tau_\gamma = 1$. When γ is a quaternionic circle of curvature k, Equation (2.3) reduces to

$$\nabla_{\dot\gamma}\dot\gamma = k(\lambda I\dot\gamma + \mu J\dot\gamma + \nu K\dot\gamma), \tag{4.4}$$

where λ, μ and ν are functions along γ satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. The extrinsic shape of quaternionic circles on a quaternionic projective space under a parallel immersion f into a real space form $\widetilde{M}^m(\tilde c)$ is known as follows:

Proposition 3. *Let $f_1 : \mathbb{H}P^n(2nc/(n+1)) \to S^{n(2n+3)-1}(c)$ be the first standard minimal imbedding and $f_2 : S^{n(2n+3)-1}(c) \to \widetilde{M}^m(\tilde c)$ a totally umbilic imbedding into a real space form, where $c \geqq \tilde c$. Then the isometric parallel imbedding $f = f_2 \circ f_1 : \mathbb{H}P^n(2nc/(n+1)) \to \widetilde{M}^m(\tilde c)$ maps every quaternionic circle of curvature k on $\mathbb{H}P^n(2nc/(n+1))$ to a circle of curvature $\sqrt{\{(n+1)(k^2 - \tilde c) + 2nc\}/(n+1)}$ in the ambient space $\widetilde{M}^m(\tilde c)$, so that it is a curve of order* 2.

We consider converses of Propositions 2 and 3 to obtain some characterizations of parallel imbeddings of complex and quaternionic projective spaces into a real space form $\widetilde{M}^m(\tilde c)$.

Theorem 2. *Let f be an isometric immersion of a nonflat real $2n(\geqq 4)$-dimensional Kähler manifold M into a complete simply connected real space form $\widetilde{M}^m(\tilde{c})$. Then the following conditions are equivalent*:

(1) *M is locally congruent to a complex projective space imbedded into some sphere in $\widetilde{M}^m(\tilde{c})$ through the first standard minimal imbedding.*

(2) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M to a circle in $\widetilde{M}^m(\tilde{c})$.*

(3) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M to a plane curve in $\widetilde{M}^m(\tilde{c})$.*

(4) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M to a curve of order 2 in $\widetilde{M}^m(\tilde{c})$.*

Proof. It suffices to show that (4) implies (1). We shall verify that the submanifold M is locally congruent to a complex space form (i.e a Kähler manifold of constant holomorphic sectional curvature) and the immersion f is parallel.

Since for an arbitrary unit vector $X \in TM$ we have Kähler circles γ_1, γ_2 with the initial conditions $\dot{\gamma}_1(0) = \dot{\gamma}_2(0) = X, \nabla_{\dot{\gamma}_1}\dot{\gamma}_1(0) = kJX, \nabla_{\dot{\gamma}_2}\dot{\gamma}_2(0) = -kJX$, we find by (3.3) in Lemma 2 that

$$\begin{aligned}&\Big(3k^2 + \|\sigma(X,X)\|^2\Big)\langle\sigma(X,X),\sigma(X,JX)\rangle \\ &\qquad\qquad + k\langle\sigma(X,X),(\bar{\nabla}_X\sigma)(X,X)\rangle = 0,\end{aligned} \tag{4.5}$$

$$\begin{aligned}-&\Big(3k^2 + \|\sigma(X,X)\|^2\Big)\langle\sigma(X,X),\sigma(X,JX)\rangle \\ &\qquad\qquad + k\langle\sigma(X,X),(\bar{\nabla}_X\sigma)(X,X)\rangle = 0.\end{aligned} \tag{4.6}$$

Hence, we see $\langle\sigma(X,X),\sigma(X,JX)\rangle = 0$.

By (3.1) and (3.2) we have

$$\kappa_i\Big\{-A_{\sigma(\dot{\gamma}_i,\dot{\gamma}_i)}\dot{\gamma}_i + \|\sigma(\dot{\gamma}_i,\dot{\gamma}_i)\|^2\dot{\gamma}_i\Big\} = \pm k\kappa_i' J\dot{\gamma}_i, \tag{4.7}$$

$$\kappa_i\Big\{\pm 3k\sigma(\dot{\gamma}_i,J\dot{\gamma}_i) + (\bar{\nabla}_{\dot{\gamma}_i}\sigma)(\dot{\gamma}_i,\dot{\gamma}_i)\Big\} = \kappa_i'\sigma(\dot{\gamma}_i,\dot{\gamma}_i), \tag{4.8}$$

