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Applicable Range of the Modified Timoshenko's Bending Equation on the Static Bending Tests

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Abstract

This paper describes the examination about the proper range of depth/span ratio which gives the Young's modulus and shear modulus in the three-point static bending test, and obtained the following results.

(1) The applicable depth/span ratio range obtained from our proposal is wider than that given by Timoshenko's theory. However, the Young's modulus cannot be given when the depth/span ratio is large, whereas the shear modulus cannot be derived when the depth/span ratio is small. Hence, the range of depth/span ratio should be wide in obtaining the both values of Young's modulus and shear modulus by our proposal properly.

(2) The accuracy of shear modulus obtained from our proposal is dependent on the value of the coefficient used for the modification of original Timoshenko's bending theory. For obtaining the proper shear modulus, this coefficient should be determined carefully by considering the various factors including the mechanical properties of the specimen themselves and the experimental conditions such as the loading nose used and the measured point of deflection.

(3) It was thought that the shear modulus can be derived from the bending test of one specimen as well as the Young's modulus. Nevertheless, the feasibility is restricted by the depth/span ratio.

Introduction

In a previous paper, the measurement method of the shear modulus of wood by three-point static bending was examined, and modified the Timoshenko's bending equation, which is usually used for giving the shear modulus by bending, was proposed under the assumption that the indentation of loading nose produces the extra deflection.[1]

In measuring the shear modulus by bending, the deflection caused by the shearing force should be evaluated properly. This deflection is small when the test specimen has a small depth/span ratio. Thus, it was thought that the shear modulus cannot be measured properly in the small range of depth/span ratios. On the contrary, the Young's modulus would be wrongly measured when the beam has an extremely large depth/span ratio.

In this paper, the determination method of the depth/span ratio range which gives the Young's modulus and shear modulus properly was examined.

Theories

When the load P is imposed at a center of the beam with the span of l, the deflection at the loading point caused y is derived by the elementary beam theory as follows:[2]

$$y=\frac{Pl^3}{48E_sI}, \tag{1}$$

where E_s is the Young's modulus calculated by the elementary theory, and I is the second moment of cross-sectional area of the beam. In the elementary beam theory, the deflection caused by the shearing force is not considered. According to the original Timoshenko's beam theory, the apparent young's modulus consists of the Young's modulus E_t and shear modulus G_t as follows:

$$\frac{1}{E_s}=\frac{1}{E_t}+\frac{s}{G_t}\cdot\left(\frac{h}{l}\right)^2, \tag{2}$$

where s is the Timoshenko's shear factor originally given as 1.2, and h and l are the depth and span of the beam, respectively. Suffix t represents that these elastic moduli are obtained from the Timoshenko's theory. By varying the value of h/l, the Young's modulus and the shear modulus can be obtained by the Timoshenko's bending theory. However, the obtained shear modulus was extremely small, and in the previous work, the original equation was modified as follows:[1]

$$\frac{1}{E_s}=\frac{1}{E_p}+\frac{s'}{G_p}\cdot\left(\frac{h}{l}\right)^2, \tag{3}$$

where s' depends on the depth/span ratio h/l as:

$$s'=1.2+\alpha\frac{h}{l}, \tag{4}$$

and suffix p represents that the moduli are derived by the modified equation. In the work, the value of 35 was derived to α as a rough approximation. To obtain the accurate moduli, however, α should be derived with considering the various factors including the mechanical properties of the specimen themselves and the experimental conditions such as the loading nose used and the measured point of deflection.

Experiment

Specimens

Akamatsu (Japanese red pine, *Pinus densiflora* D. Don) and balsa (*Ochroma lagopus* Sw.) were used for the specimens. Specimens were conditioned at 20°C and 65% relative humidity before and during the tests.

Static bending tests

Beam specimens were cut with the dimensions of 500 mm (longitudinal direction) × 30 mm (radial direction) × 10, 20, and 30 mm (tangential direction). Six specimens were used for one test condition.

