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Commutative Ideal Extensions of Null Semigroups

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§1 Introduction.

Let G and S be commutative semigroups with zero, and I an ideal of G. Let 0 and $\mathbf{0}$ be the zero elements of G and S respectively. Consider a mapping $\varphi : S \to G$ satisfying the following (1.1) :

$$(1.1)\quad \begin{cases} (1) & \varphi(\mathbf{0})=0, \\ (2) & \varphi(a)\varphi(b)=\varphi(b)\varphi(a)\in I \quad \text{if } ab=\mathbf{0} \text{ (hence, also } ba=\mathbf{0}), \\ (3) & \varphi(ab)=\varphi(a)\varphi(b) \quad \text{if } ab\neq\mathbf{0}. \end{cases}$$

If we define a mapping $\Psi : S \longrightarrow G/I$ (where G/I is the Rees factor Semigroup of G modulo I) by

$$(1.2)\quad \Psi(a)=\begin{cases} \varphi(a) & \text{if } \varphi(a)\notin I, \\ 0 & \text{if } \varphi(a)\in I, \end{cases}$$

then Ψ is clearly a homomorphism of S into G/I. Hereafter, we shall call a mapping $\varphi : S \longrightarrow G$ a *homomorphism of S into G modulo I* (abbrev., *I-homomorphism of S into G*) if it satisfies (1.1). It is obvious that a mapping $\eta : S \longrightarrow G$ is a homomorphism if η is a $\{0\}$-homomorphism. A halfgroupoid H in the sense of $R.H.$ Bruck [1] (i. e., a partial groupoid in the sense of $A.$ $H.$ Clifford and $G.$ $B.$ Preston [3]) is said to be *commutative* if it satisfies the following (1.3) :

(1.3) If $x, y\in H$ and xy is defined, then yx is defined and $xy=yx$.

suppose that M is a subsemigroup of a commutative halfgroupoid H. [1)]
Then, we shall say that a mapping $\varphi : H \longrightarrow H$ is a *translation on* $H(M)$ (abbrev., *$H(M)$-translation*) if it satisfies the following (1.4) :

(1.4) $\varphi(M) \subset M$, and the restriction $\varphi \mid M$ of φ to M is a translation of M.

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1) A subset K of a halfgroupoid H is called a subsemigroup of H if K is a semigroup with respect to the binary operation of H.

If H itself is a semigroup, it is obvious that an $H(H)$-translation is a translation of H. An $H(M)$-translation φ is said to be an *inner* $H(M)$*-translation* if $\varphi \mid M$ is an inner translation of M. The set $\mathscr{T}(H, M)$ [$\mathscr{I}(H, M)$] of all $H(M)$-translations [all inner $H(M)$-translations] is a semigroup with respect to the resultant composition. We shall call $\mathscr{T}(H, M)$ [$\mathscr{I}(H, M)$] *the semigroup of* $H(M)$*-translations* [*the semigroup of inner* $H(M)$*-translations*]. It is easily seen that $\mathscr{I}(H, M)$ is an ideal of $\mathscr{T}(H, M)$. In particular, for the case where M is a null semigroup contained in H we can easily verify the following (1.5) :

$$(1.5)\quad \begin{cases} \mathscr{T}(H, M)=\{\varphi : \varphi \text{ is a mapping of } H \text{ into } H \text{ such that} \\ \qquad\qquad \varphi(M)\subset M,\ \varphi(0)=0\},\text{ and} \\ \mathscr{I}(H, M)=\{\varphi : \varphi \text{ is a mapping of } H \text{ into } H \text{ such that} \\ \qquad\qquad \varphi(M)=\{0\}\}, \end{cases}$$

where 0 denotes the zero element of M.