with the positive curvature function κ_i of the curve $f \circ \gamma_i$ of order 2, where we take plus signature when $i = 1$ and minus signature when $i = 2$ in Equations (4.7) and (4.8). Taking the inner product of the both sides of (4.7) with $J\dot{\gamma}_i$ and evaluating this at $s = 0$, we obtain $\kappa_i'(0) = 0$. It follows from (4.7) at $s = 0$ that $A_{\sigma(X,X)}X = \|\sigma(X,X)\|^2X$ for each unit vector $X \in T_xM$ at an arbitrary point $x \in M$, so that M is an isotropic submanifold of $\widetilde{M}^m(\tilde{c})$ (see Lemma A). Moreover, from (4.8) at $s = 0$ we see

$$\pm 3k \cdot \sigma(X,JX) + (\bar{\nabla}_X\sigma)(X,X) = 0$$

for each unit vector $X \in TM$. Thus we have the following two equations:

$$(\bar{\nabla}_X \sigma)(X, X) = 0, \tag{4.9}$$

$$\sigma(X, JX) = 0. \tag{4.10}$$

Thanks to the Codazzi equation $(\bar{\nabla}_X \sigma)(Y, Z) = (\bar{\nabla}_Y \sigma)(X, Z)$, Equation (4.9) yields $\bar{\nabla}\sigma = 0$. Replacing X by $X + JX$ in (4.10), we get

$$\sigma(JX, JX) = \sigma(X, X) \tag{4.11}$$

for all $X \in TM$. Let R denote the curvature tensor of M. Then it follows from (4.10), (4.11) and the equation of Gauss that

$$\begin{aligned}\langle R(X, JX)JX, X\rangle &= \tilde{c} + \langle \sigma(X, X), \sigma(JX, JX)\rangle - \|\sigma(X, JX)\|^2 \\ &= \tilde{c} + \|\sigma(X, X)\|^2\end{aligned}$$

holds for each unit vector X. Since M is isotropic, this implies that M is a complex space form. Thus our assertion follows from the results of [F] and [T]. □

For a quaternionic Kähler manifold we obtain the following result similar to Theorem 2.

Theorem 3. *Let f be an isometric immersion of a nonflat real $4n(\geqq 8)$-dimensional quaternionic Kähler manifold M into a complete simply conneted real space form $\widetilde{M}^m(\tilde{c})$. Then the following conditions are equivalent*:

(1) *M is locally congruent to a quaternionic projective space imbedded into some sphere in $\widetilde{M}^m(\tilde{c})$ through the first standard minimal imbedding.*
(2) *There exists $k > 0$ satisfying that f maps every quaternionic circle of curvature k on M to a circle in $\widetilde{M}^m(\tilde{c})$.*
(3) *There exists $k > 0$ satisfying that f maps every quaternionic circle of curvature k on M to a plane curve in $\widetilde{M}^m(\tilde{c})$.*
(4) *There exists $k > 0$ satisfying that f maps every quaternionic circle of curvature k on M to a curve of order 2 in $\widetilde{M}^m(\tilde{c})$.*

Proof. It is enough to show the condition (4) implies the condition (1). By the same argument as in the proof of Theorem 2 we get

$$(\bar{\nabla}_X \sigma)(X, X) = 0 \quad \text{and} \quad \sigma(X, \lambda IX + \mu JX + \nu KX) = 0$$

for each unit vector $X \in TM$ and arbitrary real numbers λ, μ, ν satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. These imply that M is a parallel submanifold and

$$\sigma(X, IX) = \sigma(X, JX) = \sigma(X, KX) = 0 \tag{4.12}$$

holds for all $X \in TM$. In Equation (4.12), replacing X by $X + IX$ and so on, we get

$$\sigma(IX, IX) = \sigma(JX, JX) = \sigma(KX, KX) = \sigma(X, X), \tag{4.13}$$

and replacing X by IX and so on, we have

$$\sigma(IX, JX) = \sigma(JX, KX) = \sigma(KX, IX) = 0. \tag{4.14}$$

Let R denote the curvature tensor of M. Then it follows from (4.12), (4.13), (4.14) and the formula of Gauss that

$$\begin{aligned}
&\langle R(X, \lambda IX + \mu JX + \nu KX)(\lambda IX + \mu JX + \nu KX), X\rangle \\
&\qquad = \tilde{c} + \langle \sigma(X, X), \sigma(\lambda IX + \mu JX + \nu KX, \lambda IX + \mu JX + \nu KX)\rangle \\
&\qquad\qquad\qquad - \|\sigma(X, \lambda IX + \mu JX + \nu KX)\|^2 \\
&\qquad = \tilde{c} + \|\sigma(X, X)\|^2
\end{aligned}$$

holds for an arbitrary unit vector X and arbitrary λ, μ, ν satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. Since M is isotropic, this shows that the submanifold M is a quaternionic space form. Thus our assertion follows from the results of [F] and [T]. $\square$

