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A Solvable Composite Model in the Ladder Approximation and Its Implications about the Approximation

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Abstract

We make a study to assess the accuracy of the ladder approximation, by employing a solvable two-body positron-theoretical composite model which is defined in terms of the Bethe-Salpeter equation in the ladder approximation with various Fermi-type direct instantaneous interactions in $1+1$ dimensional world. In this model, we obtain two sets of even- and odd-parity bound-state solutions. One set is obtained with a kind of renormalization of the coupling constant, and the other is gotten without any renormalization. We examine the wave functions of all the obtained bound-state solutions in detail, as they allow the probabilistic interpretation. Judging from some considerations, we have the following consequences: (A) the wave function of the odd-parity bound-state solution obtained with a renormalization includes a remarkable and undesired feature (against a natural requirement) in the cases where the binding energy is large. This remarkable and undesired feature is owing to the inadeqeacy of the ladder approximation. (B) As for the remaining three bound-state solutions, qualitative features of their wave functions are acceptable. We suppose and emphasize that the ladder approximation (in the real world) may yield even undesired features against the physical requirements concerned with the wave functions (including the $3+1$–dimensional generalization of a natural requirement stated above), at least in the case where the coupling constants of short-range interactions are renormalized.

§1. Introduction

Glöckle, Nogami and Fukui[1)] (GNF) have presented an analytically solvable model of composite system, which satisfies all the requirements of quantum mechanics and special relativity, including the Lorents contraction of the composite system. The GNF model is defined in terms of the two-body Dirac equation in $1+1$ dimensional space-time with a direct instantaneous interaction. The GNF equation is not manifestly covariant as it refers to a specific Lorentz frame, but its relativistic covariance is guaranteed by the existence of the Lorentz boost operator. Unfortunately, that model is based on the single electron theory instead of the positron theory.

Munakata, Nagamura and the author[2)] have argued the problem of rewriting the GNF model in a manifestly covariant way and in a positron-theoretical treatment. They have shown that the GNF model is rewritten in a manifestyle covariant form of the Bethe-Salpeter equation[3] in the ladder approximation with the Fermi-type interaction, provided that the single-electron-theoretical treatment is adopted. They have found that when the positron-theoretical treatment is adopted, another problem due to a divergence occurs.

Recently, the author[4] has found an invariant manner (that is, a kind of renormalization of the coupling constant) to deal with the above problem due to a divergence and obtained an exactly solvable positron-theoretical composite model by taking account of the conjecture about the above problem (due to a divergence) presented by Glöckle, Nogami and Toyama[5] (GNT). (The work of Ref. 4) is referred to as I hereafter.) The conjecture by GNT is that even if the coupling constant is small, the instantaneous contact interaction in the form of the space-time delta-function is very strong.*)

Solvable models are, in general, useful in some aspects. As for the composite model (in 1+1 dimensions) obtained in I, its bound-state solution has provided a prescription for modifying a form factor from a static model (in the real world) to include the Lorentz contraction effect, as seen in I, by the help of the fact that the Lorentz contraction occurs only in the direction of the motion of the bound state.**) The model on I may serve also for other aspects. About the accuracy of the ladder approximation, the uneasiness has been felt.[6] It is necessary to examine and assess not only eigenvalues but also wave functions resulting from the ladder approximation.

In this paper, we employ the composite model in I to make a study for assessing the accuracy of the ladder approximation. By the help of some considerations, wave functions of bound-state solutions are examined and judged.

In §2, we state the model used here. Differently from I where the vector-type Fermi interaction is adopted, various Fermi interactions are taken to allow also formations of bound-state solutions without any renormalization. In §3, we obtain two sets of even- and odd-parity bound-state solutions and examine wave functions of obtained bound-state solutions by the help of some considerations. We have some consequences. The consequences are summarized and discussed in §4.

§2. The model

We study the positron-theoretical composite systems defined in terms of the Bethe-Salpeter equation in the ladder approximation in 1+1 dimensional world

$$\psi(x_1, x_2)=\int S_F{}^a(x_1-x_3)S_F{}^b(x_2-x_4)I(x_3, x_4)\psi(x_3, x_4)d^2x_3d^2x_4, \tag{1}$$

where the interaction between particles a and b is taken to be the sum of all the Fermi-type

*) The GNT conjecture has been presented by examining how the solutions of the Breit equation and the Salpeter equation behave as the interaction range is varied. As for the relativistic covariance of the equations with a finite-range interaction used by GNT, it is not guaranteed.

**) This prescription is provided by comparing the form factor calculated from the obtained bound-state solution with its static approximation.

direct instantaneous interactions in 1+1 dimensions

$$I(x_3, x_4) = i\delta^2(x_3 - x_4)\,(g_S + g_P \gamma_5{}^a \gamma_5{}^b + g_V \gamma_\mu{}^a \gamma^{\mu b}/2).^{*)} \tag{2}$$

For the Dirac matrices we use

$$\gamma^0 = \gamma_0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \beta\alpha = -\gamma_1 \quad \text{with} \quad \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\gamma^5 = \gamma^0\gamma^1 = \alpha$. The masses of the particles a and b are denoted by m_a and m_b respectively. The two-dimensional Feynman propagator $S_F(x)$ is defined by

$$S_F(x) = (i\not\partial + m)\Delta_F(x), \quad i\Delta_F(x) = \frac{1}{(2\pi)^2}\int \frac{e^{-ikx}}{-k^2 + m^2 - i\delta}\, d^2k, \tag{3}$$

where $kx = k^0 t - k^1 x$ and $k^2 = (k^0)^2 - (k^1)^2$.