In this case, the mapping φ_0 defined by $\varphi_0(x)=0$, $x\in H$, is an element of $\mathscr{I}(H, M)$ and is the zero element of $\mathscr{T}(H, M)$. We shall call φ_0 the 0-*mapping on* H (*with respect to* M). Now, let T and G be commutative semigroups, having $\mathbf{0}$ and 0 as their zero elements respectively. Let $T^*=T\backslash\mathbf{0}$ and $S=T^*\dot{+}G$ (where $\dot{+}$ means the disjoint sum), and define $*$ in S as follows :

$$(1.6)\quad x*y=\begin{cases} xy \text{ if } x, y\in G \text{ or if } xy\neq\mathbf{0},\ x, y\in T^*, \\ \text{not defined for the other cases.}\end{cases}$$

Then $S(*)$ is obviously a commutative halfgroupoid and G is embedded in $S(*)$. This $S(*)$ is called *the adjunction of* T *to* G. Further, the set S with a binary operation $\circ$ is called an (*ideal*) *extension of* G *by* T if it satisfies the following (1.7) (see also [2]) :

$$(1.7)\quad x\circ y=\begin{cases} x*y \quad \text{if } x, y\in S(*) \text{ and } x*y \text{ is defined,} \\ \in G \quad \text{otherwise.}\end{cases}$$

In this case, it is easily seen that G is embedded in $S(\circ)$ as an ideal of $S(\circ)$ and the Rees factor semigroup $S(\circ)/G$ of $S(\circ)$ modulo G is isomorphic with T. An (ideal) extension $S(\circ)$ of G by T is said to be a *commutative* (ideal) extension if $S(\circ)$ is commutative. Further, an (ideal) extension $S(\circ)$ of G by T is called a 0-*extension of* G *by* T if it satisfies the following (1.8) :

(1.8) $T^* \circ G = G \circ T^* = \{0\}$. [2)]

It is easily seen that there exists at least one 0-extension of G by T. Hereafter, through this paper, "extension" always means "ideal extension".

Next, let G be a commutative semigroup with zero and N a null semigroup. Let $\mathbf{0}$ and 0 be the zero elements of G and N respectively. Let $S(*)$ be the adjunction of G to N. By the definition of adjunctions, $S(*)$ is a commutative halfgroupoid containing N as its subsemigroup and the 0-mapping φ_0 of $S(*)$ with respect to N is the zero element of $\mathscr{T}(S(*), N)$.
Now let η be an $\mathscr{F}(S(*), N)$-homomorphism of G into $\mathscr{T}(S(*), N)$, and put $\eta(A) = \lambda_A$ for every $A \in G$. Since η is an $\mathscr{F}(S(*), N)$-homomorphism, we have the following (1.9):

$$(1.9)\quad \begin{cases} (1)\ \lambda_{\mathbf{0}} = \varphi_0, \\ (2)\ \lambda_A \lambda_B = \lambda_B \lambda_A \in \mathscr{F}(S(*), N) \text{ if } AB = \mathbf{0} \text{ in } G, \\ (3)\ \lambda_{AB} = \lambda_A \lambda_B \text{ if } AB \neq \mathbf{0} \text{ in } G. \end{cases}$$

If η further satisfies the following additional condition (1.10), then η is called a *complete* $\mathscr{F}(S(*), N)$*-homomorphism* of G into $\mathscr{T}(S(*), N)$:

$$(1.10)\quad \begin{cases} (4)\ \lambda_A(B) = \lambda_B(A) \quad \text{for all } A, B \in G \setminus \mathbf{0} \\ (5)\ \lambda_A(B) \in N \quad \text{if } A, B \in G \setminus \mathbf{0} \text{ and } AB = \mathbf{0} \text{ in } G. \end{cases}$$

In this paper, at first we shall present some construction theorems for commutative extensions and commutative 0-extensions of null semigroups. Especially, in § 2 and § 3 all the commutative 0-extensions of a given null semigroup N by a given commutative semigroup G with zero will be completely determined.[3)] Finally, in § 4 we shall show several applications of the construction theorems given in § 2. In particular, we shall discuss the construction of commutative nilpotent semigroups and that of commutative semigroups satisfying the ascending chain condition and the descending chain condition for ideals.