At the last stage of this section we give a characterization of a parallel imbedding of Cayley projective plane which corresponds to Theorems 2 and 3. For a Frenet curve γ of order 2 in a wide sense which is not a geodesic in a locally symmetric space M, the sectional curvature $K(\dot{\gamma}, V)$ of the plane spanned by $\dot{\gamma}$ and V is constant along γ. When M is a complex space form $\widetilde{M}_m(c)$, it is an equivalent invariant to τ_γ by the following relation: $K(\dot{\gamma}, V) = c(1 + 3\tau_\gamma^2)/4$. For a Frenet curve of order 2 in a wide sense which is not a geodesic on a Cayley projective plane $CaP^2(c)$ of maximal sectional curvature c, we call it *Cayley* if it satisfies $K(\dot{\gamma}, V) \equiv c$, and *totally real* if it satisfies $K(\dot{\gamma}, V) \equiv c/4$. It is clear that every Frenet curve of order 2 on $CaP^2(c)$ is contained in some totally geodesic $\mathbb{C}P^2(c)$ in $CaP^2(c)$ (c.f. [MT]). We here restrict ourselves on circles on $CaP^2(c)$. This, together with the results of [AMU], guarantees that every Cayley circle on $CaP^2(c)$ is contained in some totally geodesic $\mathbb{C}P^1(c)$ in $CaP^2(c)$ and every totally real circle on $CaP^2(c)$ is contained in some totally geodesic $\mathbb{R}P^2(c/4)$ in $CaP^2(c)$.

It is known that the parallel imbedding f of $CaP^2(4c/3)$ into a real space form $\widetilde{M}^m(\tilde{c})$ is decomposed as:

$$f : CaP^2\Big(\frac{4}{3}c\Big) \xrightarrow{f_1} S^{25}(c) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c}), \tag{4.15}$$

where $c \geqq \tilde{c}$, f_1 is the first standard minimal imbedding of $CaP^2(4c/3)$ into $S^{25}(c)$ and f_2 is a totally umbilic imbedding of $S^{25}(c)$ into $\widetilde{M}^m(\tilde{c})$. The extrinsic shape of circles on $CaP^2(c)$ in the ambient manifold $\widetilde{M}^m(\tilde{c})$ through the parallel imbedding f was studied in [AM].

Proposition 4. *The extrinsic shape $f \circ \gamma$ of each circle γ of curvature $k(>0)$ on $CaP^2(4c/3)$ in $\widetilde{M}^m(\tilde{c})$ through the isometric imbedding f defined by* (4.15) *is as follows*:

(1) *When γ is Cayley, the curve $f \circ \gamma$ is a plane curve in $\widetilde{M}^m(\tilde{c})$. In particular, it is a circle of curvature $\sqrt{(3k^2+4c-3\tilde{c})/3}$ in this space.*
(2) *When γ is totally real, the curve $f \circ \gamma$ is a helix of proper order* 3 *or proper order* 4 *in $\widetilde{M}^m(\tilde{c})$.*
(3) *When γ is neither Cayley nor totally real, the curve $f \circ \gamma$ is a helix of proper order* 5 *or proper order* 6 *in $\widetilde{M}^m(\tilde{c})$.*

This proposition tells us that if a circle γ is not Cayley, the curve $f \circ \gamma$ is not a curve of order 2 in the ambient manifold $\widetilde{M}^m(\tilde{c})$. Motivated by this fact, we shall characterize this parallel imbedding of a Cayley projective plane into real space forms by Cayley circles.