We introduce the c.m. and relative coordinates and momenta

$$X^\mu = \frac{m_a x_1{}^\mu + m_b x_2{}^\mu}{m_a + m_b}, \quad x^\mu = x_1{}^\mu - x_2{}^\mu, \tag{4a}$$

$$P^\mu = p_1{}^\mu + p_2{}^\mu, \quad p^\mu = \frac{m_b p_1{}^\mu - m_a p_2{}^\mu}{m_a + m_b}, \tag{4b}$$

and write $\psi(x_1, x_2)$ to be

$$\psi(x_1, x_2) = \psi(x)\, e^{-iPX} \equiv \int e^{-ipx} \phi_P(p)\, \frac{d^2p}{(2\pi)^{1/2}}\, e^{-iPX}, \tag{5}$$

separating the c.m. coordinates. Then, the Bethe–Salpeter equation for $\phi_P(p)$ in momentum space is given by

$$\phi_P(p) = -\int \frac{\not p_1 + m_a}{p_1{}^2 - m_a{}^2} \frac{\not p_2 + m_b}{p_2{}^2 - m_b{}^2} I(p-k)\phi_P(k)\, \frac{d^2k}{(2\pi)^2}, \tag{6}$$

where $p_1{}^\mu = m_a P^\mu/(m_a + m_b) + p^\mu$, $p_2{}^\mu = m_b P^\mu/(m_a + m_b) - p^\mu$. Here $I(q)$ is defined by

$$I(x_3, x_4) = \frac{1}{(2\pi)^2}\int I(q)\, e^{-iq(x_3 - x_4)}\, d^2q, \tag{7}$$

and has no q-dependence;

$$I(q) = i\{g_S + g_P \gamma_5{}^a \gamma_5{}^b + g_V \gamma_\mu{}^a \gamma^{\mu b}/2\}. \tag{8}$$

It should be noted that the denominators of the R.H.S. of Eq. (6) have appropriate small imaginary parts in accordance with Eq. (3).

Hereafter we write P^0, P^1, p^0 and p^1 tobe

$$P^0 = E, \quad P^1 = P, \quad p^0 = \varepsilon, \quad p^1 = p, \tag{9}$$

*) It is noted that as there exist only three independent Fermi-type interactions in 1+1 dimensions, the interaction types of axialvector and tensor are omitted.

and rewrite Eq. (6) as

$$\phi_P(\varepsilon, p) = -i \frac{\{m_a E/(m_a+m_b)+\varepsilon+H^a(p_1)\}\{m_b E/(m_a+m_b)-\varepsilon+H^b(p_2)\}}{[\{m_a E/(m_a+m_b)+\varepsilon\}^2-p_1^2-m_a^2][\{m_b E/(m_a+m_b)-\varepsilon\}^2-p_2^2-m_b^2]}$$
$$\times\{g_S\beta^a\beta^b+g_P\alpha^a\beta^a\alpha^b\beta^b+g_V(1-\alpha^a\alpha^b)/2\}\int \phi_P(\varepsilon', p')\frac{d\varepsilon'\, dp'}{(2\pi)^2}, \tag{10}$$

where $p_1=m_aP/(m_a+m_b)+p$, $p_2=m_bP/(m_a+m_b)-p$ and

$$H^a(p_1)=\alpha^a p_1+\beta^a m_a, \quad H^b(p_2)=\alpha^b p_2+\beta^b m_b. \tag{11}$$

We integrate both sides of Eq. (10) over ε, after we give m a small negative imaginary part. Then, this integration makes Eq. (10) become the following equation:

$$\{E-H^a(p_1)-H^b(p_2)\}\chi_P(p) = -\frac{1}{2\pi}(\Lambda_+{}^a(p_1)\Lambda_+{}^b(p_2)-\Lambda_-{}^a(p_1)\Lambda_-{}^b(p_2)\}$$
$$\times\{g_S\beta^a\beta^b+g_P\alpha^a\beta^a\alpha^b\beta^b+g_V(1-\alpha^a\alpha^b)/2\}\int_{-\infty}^{\infty}\chi_P(p')\,dp'^{*)}, \tag{12}$$

where $\Lambda_\pm{}^a(p_1)$ and $\Lambda_\pm{}^b(p_2)$ are the projection operators to the positive and negative energy states for the particles a and b respectively,

$$\Lambda_\pm{}^a(p_1)=\frac{E_1\pm H^a(p_1)}{2E_1}, \quad E_1=(m_a^2+p_1^2)^{1/2}, \tag{13a}$$
$$\Lambda_\pm{}^b(p_2)=\frac{E_2\pm H^b(p_2)}{2E_2}, \quad E_2=(m_b^2+p_2^2)^{1/2}, \tag{13b}$$

and $\chi_P(p)$ is

$$\chi_P(p)\equiv\int_{-\infty}^{\infty}\phi_P(\varepsilon, p)\,d\varepsilon. \tag{14}$$