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2) $T^* \circ G$ means the set of all elements $x \circ y \in S(\circ)$ such that $x \in T^*$ and $y \in G$; *i. e.*, $T^* \circ G = \{x \circ y : x \in T^*, y \in G\}$. Also, $G \circ T^* = \{y \circ x : y \in G, x \in T^*\}$.
There exists at least one 0-extension of G by T. For example, define a binary operation $\circ$ in $S = T^* \dot{+} G$ as follows:

$$\begin{cases} A \circ B = AB & \text{if } A, B \in T^*, AB \neq \mathbf{0} \text{ in } T, \\ A \circ B = 0 & \text{if } A, B \in T^*, AB = \mathbf{0} \text{ in } T, \\ A \circ u = u \circ A & \text{if } A \in T^*, u \in G, \\ u \circ v = uv & \text{if } u, v \in G. \end{cases}$$

Then, the resulting system $S(\circ)$ is a 0-extension of G by T.

3) This is a generalization of one of the results obtained by the author [6].

§ 2. Construction theorems.

Throughout this paragraph, G will denote a commutative semigroup with zero and N will denote a null semigroup. Let $\mathbf{0}$ and 0 be the zero elements of G and N respectively. Let $S=G^*\dot{+}N$, where $G^*=G\backslash\mathbf{0}$, and let $S(*)$ be the adjunction of G to N. Hereafter we shall denote elements of G^* by capital letters A, B, C etc. and elements of N by small letters a, b, c etc., unless otherwise stated.

Theorem 1. *Let η be any complete $\mathscr{F}(S(*), N)$-homomorphism of G into $\mathscr{T}(S(*), N)$. Then S becomes a commutative extension of N by G with respect to the binary operation defined as follows :*

$$(2.1)\quad \begin{cases} (1)\ A\circ B = AB\ (=A*B) & \text{if } AB\neq\mathbf{0} \text{ in } G,\ A,\ B\in G^*,\\ (2)\ A\circ B = \lambda_A(B) & \text{if } AB=\mathbf{0} \text{ in } G,\ A,\ B\in G^*,\\ (3)\ A\circ a = a\circ A=\lambda_A(a) & \text{if } A\in G^*,\ a\in N,\\ (4)\ a\circ b = ab\ \ (=a*b)=0 & \text{if } a,\ b\in N,\end{cases}$$

where $\lambda_A = \eta(A)$, $A\in G$.

Further, every commutative extension of N by G is found in this fashion.

Proof. *The first half of the theorem.* To prove S to be a commutative extension of N by G with respect to the binary operation defined by (2. 1), we need only to show that $S(\circ)$ satisfies the associative law, i. e., $\alpha\circ(\beta\circ\gamma) = (\alpha\circ\beta)\circ\gamma$ for any α, β, $\gamma\in S$. Since we can easily check the relation $\alpha\circ(\beta\circ\gamma)=(\alpha\circ\beta)\circ\gamma$, we omit to give its proof.

The second half of the theorem. Suppose that $S(\circledcirc)$ is a commutative extension of N by G (with a binary operation $\circledcirc$). For every $A\in G^*$, define a mapping $\lambda_A : S \longrightarrow S$ as follows : $\lambda_A(\alpha)=A\circledcirc\alpha$, $\alpha\in S$. Let λ_0 be the 0-mapping on $S(\circledcirc)$ (with respect to N). Since $\lambda_A(0)=A\circledcirc 0=(A\circledcirc 0)\circledcirc 0=0$ and since $\lambda_A(a)=A\circledcirc a\in N$ and $\lambda_0(a)=0$ for any $a\in N$, both λ_A and λ_0 are elements of $\mathscr{T}(S(*), N)$. Define a mapping $\eta : G\longrightarrow (S(*), N)$ as follows : $\eta(A)=\lambda_A$, $A\in G^*$ and $\eta(\mathbf{0})=\lambda_0$. Then it can be easily proved that η is a complete $\mathscr{F}(S(*), N)$-homomorphism of G into $\mathscr{T}(S(*), N)$. Now define a binary operation $\circ$ in S by (2. 1), and let $S(\circ)$ be the resulting system. Then we can prove $S(\circ)=S(\circledcirc)$. In fact : $A\circ B=AB=A\circledcirc B$ if $A, B\in G^*$, $AB\neq\mathbf{0}$ in G ; $A\circ B=\lambda_A(B)=A\circledcirc B$ if A, $B\in G^*$, $AB=\mathbf{0}$ in G ; $A\circ a=\lambda_A(a)=A\circledcirc a$ and $a\circ A=\lambda_A(a)=A\circledcirc a=a\circledcirc A$ if $A\in G^*$, $a\in N$; and $a\circ b=ab=a\circledcirc b$ if a, $b\in N$. Hence $S(\circ)=S(\circledcirc)$. Thus, every commutative extension of N by G is obtained by the method stated in the theorem.