Theorem 4. *Let M be an open set of $CaP^2(4c/3)$ and $f : M \longrightarrow \widetilde{M}^m(\tilde{c})$ be an isometric immersion into a complete simply conneted real space form $\widetilde{M}^m(\tilde{c})$. Then the following conditions are equivalent*:

(1) *M is a parallel submanifold of $\widetilde{M}^m(\tilde{c})$, so that our isometric immersion f of M into $\widetilde{M}^m(\tilde{c})$ is given by* (4.15).
(2) *There exists $k>0$ satisfying that f maps every Cayley circle of curvature k on M to a circle in the ambient manifold $\widetilde{M}^m(\tilde{c})$.*
(3) *There exists $k>0$ satisfying that f maps every Cayley circle of curvature k on M to a plane curve in the ambient manifold $\widetilde{M}^m(\tilde{c})$.*
(4) *There exists $k>0$ satisfying that f maps every Cayley circle of curvature k on M to a curve of order* 2 *in the ambient manifold $\widetilde{M}^m(\tilde{c})$.*

Proof. It is sufficient to show the condition (4) implies the condition (1). For each orthonormal pair $X, Y \in T_xM$ of unit vectors with $K(X,Y) = \frac{4c}{3}$ at an arbitrary point $x \in M$ we have a Cayley circle γ with initial condition that $\gamma(0) = x, \dot{\gamma}(0) = X$ and $\nabla_{\dot{\gamma}}\dot{\gamma}(0) = kY$. By (3.3) in Lemma 2 we have

$$\Big(3k^2 + \|\sigma(X,X)\|^2\Big)\langle\sigma(X,X),\sigma(X,Y)\rangle + k\langle\sigma(X,X),(\bar{\nabla}_X\sigma)(X,X)\rangle = 0.$$

For $X, -Y$ we also have

$$\Big(3k^2 + \|\sigma(X,X)\|^2\Big)\langle\sigma(X,X),\sigma(X,Y)\rangle - k\langle\sigma(X,X),(\bar{\nabla}_X\sigma)(X,X)\rangle = 0.$$

These equalities guarantee $\langle\sigma(X,X),\sigma(X,Y)\rangle = 0$ for each orthonormal pair $X, Y \in T_xM$ with $K(X,Y) = \frac{4c}{3}$. Along the same lines as in the proof of Theorem 2 we find

$$-3k \cdot \sigma(X,Y) = (\bar{\nabla}_X\sigma)(X,X) = 3k \cdot \sigma(X,Y),$$

which leads us to $(\bar{\nabla}_X \sigma)(X, X) = 0$ for an arbitrary unit vector $X \in TM$. Thanks to the equation of Codazzi, we obtain $\bar{\nabla}\sigma = 0$. Therefore our assertion follows from the results of [F] and [T]. $\square$

5. Characterization of totally geodesic Kähler immersions

In his paper Nomizu[N] gives the following characterization of totally geodesic complex projective spaces among Kähler submanifolds in a complex projective space by an extrinsic property of geodesics: A Kähler submanifold M_n in a complex projective space $\mathbb{C}P^{n+p}$ is locally congruent to $\mathbb{C}P^n$, which is a totally geodesic Kähler submanifold of $\mathbb{C}P^{n+p}$, if every geodesic on M_n is locally contained in a complex projective line $\mathbb{C}P^1$ in $\mathbb{C}P^{n+p}$. Motivated by this characterization, we shall characterize totally geodesic Kähler immersions into an arbitrary Kähler manifold by the extrinsic shape of Kähler circles on the submanifold.

Theorem 5. *Let f be a Kähler isometric immersion of a Kähler manifold M into an arbitrary Kähler manifold $\widetilde{M}$. Then the following conditions are equivalent*:

(1) *f is a totally geodesic immersion.*
(2) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M to a circle in $\widetilde{M}$.*
(3) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M to a curve of order 2 in $\widetilde{M}$.*

Proof. What we have to show is that (3) implies (1). Let γ be a Kähler circle of curvature k on M satisfying $\nabla_{\dot\gamma}\dot\gamma = \pm kJ\dot\gamma$. By the assumption we have

$$\kappa(-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + \|\sigma(\dot\gamma,\dot\gamma)\|^2\dot\gamma) = \pm k\kappa' J\dot\gamma, \tag{5.1}$$

where $\kappa(>0)$ denotes the curvature function of $f \circ \gamma$. Taking the inner product of both sides of (5.1) with $J\dot\gamma$, we find

$$\pm k\kappa' = -\kappa\langle A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma, J\dot\gamma\rangle = -\kappa\langle \sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, J\dot\gamma)\rangle = -\kappa\langle \sigma(\dot\gamma,\dot\gamma), J(\sigma(\dot\gamma,\dot\gamma))\rangle = 0.$$

This implies that $f \circ \gamma$ is a circle of positive curvature in the ambient manifold $\widetilde{M}$. This shows (3) implies (2).

