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The Extended Bukhvostov-Lipatov Model in the Path Integral Framework

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Abstract

In the path-integral framework we study a massive Thirring-like model in 2-dimensional space-time, which contains fermions with arbitrary number (N) of different species. This model is an extension of that of a previous paper (the Bukhvostov-Lipatov model), where we have considered two-species case. By this extension we expect that we can expose more general structures of this kind of model. We obtain the equivalent boson model with N species to our fermion model by mass perturbation expansion. We see that the boson Lagrangian corresponds with the one which is directly obtained from the original fermion Lagrangian by Coleman's correspondences of free fermion bilinear operators with boson fields.

§1. Introduction

In previous papers we have studied relativistic bound states of a 1-space quantum mechanical system containing different species of massive fermions in order to investigate the relativistic effects for such composite systems.[1)] This model is an extended one from the model of two kinds of fermions originally proposed by Glöckle, Nogami and Fukui (GNF).[2)] The Hamiltonian of this model is given by

$$H=\sum_{i=1}^{n}\{-i\alpha_i P_i+m\beta_i\}-\frac{g}{2}\sum_{i\neq j}^{n}(1-\alpha_i\alpha_j)\delta(x_i-x_j), \tag{1.1}$$

where i and j denote fermion species. It is essential in this model that different fermions interact with each other through the δ-function potentials, while fermions of the same kind do not interact with each other directly. All the requirements of quantum mechanics and special relativity are satisfied. We found an exact solution for n-body bound state which contains n different particles.

The GNF model is, however, based on the single-electron theory, where anti-particles are not supposed to exist, and necessarily its Hamiltonian $(1\cdot 1)$ is not positive-definite. One way to overcome this defect would be to go into field theory. It is seen that the GNF model can be derived from a massive Thirring-like model in 2-dimensional space-time, which we will give in the

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next section. This model for the case of two-species is known as Bukhvostov-Lipatov (BL) model[3] in the context of the bosonization.

The bosonization technique is one of the powerful approaches to study a 2-dimensional fermion system.[4] As Coleman has done in his pioneering work,[5] with this technique one can expose hidden properties of such a fermion system though it may be applicable only for charge-zero sectors of fermions, i.e. for sectors of pairs of fermion and anti-fermion. In a previous paper[6] (I) using the path-integral method we have studied the bosonization of the BL model with two-species fermions.

In this article we consider the bosonization of the N-species fermion model, which is a direct extension of the BL model, in the path-integral framework. It is an easier but non-trivial extension of the freedom of the bosonization technique. By this extension we expect that we can expose more general structures of this model. As is shown below, BL model has a paradoxical property that by a chiral transformation the fermion fields seem to get free and to have no interactions among different species. So we should calculate directly the generating functional to check the Coleman's bosonization correspondences for this model.

We use the same notations as in I, i.e. in Minkowski space-time, $g_{\mu\nu}=(-1,\ +1)$ and $\varepsilon^{01}=-\varepsilon_{01}=1$. Gamma-matrices are given as $\gamma^0=-\gamma_0=i\sigma_x$, $\gamma^1=\gamma_1=\sigma_y$, $\gamma_5=\gamma_0\gamma_1=\sigma_z$, where σ_x, σ_y and σ_z are the Pauli matrices.

§2. Model

Our initial Lagrangian is given by

$$\mathscr{L}=\sum_{i=1}^{N}\bar{\psi}_i(\not{\partial}-m)\psi_i+\sum_{i>j=1}^{N}\frac{1}{2}gj_{i\mu}j_j{}^{\mu}, \tag{2.1}$$

where i, j denote the fermion species and vector current $j_{i\mu}$ is given by

$$j_{i\mu}=i\bar{\psi}_i\gamma_\mu\psi_i. \tag{2.2}$$

In $(2\cdot1)$ a fermion interacts with those of the different species and never with itself directly. When $N=2$, we can change the sign of g by taking charge conjugation of one of the fermion species. In fact we saw in I that the consequences for $N=2$ are symmetric for $g\leftrightarrow -g$. On the contrary, for the case $N\geq 3$, there is not such a symmetry, and therefore the sign of g has a physical meaning for this case.