As special cases of Theorem 1, we obtain the following results :

(I) *The case where $G^*=G\backslash\mathbf{0}$ is a subsemigroup of G (i. e., the case where G*

has no non-zero zero divisor) In this case, a mapping $\eta : G \longrightarrow \mathscr{T}(S(*), N)$ is a complete $\mathscr{I}(S(*), N)$-homomorphism if and only if it satisfies the following (2. 2) :

$$(2.2)\quad \begin{cases} (1)\ \lambda_0 = 0\text{-mapping on } S(*) \text{ (with respect to } N), \\ (2)\ \lambda_A \lambda_B = \lambda_{AB} \text{ for } A, B \in G^*, \\ (3)\ \lambda_A(B) = \lambda_B(A) \text{ for } A, B \in G^*, \end{cases}$$

where $\lambda_A = \eta(A)$ for every $A \in G$.

Further, we need not to use (1) and (3) of (2. 2) when we define a binary operation $\circ$ in S by using (2. 1). Hence, in this case, η in Theorem 1 should satisfy only the condition (2) of (2. 2). Consequently, we have the following result :

Theorem 2. *If G has no non-zero zero divisor, then every commutative extension of N by G is constructed by the following manner : Let $\eta : G^* \longrightarrow N$ be a homomorphism of G^* into the semigroup $\mathscr{T}(N)$ of all translations of N. Define a binary operation $\circ$ in S by*

$$(2.3)\quad \begin{cases} (1)\ A \circ B = AB \quad \text{if } A, B \in G^*, \\ (2)\ A \circ a = a \circ A = \lambda_A(a) \quad \text{if } a \in N, A \in G^*, \\ (3)\ a \circ b = ab = 0 \quad \text{if } a, b \in N, \end{cases}$$

where $\lambda_A = \eta(A)$, $A \in G^*$.

Then, the resulting system $S(\circ)$ becomes a commutative extension of N by G.

This result also has been shown by T. Tamura [5]. It is also obvious that $\mathscr{T}(N)$ is the set of all mappings $\varphi : N \longrightarrow N$ such that $\varphi(0) = 0$.

(II) *The case where G satisfies the following* (2. 4) *and the order of N is* 2.

(2. 4) For any element A, there exists a positive integer m such that $A^m = 0$.

(Hereafter, we shall say that a commutative semigroup with zero is *point-wise nilpotent* if it satisfies (2. 4)).

In this case, every λ_A in Theorem 1 is an element of $\mathscr{I}(S(*), N)$. In fact : Let $N = \{u, 0\}$, where $u \neq 0$ and $u^2 = 0$. Since G is point-wise nilpotent, if $A \neq 0$ then $A^{n-1} \neq 0$ and $A^n = 0$ for a positive integer $n \geqq 2$. It is obvious that $\lambda_{A^{n-1}}\lambda_A$ is an element of $\mathscr{I}(S(*), N)$. On the other hand, if $\lambda_A(u) = u$ then $\lambda_A(\lambda_{A^{n-1}}(u)) = \lambda_A(\lambda_A^{n-1}(u)) = u$. Hence $\lambda_{A^{n-1}}\lambda_A \notin \mathscr{I}(S(*), N)$. This is a contradiction. Thus $\lambda_A(u) = 0$, that is, $\lambda_A(N) = \{0\}$. This implies that λ_A is an element of $\mathscr{I}(S(*), N)$. Moreover, it is obvious that the 0-mapping λ_0 on $S(*)$ (with respect to N) is an element of $\mathscr{I}(S(*), N)$. Since η in Theorem 1 is a complete $\mathscr{I}(S(*), N)$-homomorphism, we have the following result for elements A, B of G^* with $AB = 0$: For any $C \in G^*$, $\lambda_A \lambda_B(C) = \lambda_B \lambda_A(C)$