For an arbitrary unit tangent vector $X \in TM$ we denote by γ_i $(i = 1, 2)$ Kähler circles of curvature k with $\dot\gamma_i(0) = X$ and $\nabla_{\dot\gamma_1}\dot\gamma_1 = kJ\dot\gamma_1$, $\nabla_{\dot\gamma_2}\dot\gamma_2 = -kJ\dot\gamma_2$. Following (3.2), we have

$$3kJ(\sigma(\dot\gamma_1,\dot\gamma_1)) + (\bar{\nabla}_{\dot\gamma_1}\sigma)(\dot\gamma_1,\dot\gamma_1) = 0,$$
$$-3kJ(\sigma(\dot\gamma_2,\dot\gamma_2)) + (\bar{\nabla}_{\dot\gamma_2}\sigma)(\dot\gamma_2,\dot\gamma_2) = 0,$$

because $f \circ \gamma_i$ is a circle. Evaluating these at $s = 0$, we obtain

$$3kJ(\sigma(X,X)) + (\bar{\nabla}_X\sigma)(X,X) = 0,$$
$$-3kJ(\sigma(X,X)) + (\bar{\nabla}_X\sigma)(X,X) = 0,$$

which lead us to $\sigma(X,X)=0$. As X is an arbitrary unit vector, we find M is totally geodesic in $\widetilde{M}$. $\square$

Remark. In the statement of Theorem 5, we can not relax the condition on k to $k \geq 0$. For example, we consider the imbedding $f : \mathbb{C}P^n(c) \longrightarrow \mathbb{C}P^{(n^2+3n)/2}(2c)$ which is given by all homogeneous monomials of degree 2

$$(z_0, \dots, z_n) \mapsto (z_0^2, \sqrt{2}z_0z_1, \dots, z_n^2)$$

in homogeneous coordinates. This (non totally geodesic) Kähler isometric imbedding maps every geodesic on $\mathbb{C}P^n(c)$ to a totally real circle of the same positive curvature $\sqrt{c/2}$ in the ambient space $\mathbb{C}P^{(n^2+3n)/2}(2c)$.

As an immediate consequence of Theorem 5 we obtain the following characterization of totally geodesic Kähler submanifolds in a complete simply connected complex space form $\widetilde{M}_m(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$, which is either a complex projective space $\mathbb{C}P^m(\tilde{c})$, a complex Euclidean space $\mathbb{C}^m$ or a complex hyperbolic space $\mathbb{C}H^m(\tilde{c})$.

Theorem 6. *Let f be a Kähler isometric immersion of a Kähler manifold M into a complete simply connected complex space form $\widetilde{M}_m(\tilde{c})$. Then the following conditions are equivalent:*

1. *f is a totally geodesic immersion.*
2. *There exists $k>0$ satisfying that f maps every Kähler circle of curvature k on M to a circle in $\widetilde{M}_m(\tilde{c})$.*
3. *There exists $k>0$ satisfying that f maps every Kähler circle of curvature k on M to a plane curve in $\widetilde{M}_m(\tilde{c})$.*
4. *There exists $k>0$ satisfying that f maps every Kähler circle of curvature k on M to a curve of order 2 in $\widetilde{M}_m(\tilde{c})$.*

6. Veronese imbeddings

In this section we study the following problem: For an isometric immersion $f : M \to \widetilde{M}$ if any *geodesics* on M are mapped to curves of order 2 in $\widetilde{M}$ through f, what can we say on the submanifold M?

We here pay attention to the extrinsic shape of geodesics on a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c in a complex projective space $\mathbb{C}P^N(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$ through a Kähler isometric full immersion. By virtue of the classification theorem ([C, NO]) this Kähler immersion is nothing but a Kähler imbedding $f_k : \mathbb{C}P^n(c/k) \to \mathbb{C}P^N(c)$ given by

$$[z_i]_{0\leqq i\leqq n} \mapsto \Big[\sqrt{\frac{k!}{k_0!\cdots k_n!}}\, z_0^{k_0}\cdots z_n^{k_n}\Big]_{k_0+\cdots+k_n=k},$$