We expand the amplitude $\chi_P(p)$ in terms of the eigenspinors of $\beta^a\beta^b$,

$$\chi_P(p)=\chi_{++}{}^P(p)\begin{pmatrix}1\\0\end{pmatrix}^a\begin{pmatrix}1\\0\end{pmatrix}^b+\chi_{+-}{}^P(p)\begin{pmatrix}1\\0\end{pmatrix}^a\begin{pmatrix}0\\1\end{pmatrix}^b+\chi_{-+}{}^P(p)\begin{pmatrix}0\\1\end{pmatrix}^a\begin{pmatrix}1\\0\end{pmatrix}^b+\chi_{--}{}^P(p)\begin{pmatrix}0\\1\end{pmatrix}^a\begin{pmatrix}0\\1\end{pmatrix}^b, \tag{15}$$

and introduce the combinations

$$\begin{bmatrix}\chi_1{}^P(p)\\ \chi_2{}^P(p)\\ \chi_3{}^P(p)\\ \chi_4{}^P(p)\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\chi_{++}{}^P(p)+\chi_{--}{}^P(p)\\ \chi_{++}{}^P(p)-\chi_{--}{}^P(p)\\ \chi_{+-}{}^P(p)+\chi_{-+}{}^P(p)\\ \chi_{+-}{}^P(p)-\chi_{-+}{}^P(p)\end{bmatrix}. \tag{16}$$

Then Eq. (12) leads to the following four integral equations:

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*) The instantaneous interaction in the Bethe-Salpeter equation was first treated by Salpeter.[7]

$$\chi_1{}^P(p) = -\frac{m_a+m_b}{16\pi E_0{}^2\{(p+d_P)^2+\kappa^2\}}$$
$$\times\Bigg[\frac{E}{m_a+m_b}\left[E^2-(m_a-m_b)^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right][g_1A_3{}^PJ_+(p)-g_2A_2{}^PK_+(p)]$$
$$-\left[E^2-(m_a-m_b)^2-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)P\right][g_1A_1{}^PK_+(p)-g_3A_4{}^PJ_-(p)]$$
$$-\left[\left\{E^2-(m_a-m_b)^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right\}\frac{P}{m_a+m_b}-2(m_a-m_b)p\right]$$
$$\times[g_1A_1{}^PJ_+(p)+g_3A_4{}^PK_-(p)]-2Ep[g_1A_3{}^PK_-(p)+g_2A_2{}^PJ_-(p)]\Bigg], \quad (17a)$$

$$\chi_2{}^P(p) = -\frac{E}{16\pi E_0{}^2\{(p+d_P)^2+\kappa^2\}}$$
$$\times\Bigg[\frac{m_a+m_b}{E}\left[E^2-(m_a-m_b)^2-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)P\right]$$
$$\times[g_1A_3{}^PJ_+(p)-g_2A_2{}^PK_+(p)]-[E_0{}^2-(m_a-m_b)^2][g_1A_1{}^PK_+(p)-g_3A_4{}^PJ_-(p)]$$
$$-(m_a+m_b)\left[P-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)\right][g_1A_1{}^PJ_+(p)+g_3A_4{}^PK_-(p)]$$
$$-\frac{m_a+m_b}{E}\left[\frac{E_0^2}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)-(m_a-m_b)P\right][g_1A_3{}^PK_-(p)+g_2A_2{}^PJ_-(p)]\Bigg],$$
$$(17b)$$

$$\chi_3{}^P(p) = -\frac{E}{16\pi E_0{}^2\{(p+d_P)^2+\kappa^2\}}$$
$$\times\Bigg[\frac{1}{E}\left[P\left\{E^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right\}-(m_a{}^2-m_b{}^2)\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)\right]$$
$$\times[g_1A_3{}^PJ_+(p)-g_2A_2{}^PK_+(p)]-(m_a+m_b)$$
$$\times\left[P-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)\right][g_1A_1{}^PK_+(p)-g_3A_4{}^PJ_-(p)]$$
$$-\left[E^2-(m_a+m_b)^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right][g_1A_1{}^PJ_+(\mathrm{p})+g_3A_4{}^PK_-(p)]$$
$$-\frac{m_a+m_b}{E}\left[2Pp+\frac{m_a-m_b}{m_a+m_b}\{(m_a+m_b)^2-E_0{}^2\}\right][g_1A_3{}^PK_-(p)+g_2A_2{}^PJ_-(p)]\Bigg],$$
$$(17c)$$

$$\chi_4{}^P(p) = \frac{m_a+m_b}{16\pi E_0{}^2\{(p+d_P)^2+\kappa^2\}}\Bigg[2Ep[g_1A_3{}^PJ_+(p)-g_2A_2{}^PK_+(p)]$$
$$-\left[\frac{E_0^2}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)-(m_a-m_b)P\right][g_1A_1{}^PK_+(p)-g_3A_4{}^PJ_-(p)]$$
$$-\left[2Pp+\frac{m_a-m_b}{m_a+m_b}\{(m_a+m_b)^2-E_0{}^2\}\right][g_1A_1{}^PJ_+(p)+g_3A_4{}^PK_-(p)]$$
$$+\frac{E}{m_a+m_b}[(m_a+m_b)^2-E_0{}^2][g_1A_3{}^PK_-(p)+g_2A_2{}^PJ_-(p)]\Bigg], \quad (17d)$$

where

$$g_1 \equiv g_S + g_P,\ g_2 \equiv g_S - g_P + g_V,\ g_3 \equiv -g_S + g_P + g_V, \tag{18}$$

$$A_i^P \equiv \int_{-\infty}^{\infty} \chi_i^P(p)\,dp, \quad (i=1,2,3,4), \tag{19}$$

$$J_\pm(p) \equiv \frac{\dfrac{m_a P}{m_a+m_b}+p}{\left\{m_a^2+\left(\dfrac{m_a P}{m_a+m_b}+p\right)^2\right\}^{1/2}} \pm \frac{\dfrac{m_b P}{m_a+m_b}-p}{\left\{m_b^2+\left(\dfrac{m_b P}{m_a+m_b}-p\right)^2\right\}^{1/2}}, \tag{20}$$

$$K_\pm(p) \equiv \frac{m_a}{\left\{m_a^2+\left(\dfrac{m_a P}{m_a+m_b}+p\right)^2\right\}^{1/2}} \pm \frac{m_b}{\left\{m_b^2+\left(\dfrac{m_b P}{m_a+m_b}-p\right)^2\right\}^{1/2}}. \tag{21}$$