Specimen was supported by the spans varied as 60, 80, 130,180, 230, 280, 330, 380, 430, and 480 mm at the interval of 50 mm, and the vertical load whose velocity was 5 mm/min was applied to the center of the longitudinal-radial (LR) surface with a loading head whose radius was 15 mm. The deflection was measured from the crosshead movement of the test machine. The load-deflection diagram was recorded by a X-Y recorder.

From the linear segment of load-deflection diagram, the value of E_s corresponding to each h/l was obtained. The Young's moduli, E_t and E_p, and shear moduli, G_t and G_p, were calculated by regressing the $1/E_s-h/l$ relationships into Eqs. (2) and (3), respectively.

Flexural vibration tests

For examining the Young's moduli and shear moduli obtained by our proposal, these moduli were independently measured by the free-free flexural vibration tests.

The test beam was suspended by two threads at the nodal positions of the free-free vibration corresponding to its resonance mode. Specimen was excited in the direction of the thickness at one end by a hammer. The resonance frequencies whose mode was from 1st to 4th were measured by the FFT (fast frequency transform) digital signal analyzer, and the Young's modulus E_v and shear modulus G_v were obtained from the Timoshenko-Goens-Hearmon method whose detail was described in several previous papers.[3,4]

Results and discussion

Table 1 shows the comparisons of the Young's moduli and shear moduli obtained by the flexural vibration tests and the static bending tests. In the modified equation the value of α is 35. As summarized in the previous paper, both of the Young's modulus and shear modulus are obtained by regressing the $1/E_s-h/l$ relationships into the modified equation properly. On the contrary, the original Timoshenko's theory is not effective in obtaining the shear modulus. The shear modulus derived by the vibration tests, G_v, were substituted into the original Timoshenko's equation and our proposal again, and the Young's moduli corresponding to the depth/span ratio, E_t' and E_p', were calculated. Similarly, the shear moduli corresponding to the depth/span ratio, G_t' and G_p', were obtained by substituting the Young's modulus E_v into Eqs.

Table 1 Young's moduli and shear moduli calculated by the vibration tests and the static bending tests (unit: GPa).

Species	Vibration test		Timoshenko's theory		Modified equation	
	E_v	G_v	E_t	G_t	E_p	G_p
Akamatsu	16.5	1.30	16.3	0.13	16.0	1.54
Balsa	3.18	0.20	2.90	0.05	2.84	0.17

Note: E and G represent the Young's modulus and shear modulus, respectively. Suffixes v, t, and p represent the vibration tests, original Timoshenko's bending theory, and modified equation respectively. Coefficient $\alpha=35$ in our proposal.

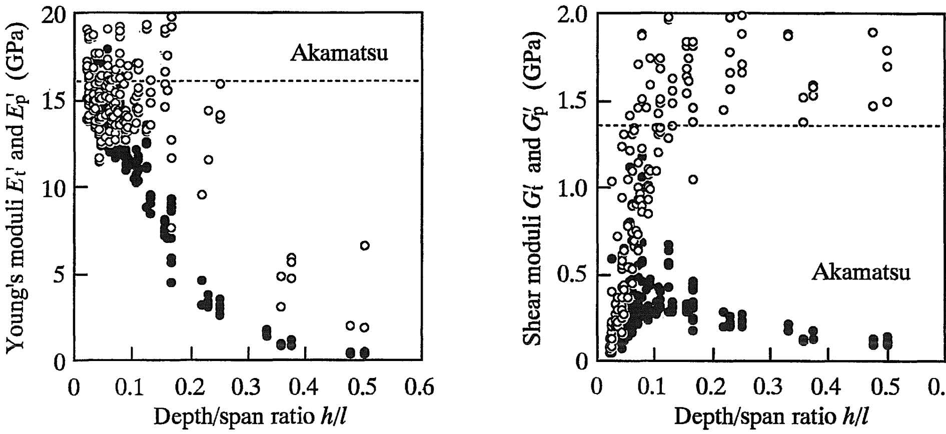

Fig. 1. Comparisons of the Young's moduli and the shear moduli obtained from original Timoshenko's theory and modified equation corresponding to the depth/span ratios.
Legend: Solid and blank circles are obrained from Timoshenko's theory and our proposal, respectively, and dashed lines represent the Young's and shear moduli obtained by the flexural vibration tests.
Note: Suffixes t and p represent original Timoshenko's bending theory and modified equation, respectively.