where $[*]$ means the point of the projective space with the homogeneous coordinates $*$ and $N = (n+k)!/(n!k!) - 1$. We usually call f_k the k-th *Veronese imbedding.* It is known that the second fundamental form of f_k is parallel if and only if $k = 1$ or $k = 2$. These parallel imbeddings f_k $(k = 1, 2)$ have various geometric properties. For example, the second Veronese imbedding f_2 maps each geodesic on the submanifold $\mathbb{C}P^n(c/2)$ to a circle of curvature $\sqrt{c}/2$ in a real projective plane $\mathbb{R}P^2(c/4)$ of curvature $c/4$ which is a totally real totally geodesic submanifold of the ambient manifold $\mathbb{C}P^{n(n+3)/2}(c)$. Using such a property, Nomizu[N] gives the following characterization. Let $f : M_n \to \mathbb{C}P^N(c)$ be a Kähler isometric full immersion of an n-dimensional Kähler manifold into an N-dimensional complex projective space of constant holomorphic sectional curvature c. He shows that either $M_n = \mathbb{C}P^n(c)$ and $N = n$, or $M_n = \mathbb{C}P^n(c/2)$, $N = n(n+3)/2$ and f is locally equivalent to the second Veronese imbedding f_2 if and only if for each geodesic γ on M_n the curve $f \circ \gamma$ is a circle in $\mathbb{C}P^N(c)$.

The main purpose of this section is to improve this characterization. We relax the condition that $f \circ \gamma$ is a circle to the condition that it is a curve of order 2. We here briefly recall some fundamental results on Veronese imbeddings f_k $(k = 1, 2, \dots)$ (see [PS]). An isometric immersion f of a Riemannian manifold M into an ambient Riemannian manifold $\widetilde{M}$ is called a *d-planar geodesic immersion* if for each geodesic γ on M the curve $f \circ \gamma$ is locally contained in a d-dimensional totally geodesic submanifold of $\widetilde{M}$. In particular, a curve ρ is called d-planar if it is locally contained in a d-dimensional totally geodesic submanifold. A d-planar curve ρ is said to be *proper d-planar* if it is not $(d-1)$-planar. We call a d-planar geodesic immersion $f : M \to \widetilde{M}$ *proper* if the curve $f \circ \gamma$ is proper d-planar for each geodesic γ of the submanifold M.

Proposition B. *The k-th Veronese imbedding $f_k : \mathbb{C}P^n(c/k) \to \mathbb{C}P^N(c)$ is proper k-planar geodesic.*

In their paper[PS] J.S. Pak and K. Sakamoto considered the converse of Proposition B to obtain a characterization of each f_k:

Theorem C. *Let $f : M_n \to \mathbb{C}P^N(c)$ be a proper k-planar geodesic Kähler isometric full immersion of an n-dimensional Kähler manifold into an N-dimensional complex projective space of constant holomorphic sectional curvature c. Suppose that for each geodesic γ on M_n the curve $f \circ \gamma$ is locally contained in a k-dimensional totally real totally geodesic submanifold $\mathbb{R}P^k(c/4)$ of $\mathbb{C}P^N(c)$. Then M_n is locally congruent to $\mathbb{C}P^n(c/k)$, $N = (n+k)!/(n!k!) - 1$ and f is locally equivalent to the k-th Veronese imbedding f_k.*

We remark that for each geodesic γ on $\mathbb{C}P^n(c/k)$ the curve $f_k \circ \gamma$ is a helix of proper order k in $\mathbb{R}P^k(c/4)$ with the curvatures $\kappa_1, \dots, \kappa_{k-1}$ which are independent of the choice of γ.

The following is another (local) characterization of each Veronese imbedding f_k (see [C, NO]).

Theorem D. *Let $f : M_n(c) \to M_N(\tilde{c})$ be a Kähler isometric immersion of a complex space form into another complex space form. If $\tilde{c} > 0$ and f is full, then $\tilde{c} = kc$ and $N = (n+k)!/(n!k!) - 1$ for some positive integer k.*

We now state our result. The reader should confer the following result with Theorem 6.

Theorem 7. *Let $f : M_n \to \mathbb{C}P^N(c)$ be a Kähler isometric full immersion of an n-dimensional connected Kähler manifold into an N-dimensional complex projective space of constant holomorphic sectional curvature c. If the image $f \circ \gamma$ of each geodesic γ on M_n is a curve of order 2 in $\mathbb{C}P^N(c)$, then one of the following holds*:

(i) *M_n is locally congruent to $\mathbb{C}P^n(c)$ and $N = n$,*
(ii) *M_n is locally congruent to $\mathbb{C}P^n(c/2)$, $N = n(n+3)/2$ and f is locally equivalent to the second Veronese imbedding f_2.*

Proof. It is enough to prove our Theorem in case that the immersion f is not totally geodesic, namely f is not of case (i). Our steps for proof are as follows: We first show that the submanifold M is (λ-) isotropic at its each point in the ambient manifold $\mathbb{C}P^N(c)$, next we verify that the function λ is constant on M and finally we calculate the holomorphic sectional curvature of M.