When Eqs. (17a)~(17d) have the bound-state solution, they determine E_0, that is, the rest mass of the bound state, which is related to E and P by

$$E = (E_0^2 + P^2)^{1/2}. \tag{22}$$

Through E_0, they determine also κ^2 and d_P to be

$$\kappa^2 \equiv \frac{E^2\{E_0^2-(m_a-m_b)^2\}\{(m_a+m_b)^2-E_0^2\}}{4E_0^4}, \tag{23}$$

$$d_P \equiv -\frac{P}{2E_0^2}\left(\frac{m_a-m_b}{m_a+m_b}\right)\{(m_a+m_b)^2-E_0^2\}. \tag{24}$$

When bound-state solutions are found, they are normalized to be

$$\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dX \chi_{P'}^{+}(p)\chi_P(p)\, e^{-i(P'-P)X} = 2\pi(E/E_0)\delta(P'-P). \tag{25}$$

§3. Bound-state solutions and an examination of them about the accuracy of the ladder approximation

3.1 *Bound-state solutions*

Observing Eqs. (17a)~(17d), we search bound-state solutions.

An even-parity bound-state solution obtained with a renormalization

In the case of $g_1=0$ in Eqs. (17a)~(17d), we have an even-parity bound-state solution:

$$\chi_1^P(p) = \frac{g_2 A_2^P(m_a+m_b)E}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}} \times\left[\frac{1}{m_a+m_b}\left[E^2-(m_a-m_b)^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right]K_+(p)+2pJ_-(p)\right], \tag{26a}$$

$$\chi_2^P(p) = \frac{g_2 A_2^P(m_a+m_b)}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}}\left[\left[E^2-(m_a-m_b)^2-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)P\right]K_+(p)\right.$$

$$+\left[\frac{E_0^2}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)-(m_a-m_b)P\right]J_-(p)\Bigg], \tag{26b}$$

$$\chi_3^P(p)=\frac{g_2A_2^P(m_a+m_b)}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}}\Bigg[\left[\frac{P}{m_a+m_b}\left\{E^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right\}-(m_a-m_b)\right.$$
$$\left.\times\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)\right]K_+(p)+\left[2Pp+\frac{m_a-m_b}{m_a+m_b}\{(m_a+m_b)^2-E_0^2\}\right]J_-(p)\Bigg], \tag{26c}$$

$$\chi_4^P(p)=-\frac{g_2A_2^P(m_a+m_b)E}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}}\left[2pK_+(p)-\frac{1}{m_a+m_b}[(m_a+m_b)^2-E_0^2]J_-(p)\right]. \tag{26d}$$

The rest mass E_0 of the bound-state solution of Eqs. (26a)~(26d) is determined from the equation obtained by integrating both sides of Eq. (26b) over p:

$$A_2^P=\int_{-\infty}^{\infty}(\text{R.H.S. of Eq. (26b)})\,dp. \tag{27}$$

Although the interal on the R.H.S. of Eq. (27) diverges, Eq. (27) is rewritten to be

$$\frac{8\pi}{g_2}-\int_{-\infty}^{\infty}\left\{\frac{1}{(m_a^2+p^2)^{1/2}}+\frac{1}{(m_b^2+p^2)^{1/2}}\right\}dp$$
$$=\int_{-\infty}^{\infty}\left[\frac{1}{\{(p+d_P)^2+\kappa^2\}}\left[\frac{m_a+m_b}{2E_0^2}\left\{E^2-(m_a-m_b)^2-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)P\right\}\right.\right.$$
$$\left.\times K_+(p)-\left\{\frac{(p+d_p)\left(d_P-\frac{m_aP}{m_a+m_b}\right)+\kappa^2}{\left(m_a^2+\left(\frac{m_aP}{m_a+m_b}+p\right)^2\right)^{1/2}}+\frac{(p+d_P)\left(d_P+\frac{m_bP}{m_a+m_b}\right)+\kappa^2}{\left(m_b^2+\left(\frac{m_bP}{m_a+m_b}-p\right)^2\right)^{1/2}}\right\}\right]$$
$$-\frac{\frac{m_aP}{m_a+m_b}\left(\frac{m_aP}{m_a+m_b}+2p\right)}{\left(m_a^2+\left(\frac{m_aP}{m_a+m_b}+p\right)^2\right)^{1/2}(m_a^2+p^2)^{1/2}\left\{\left(m_a^2+\left(\frac{m_aP}{m_a+m_b}+p\right)^2\right)^{1/2}+(m_a^2+p^2)^{1/2}\right\}}$$
$$\left.-\frac{\frac{m_bP}{m_a+m_b}\left(\frac{m_bP}{m_a+m_b}-2p\right)}{\left(m_b^2+\left(\frac{m_bP}{m_a+m_b}-p\right)^2\right)^{1/2}(m_b^2+p^2)^{1/2}\left\{\left(m_b^2+\left(\frac{m_bP}{m_a+m_b}-p\right)^2\right)^{1/2}+(m_b^2+p^2)^{1/2}\right\}}\right]$$
$$\times dp, \tag{28}$$

and this divergence is dealt with by a kind of renormalization of the coupling constant. Owing to the manifest covariance of Eq. (1), we can choose the rest frame of the bound state and have

$$\frac{8\pi}{G}\equiv\frac{8\pi}{g_2}-\int_{-\infty}^{\infty}\left\{\frac{1}{(m_a^2+p^2)^{1/2}}+\frac{1}{(m_b^2+p^2)^{1/2}}\right\}dp$$

$$=\frac{2\{E_0^2-(m_a-m_b)^2\}^{1/2}}{\{(m_a+m_b)^2-E_0^2\}^{1/2}}$$
$$\times\left[\frac{\{E_0^2+(m_a^2-m_b^2)\}}{|E_0^2+(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2+(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right]\right.$$
$$\left.+\frac{\{E_0^2-(m_a^2-m_b^2)\}}{|E_0^2-(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2-(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right]\right]. \tag{29}$$