(2) and (3). Figure 1 shows the dependence of the values of E_t', E_p', G_t', and G_p' on the depth/span ratio, h/l. It is essential that these moduli should not depend on the depth/span ratio. Theoretically, these moduli should be constant values independently of the depth/span ratios. As for the Young's moduli, the values calculated by the original Timoshenko's theory are constant only in the range of $h/l<0.05$, whereas those derived by the modified equation are constant where $h/l<0.1$. On the other hand, the shear moduli calculated by the Timoshenko's theory decrease with the increasing the depth/span ratio, whereas those of the modified equation become constant where $h/l>0.2$. Additionally, the shear modulus obtained by the Timoshenko's theory is extremely smaller than that given by the vibration tests. From these comparisons, hence, the modified equation is superior to original Timoshenko's theory.

In the static bending tests, accurate measurement is often disturbed by several factors such as inhomogeneities of the specimens and inaccurate loading and support conditions. Here, we examined the influence of the measurement error on the accuracy of these moduli as the same manner applied on the examination of the error in the lateral vibration tests made by Kubojima and colleagues.[3,4] We supposed that the substituted shear modulus contained the measurement error of $\pm 30\%$, and the values of $0.7G_v$ and $1.3G_v$ were substituted into G_v of Eq. (3). The Young's moduli obtained by these calculations were defined as E_{p+}' and E_{p-}'. By the similar procedure, the shear moduli G_{p+}' and G_{p-}' were obtained by substituting $0.7E_v$ and $1.3E_v$, respectively, into E_p of Eq. (3). Figure 2 shows the dependence of these moduli on the depth/span ratio for the akamatsu data. The Young's modulus is not influenced by the value of shear

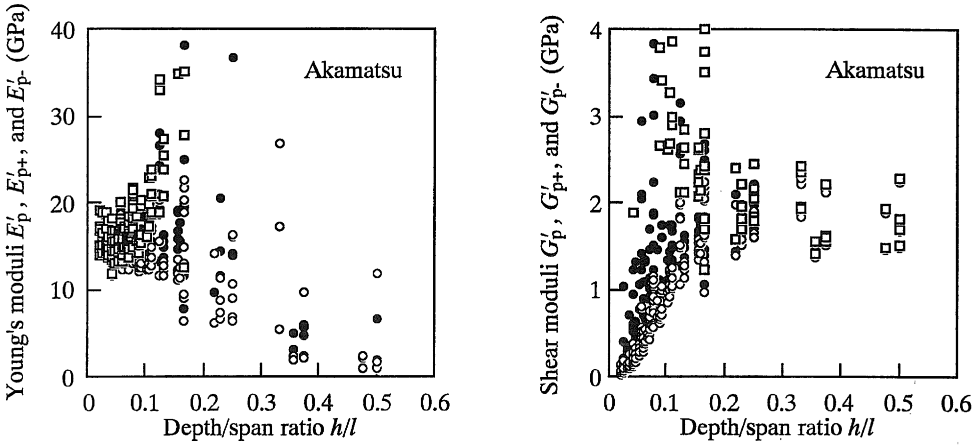

Fig. 2. Young's moduli E_p', E_{p+}', and E_{p-}' obtained by substituting G_v, $1.3G_v$ and $0.7G_v$, respectively, and shear moduli G_p', G_{p+}', and G_{p-}' obtained by substituting E_v, $1.3E_v$, and $0.7E_v$, respectively, into modified equation.
Legend: Black dotts represent E_p' and G_p', white dotts represent E_{p+}' and G_{p+}' and white rectangles represent E_{p-}' and G_{p-}'.