The quartic interaction part in $(2\cdot1)$ is rewritten as

$$\sum_{i>j=1}^{N}\frac{1}{2}gj_{i\mu}j_j{}^{\mu}=\frac{g}{4}\left(\sum_i j_{i\mu}\right)^2-\frac{g}{4}\sum_i j_{i\mu}j_i{}^{\mu}, \tag{2.3}$$

and this is equivalent to

$$\frac{g}{2}\sum_i j_{i\mu}X^\mu-\frac{g}{4}X_\mu X^\mu+\frac{g}{2}\sum_i\left(\frac{1}{2}A_{i\mu}-j_{i\mu}\right)A_i{}^{\mu}, \tag{2.4}$$

with using auxiliary vector fields X_μ and $A_{i\mu}$. Now we put

$$B_{i\mu}=X_\mu-A_{i\mu}, \tag{2.5}$$

and integrate out over X_μ by the path-integral formulation to obtain

$$\frac{g}{2}\sum_i j_{i\mu}B_i{}^\mu-\frac{g}{4(N-1)}\left(\sum_i B_{i\mu}\right)^2+\frac{g}{4}\sum_i (B_{i\mu})^2. \tag{2.6}$$

Here we should note that for $N\geq 3$ the above expression contains direct coupling among the same species of boson fields $B_{i\mu}$ while fermion fields do not interact directly with themselves in the original Lagrangian (2·1).

In 2-dimensional space-time, we can write vector fields $B_{i\mu}$ with two scalar fields ϕ_i and χ_i as

$$B_{i\mu}=\varepsilon_{\mu\nu}\partial^\nu\phi_i+\partial_\mu\chi_i. \tag{2.7}$$

With these scalar fields we transform the fermion fields as

$$\psi_i \longrightarrow \psi_i{}'=\exp\left\{\frac{ig}{2}(-\gamma_5\phi_i+\chi_i)\right\}\psi_i, \tag{2.8}$$

to rewrite the Lagrangian as

$$\mathscr{L}_{\text{eff}}=\sum_i \bar{\psi}_i{}'(\partial\!\!\!/-m\exp\{ig\gamma_5\phi_i\})\psi_i{}'-\frac{g}{4(N-1)}\left\{\left(\sum_i\partial_\mu\chi_i\right)^2-\left(\sum_i\partial_\mu\phi_i\right)^2\right\}$$
$$+\frac{g}{4}\sum_i(\partial_\mu\chi_i)^2+\left(\frac{g^2}{8\pi}-\frac{g}{4}\right)\sum_i(\partial_\mu\phi_i)^2. \tag{2·9}$$

In the above expression the term $(g^2/8\pi)(\partial\phi)^2$ comes from $\det|\exp(-ig\gamma_5\phi_i)|$ in the path-integral measure following Fujikawa.[7] In I we have missed a factor 2 for this term. It is seen that χ_i is decoupled from the other fields and can be integrated out. Then we obtain

$$\mathscr{L}_{\text{eff}}=\sum_i \bar{\psi}_i(\partial\!\!\!/-m\exp\{ig\gamma_5\phi_i\})\psi_i+\frac{g}{4(N-1)}\left(\sum_i\partial_\mu\phi_i\right)^2+\frac{g}{4}\left(\frac{g}{2\pi}-1\right)\sum_i(\partial_\mu\phi_i)^2, \tag{2·10}$$

where we write ψ_i for $\psi_i{}'$. As mentioned in the previous section, one should note that the coupling between ψ_i and ϕ_i disappears when $m=0$, and the fermion fields become free. Then it seems that the connected part of the corelation $\langle j_{a\mu}j_{b\nu}\rangle$, for an example, vanishes. This is, however, not the case. To calculate such a correlation we should use the original fermion fields and take the splitting technique to $j_{a\mu}$'s. We will discuss this point in the last section.