Here G is an effective coupling constant.

As a reinterpretation of the coupling constant is made, $g_2A_2^P/(8\pi)$ in Eqs. (26a)$\sim$(26d) is substituted by a constant N for the normalization.

An odd-parity bound-state solution obtained with a renormalization

In the case of $g_1=0$ in Eqs. (17a)$\sim$(17d), we have also an odd-parity bound-state solution:

$$\chi_1^P(p)=-\frac{g_3A_4^P(m_a+m_b)}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}}\left[\left[E^2-(m_a-m_b)^2-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)P\right]J_-(p)\right.$$
$$\left.-\left[\left\{E^2-(m_a-m_b)^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right\}\frac{P}{m_a+m_b}-2(m_a-m_b)p\right]K_-(p)\right], \tag{30a}$$

$$\chi_2^P(p)=-\frac{g_3A_4^PE}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}}\left[\left[E_0^2-(m_a-m_b)^2\right]J_-(p)\right.$$
$$\left.-(m_a+m_b)\left[P-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)\right]K_-(p)\right], \tag{30b}$$

$$\chi_3^P(p)=-\frac{g_3A_4^PE}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}}\left[(m_a+m_b)\left[P-\frac{m_a-m_b}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)\right]J_-(p)\right.$$
$$\left.-\left[E^2-(m_a+m_b)^2-\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)^2\right]K_-(p)\right], \tag{30c}$$

$$\chi_4^P(p)=\frac{g_3A_4^P(m_a+m_b)}{16\pi E_0^2\{(p+d_P)^2+\kappa^2\}}\left[\left[\frac{E_0^2}{m_a+m_b}\left(\frac{m_a-m_b}{m_a+m_b}P+2p\right)-(m_a-m_b)P\right]J_-(p)\right.$$
$$\left.-\left[2Pp+\frac{m_a-m_b}{m_a+m_b}\{(m_a+m_b)^2-E_0^2\}\right]K_-(p)\right]. \tag{30d}$$

The eigenvalue condition is given by integrating both sides of Eq. (30d) over p. Although a divergence is involved similarly to Eq. (27), this divergence is also dealt with in a similar manner to for Eq. (27). In the rest frame of the bound state, we have

$$-\frac{8\pi}{G'}\equiv\frac{8\pi}{g_3}-\int_{-\infty}^{\infty}\left\{\frac{1}{(m_a^2+p^2)^{1/2}}+\frac{1}{(m_b^2+p^2)^{1/2}}\right\}dp=-\frac{2\{(m_a+m_b)^2-E_0^2\}^{1/2}}{\{E_0^2-(m_a-m_b)^2\}^{1/2}}$$
$$\times\left[\frac{\{E_0^2+(m_a^2-m_b^2)\}}{|E_0^2+(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2+(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right]\right.$$
$$\left.+\frac{\{E_0^2-(m_a^2-m_b^2)\}}{|E_0^2-(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2-(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right]\right]. \tag{31}$$

As a reinterpretation of the coupling constant is made, $-g_3A_4^P/(8\pi)$ in Eqs. (30a)$\sim$

(30d) is substituted by a constant N' for the normalization.

An even-parity bound-state solution obtained without any renormalization

In the case of $g_2=g_3=0$ in Eqs. (17a)$\sim$(17d), we have an even-parity bound-state solution. We state this solution by choosing the rest frame of the bound state from the beginning, as it is obtained without any renormalization of the coupling constant.

Assuming $A_3^{P=0}=0$ in this case of $g_2=g_3=0$ in Eqs. (17a)$\sim$(17d), we obtain

$$\chi_1^{P=0}(p)=\frac{g_1A_1^{P=0}(m_a+m_b)}{16\pi E_0^2(p^2+\kappa_0^2)}[\{E_0^2-(m_a-m_b)^2\}\{K_+(p)|_{P=0}\}-2(m_a-m_b)p\{J_+(p)|_{P=0}\}], \tag{32a}$$

$$\chi_2^{P=0}(p)=\frac{E_0}{m_a+m_b}\chi_1^{P=0}(p), \tag{32b}$$

$$\chi_3^{P=0}(p)=-\frac{g_1A_1^{P=0}}{16\pi E_0(p^2+\kappa_0^2)} \times[2(m_a-m_b)p\{K_+(p)|_{P=0}\}+\{(m_a+m_b)^2-E_0^2+4p^2\}\{J_+(p)|_{P=0}\}], \tag{32c}$$

$$\chi_4^{P=0}(p)=-\frac{g_1A_1^{P=0}}{16\pi(p^2+\kappa_0^2)} \times\left[2p\{K_+(p)|_{P=0}\}+\frac{m_a-m_b}{E_0^2}\{(m_a+m_b)^2-E_0^2\}\{J_+(p)|_{P=0}\}\right], \tag{32d}$$

where

$$\kappa_0^2\equiv\kappa^2|_{P=0}. \tag{33}$$

The eigenvalue condition is given by integrating both sides of Eq. (32a) over p. This integral does not diverge, because $J_+(p)|_{P=0}$ is reduced to

$$J_+(p)|_{P=0}=-(m_a^2-m_b^2)\frac{p}{(m_a^2+p^2)(m_b^2+p^2)^{1/2}+(m_a^2+p^2)^{1/2}(m_b^2+p^2)}. \tag{34}$$