modulus seriously where the depth/span ratio is smaller than 0.05. In this depth/span ratio range, the deflection caused by the shearing force can be neglected. On the other hand, the shear modulus is not influenced by the Young's modulus where the depth/span ratio is larger than 0.2 because of the small effect of bending deflection. When the bending tests were made out of these proper depth/span ratio ranges, either of these moduli would be measured wrongly. To obtain both of the Young's modulus and the shear modulus by the regressing calculations, therefore, we thought that the specimens in wide depth/span ratio range should be used as far as possible, and at least, the depth/span ratio ranges mentioned above should not be reduced.

As mentioned above, the value of α should be examined for obtaining the precise shear modulus. The values of E_v and G_v were substituted into E_p and G_p of Eq. (3), respectively, and α was calculated by the method of least squares. The obtained values of a were 25 for akamatsu and 45 for balsa. We put 35 into α as a rough approximation. For obtaining the shear modulus precisely, however, α should be determined carefully by considering the various factors including the mechanical properties of the specimen themselves and the experimental conditions such as the loading nose used and the measured point of deflection.

To obtain the Young's modulus only, bending test of the specimen with a small depth/span ratio is often made. Similarly, we thought that the shear modulus can be obtained by the bending test of the specimen with a large depth/span ratio. When the depth/span ratio is small enough, the second term of Eq. (3) can be neglected, and the Young's modulus E_p approaches to the apparent Young's modulus E_s obtained by the elementary bending theory. Thus, the

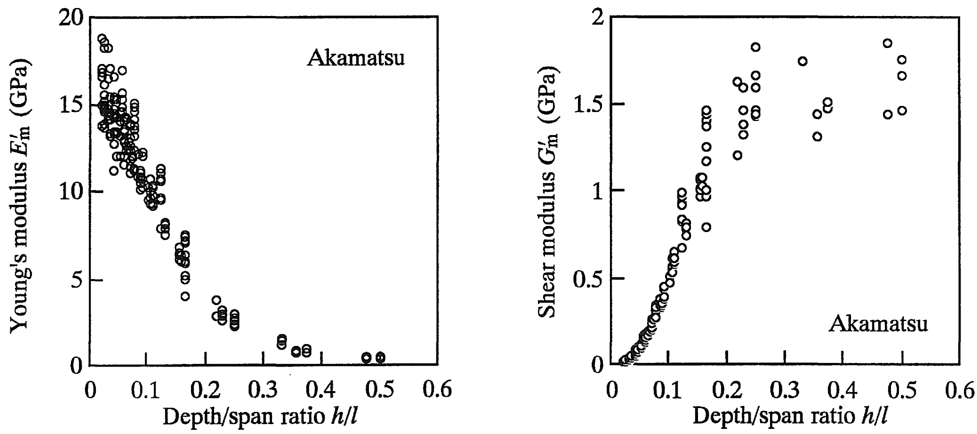

Fig. 3. Young's modulus E_m' and shear modulus G_m' calculated by neglecting the second and first terms in our proposal, respectively.