The generating functional of the Green functions is given by

$$Z=\int\prod_i d\bar{\psi}_i\,d\psi_i\,d\phi_i\exp\left\{i\int d^2x\mathscr{L}_{\text{eff}}\right\}. \tag{2·11}$$

§3. Perturbative expansion

In order to calculate (2·11) by perturbation theory we choose $\mathscr{L}_0=\mathscr{L}_{\text{eff}}(m=0)$ of (2·10) as the free Lagrangian. We find that the free fermion and boson propagators are given as

$$\langle\bar{\psi}_i(x)\psi_j(y)\rangle=\frac{\delta_{i,j}}{2\pi}\frac{\gamma\cdot(x-y)}{(x-y)^2}, \tag{3·1}$$

$$\langle\phi_i(x)\phi_j(y)\rangle=-\frac{1}{g(g-2\pi)}\left\{\frac{2\pi}{2\pi+g(N-1)}-\delta_{i,j}\right\}\ln(x-y)^2\mu^2, \tag{3·2}$$

where $\langle\cdots\rangle$ denotes the vacuum expectation value of the time-ordered product. Parameter μ is a small infrared cutoff mass, which will be set to zero after the calculations.

Now, we calculate Z of (2·11) through the perturbative expansion with respect to m. We note that all the odd order terms vanish because of traceless property of γ-matrices and the super selection rule for the boson field ϕ, i.e. $\langle\exp i\Sigma\beta_i\phi\rangle=0$ unless $\Sigma\beta_i=0$ where $\beta_i=\pm g$.[4),5)] The $2n$-th order term of the expansion of Z is given as

$$Z^{(2n)}=\frac{(im)^{2n}}{(2n)!}\int\prod d\bar{\psi}_i\,d\psi_i\,d\phi_i\left\{\int dx\sum_{i=1}^{N}\bar{\psi}_i\,e^{ig\gamma_5\phi_i}\psi_i\right\}^{2n}\exp\left\{i\int dx\,\mathscr{L}_0\right\}$$
$$=\frac{(im)^{2n}}{(2n)!}\left\langle\left\{\int dx\sum_{i=1}^{N}\bar{\psi}_i\,e^{ig\gamma_5\phi_i}\psi_i\right\}^{2n}\right\rangle. \tag{3·3}$$

Using the identity

$$\bar{\psi}\,e^{ig\gamma_5\phi}\psi=e^{ig\phi}\bar{\psi}\Gamma_+\psi+e^{-ig\phi}\bar{\psi}\Gamma_-\psi, \tag{3·4}$$

where we put $\Gamma_\pm=(1\pm\gamma_5)/2$, we expand (3·3) as

$$Z^{(2n)}=(im)^{2n}\sum_{r_1,r_2,\cdots,r_N}^{n}\frac{\delta_{n,r_1+r_2+\cdots+r_N}}{(2r_1)!(2r_2)!\cdots(2r_N)!}\left\langle\left\{\int dx(e^{ig\phi_1}\bar{\psi}_1\Gamma_+\psi_1+e^{-ig\phi_1}\bar{\psi}_1\Gamma_-\psi_1)\right\}^{2r_1}\right.$$
$$\times\left\{\int dx(e^{ig\phi_2}\bar{\psi}_2\Gamma_+\psi_2+e^{-ig\phi_2}\bar{\psi}_2\Gamma_-\psi_2)\right\}^{2r_2}\times\cdots\cdots$$
$$\left.\times\left\{\int dx(e^{ig\phi_N}\bar{\psi}_N\Gamma_+\psi_N+e^{-ig\phi_N}\bar{\psi}_N\Gamma_-\psi_N)\right\}^{2r_N}\right\rangle$$
$$=(im)^{2n}\sum_{r_1,r_2,\cdots,r_N}^{n}\frac{\delta_{n,r_1+r_2+\cdots+r_N}}{(r_1!)^2(r_2!)^2\cdots(r_N!)^2}\int\prod dx\,dy$$
$$\times\left\langle\prod_{i=1}^{r_1}\bar{\psi}_1(x_{1,i})\Gamma_+\psi_1(x_{1,i})\bar{\psi}_1(y_{1,i})\Gamma_-\psi_1(y_{1,i})\right\rangle$$
$$\times\left\langle\prod_{i=1}^{r_2}\bar{\psi}_2(x_{2,i})\Gamma_+\psi_2(x_{2,i})\bar{\psi}_2(y_{2,i})\Gamma_-\psi_2(y_{2,i})\right\rangle\times\cdots\cdots$$
$$\times\left\langle\exp ig\left[\sum_{i=1}^{r_1}\{\phi_1(x_{1,i})-\phi_1(y_{1,i})\}+\sum_{i=1}^{r_2}\{\phi_2(x_{2,i}-\phi_2(y_{2,i}))\}\right.\right.$$
$$\left.\left.+\cdots+\sum_{i=1}^{r_N}\{\phi_N(x_{N,i})-\phi_N(y_{N,i})\}\right]\right\rangle. \tag{3·5}$$