As Lemma A holds in a trivial sense at an arbitrary geodesic point of M, we have only to consider a non-geodesic point $x \in M$ and a unit vector $X \in T_xM$ with $\sigma(X,X) \neq 0$. We take the geodesic $\gamma = \gamma(s)$ ($s \in I$) on M with initial condition that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Here, I is a sufficiently small open interval on $\mathbb{R}$ satisfying $\sigma(\dot{\gamma}(s), \dot{\gamma}(s)) \neq 0$ for all $s \in I$. As $\kappa = |\sigma(\dot{\gamma}, \dot{\gamma})|$ is the curvature of the curve $f \circ \gamma$ of order 2, we see by Lemma 1 that our curve $f \circ \gamma$ is a Frenet curve of proper order 2 whose Frenet frame is $\{\dot{\gamma}, \frac{1}{\kappa}\sigma(\dot{\gamma}, \dot{\gamma})\}$. Since f is a Kähler immersion, we find that $\tau_\gamma = |\frac{1}{\kappa}\langle J\dot{\gamma}, \sigma(\dot{\gamma}, \dot{\gamma})\rangle| = 0$, where J is the complex structure of $\mathbb{C}P^N(c)$. So we can take the totally real totally geodesic $\mathbb{R}P^2(c/4)$ passing x satisfying that the vectors $\dot{\gamma}(0)$ and $\sigma(\dot{\gamma}, \dot{\gamma})(0)$ span the tangent space $T_x\mathbb{R}P^2(c/4)$. We here consider the Frenet curve ρ of proper order 2 on the surface $\mathbb{R}P^2(c/4)$ passing the point $x = \rho(0)$ with the same curvature $\kappa(s)$ (> 0) and the same initial frame $\{\dot{\gamma}(0), \frac{1}{\kappa(0)}\sigma(\dot{\gamma}, \dot{\gamma})(0)\}$. By the uniqueness of solutions for ordinary differential equations we can see that the curve $f \circ \gamma$ locally coincides with ρ, so that it is locally contained in $\mathbb{R}P^2(c/4)$.

The following discussion on the isotropic property of M is the same as in [pp. 40–41, PS]. However we here write it down in detail for readers' convenience. As $\mathbb{R}P^2(c/4)$ is a 2-dimensional totally geodesic submanifold of $\mathbb{C}P^N(c)$, the vectors $\dot{\gamma}(s)$ and $\sigma(\dot{\gamma}(s), \dot{\gamma}(s))$ span the tangent space $T_{\gamma(s)}\mathbb{R}P^2(c/4)$ for each s. This, together with $\widetilde{\nabla}_{\dot{\gamma}}(\sigma(\dot{\gamma}, \dot{\gamma})) \in T_{\gamma(s)}\mathbb{R}P^2(c/4)$, implies

$$\widetilde{\nabla}_{\dot{\gamma}}(\sigma(\dot{\gamma}, \dot{\gamma})) = \alpha\dot{\gamma} + \beta\sigma(\dot{\gamma}, \dot{\gamma}) \tag{6.1}$$

for some smooth functions $\alpha = \alpha(s)$ and $\beta = \beta(s)$ on the interval I. Let Y be an arbitrary vector at x which is perpendicular to the vector $X = \dot\gamma(0)$. We extend the vector Y to a vector field $\widetilde{Y}$ on the curve $f \circ \gamma$. Hence Equation (6.1) gives

$$\begin{aligned}\langle \sigma(X,X), \sigma(X,Y)\rangle &= \langle \sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma,\widetilde{Y})\rangle(0) = \langle \sigma(\dot\gamma,\dot\gamma), \widetilde{\nabla}_{\dot\gamma}\widetilde{Y}\rangle(0)\\ &= -\langle \widetilde{\nabla}_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma)), \widetilde{Y}\rangle(0) = -\langle \alpha(0)X + \beta(0)\sigma(X,X), Y\rangle = 0.\end{aligned}$$