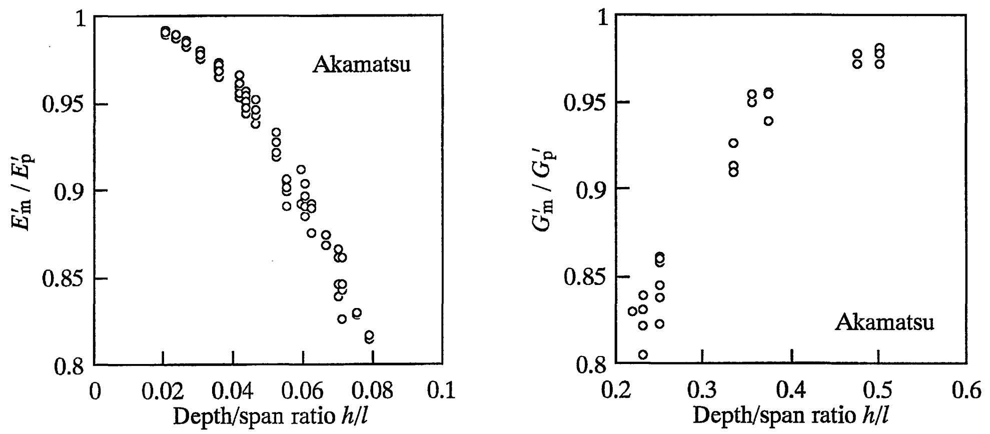

Fig. 4. Dependence of the values E_m'/E_p' and G_m'/G_p' on the depth/span ratio.
Note: E_p' and G_p': See Figs. 1 and 2, E_m' and G_m': See Fig. 3.

Young's modulus in the small depth/span ratio range, E_m', is derived by the similar form to that of the elementary bending theory as:

$$E_m' = E_s. \tag{5}$$

Inversely, the first term of Eq. (3) may be neglected where the depth/span ratio is large, and the shear modulus G_m' is approximated as follows:

Table 2 Young's moduli obtained by averaging the approximated moduli E'_m in the depth/span ratio range smaller than 0.04, and shear moduli obtained by averaging the approximated ones G'_m in the depth/span ratio range larger than 0.3 (unit: GPa).

Species	E_m	G_m
Akamatsu	15.5	1.52
Balsa	2.70	0.16

Notes: E'_m and G'_m: See Figs. 3 and 4.

$$G_m' = s'E_s\left(\frac{h}{l}\right)^2. \tag{6}$$

Figure 3 shows the dependence of the Young's modulus and shear modulus obtained by Eqs. (5) and (6), respectively, on the depth/span ratios, and the values of E_m'/E_p' and G_m'/G_p' corresponding to the depth/span ratios are shown in Figure 4. As mentioned before, the value of E_p' is constant where $h/l<0.05$, where as is constant where $h/l>0.2$. From Fig. 4, however, E_m' and G_m' are evaluated as about 90% of E_p' at $h/l=0.05$ and about 80% of G_p' at $h/l=0.2$, respectively. We thought that the approximations by Eqs. (5) and (6) are effective when the E_m' and G_m' are evaluated as about 95% of E_p' and G_p', respectively, and the ranges of depth/span ratios satisfying these demands are $h/l<0.04$ for the Young's modulus and $h/l>0.3$ for the shear modulus. Table 2 shows the Young's modulus E_m averaged by the values of E_m' in $h/l<0.04$, and the shear modulus G_m averaged by the values of G_m' in $h/l>0.3$. When the bending test is made with the specimen in these depth/span ratios it is considered that the either modulus can be obtained without the regressing calculation.

Conclusion

Here the proper range of depth/span ratio which gives the Young's modulus and shear modulus in the three-point static bending test was examined, and the following results were obtained:

(1) The applicable depth/span ratio range obtained from our proposal is wider than that given by Timoshenko's theory. However, the Young's modulus cannot be given when the depth/span ratio is large, whereas the shear modulus cannot be derived when the depth/span ratio is small. Hence, the range of depth/span ratio should be wide in obtaining the both values of Young's modulus and shear modulus by the proposal properly.

(2) The accuracy of shear modulus obtained from previously made the proposal is dependent on the value of the coefficient used for the modification of original Timoshenko's bending theory. For obtaining the proper shear modulus, this coefficient should be determined carefully by considering the various factors including the mechanical properties of the specimen themselves and the experimental conditions such as the loading nose used and the measured point of deflection.

(3) It was thought that the shear modulus can be derived from the bending test of one

specimen as well as the Young's modulus. Nevertheless, the feasibility is restricted by the depth/span ratio.

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