With the fermion propagator (3·1) we can calculate a fermion part in the above (3·5) as

$$\left\langle\prod_{i=1}^{r_k}\bar{\psi}_k(x_{k,i})\Gamma_+\psi_k(x_{k,i})\bar{\psi}_k(y_{k,i})\Gamma_-\psi_k(y_{k,i})\right\rangle=\frac{1}{(2\pi)^{2r_k}}\frac{\prod_{i>j}^{r_k}(x_{k,i}-x_{k,j})^2(y_{k,i}-y_{k,j})^2}{\prod_{i,j}^{r_k}(x_{k,i}-y_{k,j})^2}, \tag{3·6}$$

and we obtain

$$\begin{aligned}Z^{(2n)}=&\left(\frac{im}{2\pi}\right)^{2n}\sum_{r_1,r_2,\cdots,r_N}^{n}\frac{\delta_{n,r_1+r_2+\cdots+r_N}}{(r_1!)^2(r_2!)^2\cdots(r_N!)^2}\int\prod dx\,dy\\&\times\frac{\prod_{i>j}^{r_1}(x_{1,i}-x_{1,j})^2(y_{1,i}-y_{1,j})^2}{\prod_{i,j}^{r_1}(x_{1,i}-y_{1,j})^2}\times\frac{\prod_{i>j}^{r_2}(x_{2,i}-x_{2,j})^2(y_{2,i}-y_{2,j})^2}{\prod_{i,j}^{r_2}(x_{2,i}-y_{2,j})^2}\\&\times\cdots\times\frac{\prod_{i>j}^{r_N}(x_{N,i}-x_{N,j})^2(y_{N,i}-y_{N,j})^2}{\prod_{i,j}^{r_N}(x_{N,i}-y_{N,j})^2}\\&\times\left\langle\exp ig\left[\sum_{i=1}^{r_1}\{\phi_1(x_{1,i})-\phi_1(y_{1,i})\}+\sum_{i=1}^{r_2}\{\phi_2(x_{2,i})-\phi_2(y_{2,i})\}+\cdots\right]\right\rangle.\end{aligned} \tag{3·7}$$

Now, let us calculate the contractions among the boson fields with the same species in the above expression to obtain