So it follows from Lemma A that the submanifold M is (λ-)isotropic at its each point in $\mathbb{C}P^N(c)$. In order to see the function λ is constant on M, we choose an arbitary orthonomal pair of vectors Z, W at any fixed point $y \in M$. Assume that $\lambda(y) \neq 0$. We here define a smooth vector field $\widetilde{Z}$ (resp. $\widetilde{W}$) on some sufficiently small neighborhood $\mathcal{U}_y$ by using parallel displacement for the vector Z (resp. W) along each geodesic with origin y. Note that $\nabla\widetilde{Z} = \nabla\widetilde{W} = 0$ at the point y and $\langle \widetilde{Z}, \widetilde{W}\rangle = 0$ on $\mathcal{U}_y$. Hence, at the point y we find

$$\begin{aligned}Z(\lambda^2) &= Z\langle \sigma(\widetilde{W},\widetilde{W}), \sigma(\widetilde{W},\widetilde{W})\rangle = 2\langle D_Z(\sigma(\widetilde{W},\widetilde{W})), \sigma(W,W)\rangle\\ &= 2\langle (\bar\nabla_Z\sigma)(W,W), \sigma(W,W)\rangle.\end{aligned}$$

So, using the equation of Codazzi, Lemma A and an equality $\widetilde{\nabla}_W(\sigma(\widetilde{W},\widetilde{W})) = \alpha_0 W + \beta_0\sigma(W,W)$ which corresponds to (6.1), at the point y we have

$$\begin{aligned}Z(\lambda^2) &= 2\langle (\bar\nabla_W\sigma)(Z,W), \sigma(W,W)\rangle = 2\langle D_W(\sigma(\widetilde{Z},\widetilde{W})), \sigma(W,W)\rangle\\ &= 2\langle \widetilde{\nabla}_W(\sigma(\widetilde{Z},\widetilde{W})), \sigma(W,W)\rangle = -2\langle \sigma(Z,W), \widetilde{\nabla}_W(\sigma(\widetilde{W},\widetilde{W}))\rangle\\ &= -2\langle \sigma(Z,W), \alpha_0 W + \beta_0\sigma(W,W)\rangle = 0.\end{aligned}$$

This implies that if $\lambda(y) \neq 0$, then the function λ is constant on $\mathcal{U}_y$. Therefore, from the connectivity of M and the continuity of λ we can see that the function λ is (nonzero) constant on the submanifold M.

On the other hand, the holomorphic sectional curvature $K(X, JX)$ of M determined by a unit vector X is given by

$$K(X,JX) = \langle R(X,JX)JX, X\rangle = c - 2\|\sigma(X,X)\|^2,$$

hence M is a complex space form. Therefore from Theorem D and Proposition B we can see that M_n is locally congruent to $\mathbb{C}P^n(c/2)$, $N = n(n+3)/2$ and f is locally equivalent to the second Veronese imbedding f_2. $\square$

In the last stage of this section we shall make mention of submanifolds in a real space form. In his paper[Sa] Sakamoto classified 2-planar geodesic submanifolds in a complete simply connected real space form $\widetilde{M}^N(\tilde c)(= \mathbb{R}^N,\ S^N(\tilde c)$ or $H^N(\tilde c))$ of curvature $\tilde c$:

Theorem E. *Let $f : M^n \to \widetilde{M}^{n+p}(\tilde{c})$ be a 2-planar geodesic immersion of an n-dimensional Riemannian manifold into an $(n+p)$-dimensional complete simply connected real space form $\widetilde{M}^{n+p}(\tilde{c})$. Then M^n is totally umbilic in $\widetilde{M}^{n+p}(\tilde{c})$ or M^n is locally congruent to a compact symmetric space of rank one imbedded into some sphere in $\widetilde{M}^{n+p}(\tilde{c})$ through the first standard minimal imbedding.*

Combining Theorem E with our discussion in this paper, we obtain the following immediately.

Theorem 8. *Let $f : M^n \to \widetilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of an n-dimensional Riemannian manifold into an $(n+p)$-dimensional complete simply connected real space form $\widetilde{M}^{n+p}(\tilde{c})$. Suppose that for each geodesic γ on M^n the curve $f \circ \gamma$ is a curve of order 2 in the ambient space $\widetilde{M}^{n+p}(\tilde{c})$. Then M^n is totally umbilic in $\widetilde{M}^{n+p}(\tilde{c})$ or M^n is locally congruent to a compact symmetric space of rank one imbedded into some sphere in $\widetilde{M}^{n+p}(\tilde{c})$ through the first standard minimal imbedding.*

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