We have

$$\frac{2\pi E_0^2}{f_{S+P}(m_a+m_b)^2}=\frac{\{E_0^2-(m_a-m_b)^2\}^{1/2}}{\{(m_a+m_b)^2-E_0^2\}^{1/2}} \times\left[\frac{\{E_0^2+(m_a^2-m_b^2)\}}{|E_0^2+(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2+(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right] +\frac{\{E_0^2-(m_a^2-m_b^2)\}}{|E_0^2-(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2-(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right]\right] +\frac{m_a-m_b}{m_a+m_b}\ln\left(\frac{m_a}{m_b}\right), \tag{35}$$

where

$$f_{S+P}\equiv\frac{g_1}{2}=g_S=g_P \quad (>0). \tag{36}$$

The assumption of $A_3^{P=0}=0$ is satisfied self-consistently in this solution, because $\chi_3^{P=0}(p)$ of Eq. (32c) is an odd function of p.

An odd-parity bound-state solutionn obtained without any renormalization

In the case of $g_2=g_3=0$ in Eqs. (17a)～(17d), we have also an odd-parity bound-state solution without any renormalization of the coupling constant. We state this solution by choosing the rest frame of the bound state.

Assuming $A_1^{P=0}=0$ in this case of $g_2=g_3=0$ in Eqs. (17a)～(17d), we obtain

$$\chi_1^{P=0}(p)=-\frac{g_1A_3^{P=0}(m_a+m_b)}{16\pi E_0(p^2+\kappa_0^2)} \times\left[\frac{1}{m_a+m_b}\{E_0^2-(m_a-m_b)^2-4p^2\}\{J_+(p)|_{P=0}\}-2p\{K_-(p)|_{P=0}\}\right], \tag{37a}$$

$$\chi_2^{P=0}(p)=-\frac{g_1A_3^{P=0}(m_a+m_b)}{16\pi E_0^2(p^2+\kappa_0^2)}\left[\{E_0^2-(m_a-m_b)^2\}\{J_+(p)|_{P=0}\}-2\frac{E_0^2}{m_a+m_b}p\{K_-(p)|_{P=0}\}\right], \tag{37b}$$

$$\chi_3^{P=0}(p)=\frac{g_1A_3^{P=0}(m_a^2-m_b^2)}{16\pi E_0^2(p^2+\kappa_0^2)}\left[2p\{J_+(p)|_{P=0}\}+\frac{1}{m_a+m_b}\{(m_a+m_b)^2-E_0^2\}\{K_-(p)|_{P=0}\}\right], \tag{37c}$$

$$\chi_4^{P=0}(p)=\frac{g_1A_3^{P=0}(m_a+m_b)}{16\pi E_0(p^2+\kappa_0^2)}\left[2p\{J_+(p)|_{P=0}\}+\frac{1}{m_a+m_b}\{(m_a+m_b)^2-E_0^2\}\{K_-(p)|_{P=0}\}\right]. \tag{37d}$$

The assumption of $A_1^{P=0}=0$ is satisfied self-consistently in this solution, because $\chi_1^{P=0}(p)$ of Eq. (37a) is an odd function of p.

The eigenvalue condition is given by integrating both sides of Eq. (37c) over p. We have

$$\frac{2\pi E_0^2}{f_{S+P}(m_a-m_b)^2}=\frac{\{(m_a+m_b)^2-E_0^2\}^{1/2}}{\{E_0^2-(m_a-m_b)^2\}^{1/2}} \times\left[\frac{\{E_0^2+(m_a^2-m_b^2)\}}{|E_0^2+(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2+(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right]\right. \left.+\frac{\{E_0^2-(m_a^2-m_b^2)\}}{|E_0^2-(m_a^2-m_b^2)|}\text{Arctan}\left[\frac{|E_0^2-(m_a^2-m_b^2)|}{\{E_0^2-(m_a-m_b)^2\}^{1/2}\{(m_a+m_b)^2-E_0^2\}^{1/2}}\right]\right] -\frac{m_a+m_b}{m_a-m_b}\ln\left(\frac{m_a}{m_b}\right). \tag{38}$$

Here f_{S+P} is taken to be $f_{S+P}<0$ in this solution, differently from f_{S+P} in the solution of Eqs. (32a)～(32d).

It is noted that this solution disappears in the case of $m_a=m_b$. This disappearance is due to the interaction which is taken for the formation of this solution.

3.2 *An examination of the bound-state solutions about the accuracy of the ladder approximation*

The wave functions of the bound states in the present model allow the probabilistic interpretation. We examine the wave functions of the above four bound-state solutions, in order to obtain the information on the accuracy of the ladder approximation.