$$\begin{aligned}&\left\langle\exp ig\left[\sum_{i=1}^{r_k}\{\phi_k(x_{k,i})-\phi_k(y_{k,i})\}\right]\right\rangle=\exp\left\{-\frac{g^2r_k}{2}\langle\phi_k(0)^2\rangle\right\}\\&\quad\times\exp-g^2\left[\sum_{i>j}^{r_k}\{\langle\phi_k(x_{k,i})\phi_k(x_{k,j})\rangle+\langle\phi_k(y_{k,i})\phi_k(y_{k,j})\rangle\}-\sum_{i,j}^{r_k}\langle\phi_k(x_{k,i})\phi_k(y_{k,j})\rangle\right]\\&=\exp\left\{-\frac{g^2r_k}{2}\langle\phi_k(0)^2\rangle\right\}\\&\quad\times\exp\left\{\frac{g^2(N-1)}{(g-2\pi)\{2\pi+g(N-1)\}}\ln\left(\frac{\prod_{i>j}^{r_k}(x_{k,i}-x_{k,j})^2\mu^2(y_{k,i}-y_{k,j})^2\mu^2}{\prod_{i,j}^{r_k}(x_{k,i}-y_{k,j})^2\mu^2}\right)\right\},\end{aligned} \tag{3·8}$$

where we have used (3·2). Here we suppose that for a small distance the boson propagator is properly regularized. We find that each fermion part in (3·7) has the same combination of the coordinates $\{x_{k,i}, y_{k,i}\}$ as that in the above expression.

We, therefore, suppose that in the corresponding boson model the free boson propagator gives such a fermion contribution besides the original boson one (3·2) as

$$\langle\phi_i'(x)\phi_j'(y)\rangle=\langle\phi_i(x)\phi_j(y)\rangle-\frac{\delta_{i,j}}{g^2}\ln(x-y)^2\mu^2$$
$$=\frac{1}{g(g-2\pi)}\left\{\frac{2\pi}{2\pi+g(N-1)}-\frac{2\pi}{g}\delta_{i,j}\right\}\ln(x-y)^2\mu^2, \tag{3·9}$$

and that potential term is given by

$$m'^2\{\cos g\phi_1'+\cos g\phi_2'+\cdots+\cos g\phi_N'\}. \tag{3·10}$$

The $2n$-th order term of the perturbative expansion of the generating functional for such a model is calculated as

$$Z_B^{(2n)}=\frac{(im'^2)^{2n}}{(2n)!}\left\langle\left\{\int dx(\cos y\phi_1'+\cos g\phi_2'+\cdots+\cos g\phi_N')\right\}^{2n}\right\rangle$$
$$=(im'^2)^{2n}\sum_{r_1,r_2,\cdots,r_N}^{n}\frac{\delta_{n,r_1+r_2+\cdots+r_N}}{(2r_1)!(2r_2)!\cdots(2r_N)!}$$
$$\times\left\langle\left(\int dx\cos g\phi_1'\right)^{2r_1}\left(\int dx\cos g\phi_2'\right)^{2r_2}\cdots\left(\int dx\cos g\phi_N'\right)^{2r_N}\right\rangle$$
$$=\left(\frac{im'^2}{2}\right)^{2n}\sum_{r_1,r_2,\cdots,r_N}^{n}\frac{\delta_{n,r_1+r_2+\cdots+r_N}}{(r_1!)^2(r_2!)^2\cdots(r_N!)^2}\int\prod dx\,dy$$
$$\times\left\langle\exp ig\left[\sum_i^{r_1}\{\phi_1'(x_{1,i})-\phi_1'(y_{1,i})\}+\cdots+\sum_i^{r_N}\{\phi_N'(x_{N,i})-\phi_N'(y_{N,i})\}\right]\right\rangle.$$
$$=\left(\frac{im'^2}{2}\right)^{2n}\sum_{r_1,\cdots,r_N}^{n}\frac{\delta_{n,r_1+\cdots+r_N}}{(r_1!)^2\cdots(r_N!)^2}\int\prod dx\,dy$$
$$\times\frac{\prod_{i>j}^{r_1}(x_{1,i}-x_{1,j})^2\mu^2(y_{1,i}-y_{1,j})^2\mu^2}{\prod_{i,j}^{r_1}(x_{1,i}-y_{1,j})^2\mu^2}\times\cdots$$
$$\times\left\langle\exp ig\left[\sum_{i=1}^{r_1}\{\phi_1(x_{1,i})-\phi_1(y_{1,i})\}+\cdots+\sum_{i=1}^{r_N}\{\phi_N(x_{N,i})-\phi_N(y_{N,i})\}\right]\right\rangle$$
$$\times\exp ng^2\{\langle\phi_i(0)^2\rangle-\langle\phi_i'(0)^2\rangle\}. \tag{3·11}$$

If we choose the renormalization of mass as

$$m'^2=\frac{m\mu}{\pi}\exp\frac{g^2}{2}\{\langle\phi_i'^2\rangle-\langle\phi_i^2\rangle\}, \tag{3·12}$$

then Eq. (3·11) is equal to (3·7). From the free boson propagator (3·9) and the potential term (3·10) we find that the boson model with the Lagrangian

$$\mathscr{L}_B=\frac{g^2}{16\pi^2}\sum_{i,j}[-g+(g-2\pi)\delta_{i,j}]\partial_\mu\phi_i'\partial^\mu\phi_j'+m'^2\sum_i\cos g\phi_i'$$
$$=-\frac{g^2}{8\pi^2}\left\{\pi\sum(\partial\phi_i')^2+g\sum_{i\neq j}\partial\phi_i'\partial\phi_j'\right\}+m'^2\sum\cos g\phi_i' \tag{3·13}$$

is equivalent to the fermion model of (2·1). We rescale the boson fields as

$$g\phi_i'=\sqrt{4\pi}\phi_i'', \tag{3·14}$$

to obtain

$$\mathscr{L}_B=-\frac{1}{2}\sum(\partial\phi_i'')^2-\frac{g}{2\pi}\sum_{i\neq j}\partial\phi_i''\partial\phi_j''+m'^2\cos\sqrt{4\pi}\phi_i''. \tag{3·15}$$

We see that this Lagrangian is obtainable from the original fermion Lagrangian (2·1) by the Coleman's correspondences:

$$\bar{\psi}_{fi}\partial\!\!\!/\psi_{fi}\longleftrightarrow\frac{1}{2}(\partial\phi_{fi})^2, \tag{3·16}$$

$$\bar{\psi}_{fi}\psi_{fi}\longleftrightarrow -m''\cos\sqrt{4\pi}\phi_{fi}, \tag{3·17}$$

$$i\bar{\psi}_{fi}\gamma_\mu\psi_{fi}\longleftrightarrow -\frac{1}{\sqrt{\pi}}\varepsilon_{\mu\nu}\partial^\nu\phi_{fi}, \tag{3·18}$$

where suffix f denotes free fields for $g=m=0$.

§4. Discussion

We have calculated generating functional of the fermion model with n-species by mass perturbation and seen that it is equivalent to the boson model which is the extension of sine-Gordon model. As we mentioned just before, the boson Lagrangian is just obtainable from the fermion Lagrangian by the Coleman's correspondences (3·16)~(3·18) for the free massless fields. This seems to show that these correspondences still hold in the Heisenberg picture i.e. for $g\neq 0\neq m$. We have pointed out, however, that the fermion interactions among different species all vanish by the chiral transformation (2·8). Then $\langle j_{1\mu}(x)j_{2\nu}(y)\rangle$ does not seem to be equal to $\langle\varepsilon_{\mu\rho}\partial^\rho\phi_1(x)\,\varepsilon_{\nu\sigma}\partial^\sigma\phi_2(y)\rangle$ for $g\neq 0$. In the operator formalism, to calculate the former correlation we must use the original fermion fields instead of transformed ones and split the field products. We can see by such a splitting process that $j_{i\mu}$ get the explicit dependence on ϕ_i. It is an interesting subject, therefore, how to derive the correspondences (3·16)~(3·18) in the path-integral formulation.

In our path-integral formulation the massive Thirring-like model is described as the system of massless fermions interacting with massless bosons where the interaction part contains mass parameter. While the bosonization technique may be applicable only for the charge-zero sectors, it is an interesting subject to study the charged sectors, i.e. the bound states of some particles in our formulation.

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