Calculating

$$\int_{-\infty}^{\infty} |\chi_{++}^{P=0}(p)|^2 dp, \quad \int_{-\infty}^{\infty} |\chi_{--}^{P=0}(p)|^2 dp, \quad \int_{-\infty}^{\infty} |\chi_{+-}^{P=0}(p)|^2 dp, \quad \int_{-\infty}^{\infty} |\chi_{-+}^{P=0}(p)|^2 dp \quad (39)$$

in each of the bound-state solutions of Eqs. (26a)～(26d), Eqs. (30a)～(30d) and Eqs. (32a) ～(32d) in the case of $m \equiv m_a = m_b$*) (after performing the normalization by Eq. (25)), we show numerical results in Figs. 1～3. These quantities in the bound-state solution of Eqs. (37a)～(37d) in the case where $m_a - m_b = 10^{-2}$ m and $(m_a - m_b)/(m_a + m_b) = 10^{-3}$ are shown in Fig. 4.

Observing Figs. 1～4, we find qualitative features, which include a remarkable one. After some considerations, we have the following consequences.

(1) The odd-parity bound-state solution obtained with a renormalization in Fig. 2 is notewor-

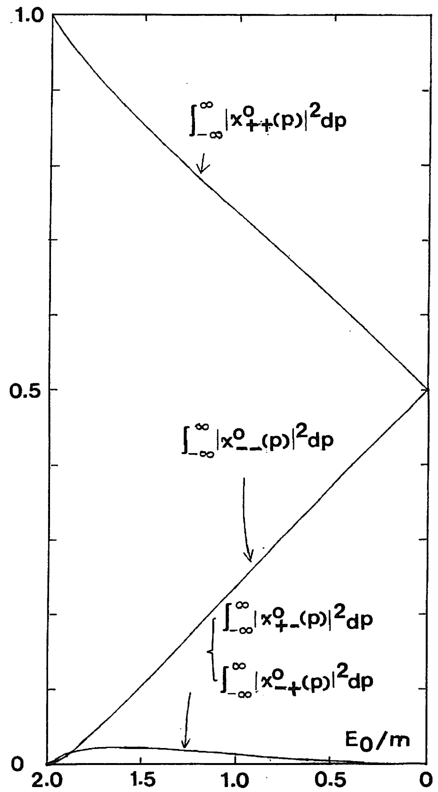

Fig. 1. The probabilities $\int_{-\infty}^{\infty} |\chi_{++}^{P=0}(p)|^2 dp$, $\int_{-\infty}^{\infty} |\chi_{--}^{P=0}(p)|^2 dp$, $\int_{-\infty}^{\infty} |\chi_{+-}^{P=0}(p)|^2 dp$ and $\int_{-\infty}^{\infty} |\chi_{-+}^{P=0}(p)|^2 dp$ in the even-parity bound-state solution of Eqs. (26a)～(26d) in the text.

*) We assume that the particles a and b belong to different kinds.

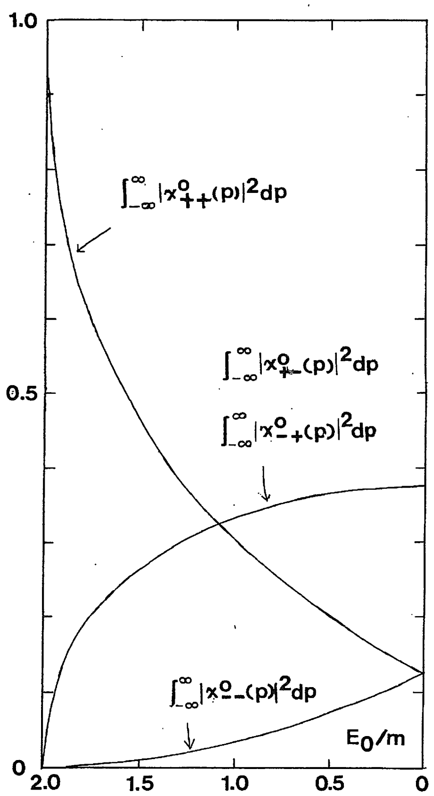

Fig. 2. The probabilities $\int_{-\infty}^{\infty}|\chi_{++}^{P=0}(p)|^2 dp$, $\int_{-\infty}^{\infty}|\chi_{--}^{P=0}(p)|^2 dp$, $\int_{-\infty}^{\infty}|\chi_{+-}^{P=0}(p)|^2 dp$ and $\int_{-\infty}^{\infty}|\chi_{-+}^{P=0}(p)|^2 dp$ in the odd-parity bound-state solution of Eqs. (30a)$\sim$(30d) in the text.

thily strange in comparison with the other solutions. Although the probability of the odd-parity component $\chi_{++}{}^0(p)$ in this solution is dominative for E_0 of $2.0 \geqq E_0/m \gtrsim 1.1$ in accordance with the natural requirement that the probability of the odd-parity component $\chi_{++}{}^0(p)$ in the odd-parity bound-state solution should be dominative, this probability in this solution does not satisfy the natural requirement for E_0 of $1.1 \gtrsim E_0/m \geqq 0$. This remarkable and undesired feature must be noticed. It is noted that the natural requirement is satisfied by the odd-parity bound-state solution in Fig. 4 and the counter natural requirement (that the probability of the even-parity component $\chi_{++}{}^0(p)$ in the even-parity bound-state solution should be dominative) is satisfied by the even-parity bound-state solutions in Figs. 1 and 3.

(2) The above remarkable and undesired feature of the odd-parity bound-state solution obtained with a renormalization is owing to the inadequacy of the ladder approximation. There are no other reasons for it in the present model.

(3) As for the other solutions in Figs. 1, 3 and 4, qualitative features of their wave functions are acceptable.

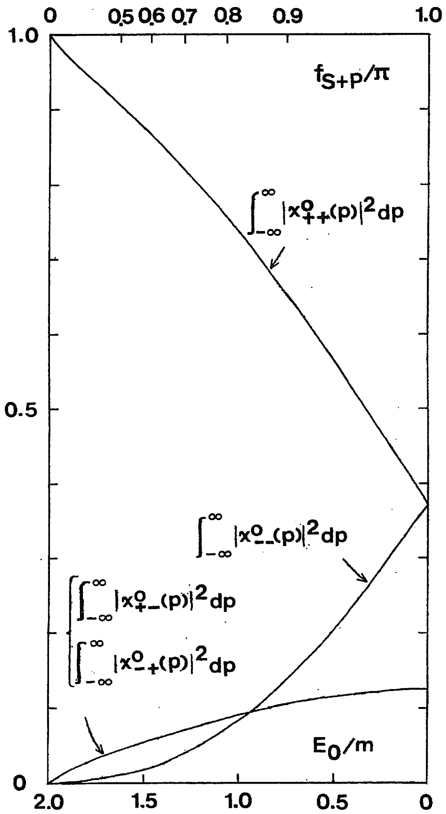

Fig. 3. The probabilities $\int_{-\infty}^{\infty} |\chi_{++}^{P=0}(p)|^2 dp$, $\int_{-\infty}^{\infty} |\chi_{--}^{P=0}(p)|^2 dp$, $\int_{-\infty}^{\infty} |\chi_{+-}^{P=0}(p)|^2 dp$ and $\int_{-\infty}^{\infty} |\chi_{-+}^{P=0}(p)|^2 dp$ in the even-parity bound-state solution of Eqs. (32a)$\sim$(32d) in the text. The values of f_{S+P}/π corresponding to those of E_0/m are also shown.

§4. Summary and discussion

In this paper, we have made a study to assess the accuracy of the ladder approximation, by employing a solvable two-body positron-theoretical composite model which is defined in terms of the Bethe-Salpeter equation in the ladder approximation with various Fermi-type direct instantaneous interactions in $1+1$ dimensional world. In this model, we have found two sets of even- and odd-parity bound-state solutions, one of which is obtained with a kind of renormalization of the coupling constant, and the other of which is gotten without any renormalization. We have examined the wave functions of all the obtained bound-state solutions in detail, as they allow the probabilistic interpretation. After some considerations, we have obtained consequences, which are summarized as follows: (A) The wave function of the odd-parity bound-state solution obtained with a renormalization contains a remarkable and undesired feature. The probability of its odd-parity component $\chi_{++}{}^0(p)$ for the cases of large binding energies does not satisfy the natural requirement that the probability of the odd-parity component $\chi_{++}{}^0$ (p) in the odd-parity bound-state solution should be dominative. This remarkable and un-

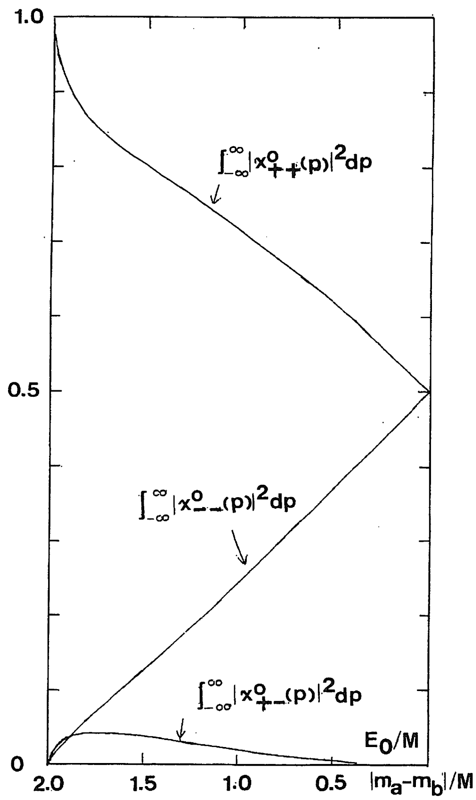

Fig. 4. The probabilities
$\int_{-\infty}^{\infty} |\chi_{++}^{P=0}(p)|^2 dp$, $\int_{-\infty}^{\infty} |\chi_{--}^{P=0}(p)|^2 dp$,
$\int_{-\infty}^{\infty} |\chi_{+-}^{P=0}(p)|^2 dp$ and $\int_{-\infty}^{\infty} |\chi_{-+}^{P=0}(p)|^2 dp$
in the odd-parity bound-state solution of Eqs. (37a)$\sim$(37d) in the test. It is noted that
$\int_{-\infty}^{\infty} |\chi_{-+}^{P=0}(p)|^2 dp$
is very close to
$\int_{-\infty}^{\infty} |\chi_{+-}^{P=0}(p)|^2 dp$
for E_0/M of $2.0 \geqq E_0/M \geqq |m_a - m_b|/M$, where M is $(m_a + m_b)/2$.

desired feature of the odd-parity bound-state solution obtained with a renormalization is owing to the inadequacy of the ladder approximation. (B) As for the other solutions, qualitative features of their wave functions are acceptable.

We make some discussions about the consequences.

(I) Judging from the present study, we suppose and emphasize that the ladder approximation (in the real world) may yield not only unsatisfactory quantitative results but also even undesired features against the physical requirements concerned with the wave functions, at least in the cases where the coupling constants of short-range interactions are renormalized. (In our opinion, the 3+1–dimensional generalization of the natural requirement stated above is a physical requirement)

(II) We suppose that when the ladder approximation is used (in the real world), the wave functions should be examined in detail at least in the cases where the coupling constants of short-range interactions are renormalized.

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