$= \lambda_A(\lambda_B(C)) = \lambda_A(\lambda_C(B)) = \lambda_C \lambda_A(B) = \lambda_A \lambda_C(B) = \lambda_C(\lambda_A(B)) = \lambda_C(u)$ or $\lambda_C(0)$. Since $\lambda_C(u)=0$ and $\lambda_C(0)=0$, we have $\lambda_A \lambda_B(C) = 0$. Consequently, $\lambda_A \lambda_B(\alpha) = 0$ for all $\alpha \in S(*)$. Therefore, $\lambda_A \lambda_B$ is the zero element λ_0 of $\mathscr{F}(S(*), N)$. Hence, η must be a $\{0\}$-homomorphism of G into $\mathscr{F}(S(*), N)$ (hence, of course a homomorphism of G into $\mathscr{F}(S(*), N)$).

By Theorem 1 and the results above, we obtain the following

Theorem 3. *If G is point-wise nilpotent and if the order of N is 2, then every commutative extension of N by G is constructed by the following manner : Let η be a $\{0\}$-homomorphism of G into $\mathscr{F}(S(*), N)$ satisfying* (4), (5) *of* (1. 10). *Define a binary operation $\circ$ in $S = G^* \dot{+} N$ by*

$$(2.5)\quad \begin{cases} A \circ B = AB & \text{if } A, B \in G^*, AB \neq \mathbf{0} \text{ in } G, \\ A \circ B = \lambda_A(B) & \text{if } A, B \in G^*, AB = \mathbf{0} \text{ in } G, \\ A \circ a = a \circ A = 0 & \text{if } A \in G^*, a \in N, \\ a \circ b = 0 & \text{if } a, b \in N, \end{cases}$$

where $\lambda_A = \eta(A)$, $A \in G^*$.

Then, $S(\circ)$ is a commutative extension of N by G.

Corollary. *If G is point-wise nilpotent and if the order of N is 2, then every commutative extension of N by G is a 0-extension of N by G.*

Proof. Obvious.

Next, we shall study commutative 0-extensions of N by G. At first, we introduce the concept of C-*factors* of G : Let Ω be the set of all ordered pairs (A, B) of elements A, B of G such that $AB=\mathbf{0}$ in G (each of A, B can be the zero element $\mathbf{0}$ of G) : $\Omega = \{(A, B) : AB = \mathbf{0} \text{ in } G, A, B \in G\}$.
A non-empty subset Γ of Ω is called a C-factor of G if Γ satisfies the following (2. 6) :

$$(2.6)\quad \begin{cases} (1) \text{ For } A, B \in G, (A, B) \in \Gamma \text{ implies } (B, A) \in \Gamma. \\ (2) \text{ For } A, B, C \in G, (AB, C) \in \Gamma \text{ implies } (A, BC) \in \Gamma. \end{cases}$$

In particular, a C-factor Γ of G is called a *principal C-factor* if it satisfies the following (2. 7) :

(2. 7) $(A, \mathbf{0}), (\mathbf{0}, A) \in \Gamma$ for all $A \in G$.

To each element a of N, assign a C-factor of G or the empty subset of Ω, say Γ_a. Let $\mathfrak{S}$ be the set of all Γ_a, $a \in N$: $\mathfrak{S} = \{\Gamma_a : a \in N\}$. Then $\mathfrak{S}$ is called a *composite systsm of C-factors of G relative to N* if it satisfies the following (2. 8) :

$$(2.8)\quad \begin{cases} (1)\ \Gamma_0 \text{ is a principal C-factor,} \\ (2)\ \Omega=\bigcup\ \{\Gamma_a:\ a\in N\}, \\ (3)\ \Gamma_a\cap\Gamma_b=\Box \text{ for } a\neq b,\ a,\ b\in N. \end{cases}$$ [4)]

Remark. In particular, let the order of N be 2 : $N=\{u, 0\}$, $u^2=0$, $u\neq 0$. If Γ_0 is a principal C-factor of G, then $\Omega\backslash\Gamma_0$ is a C-factor or the empty subset of Ω and the collection $\{\Gamma_0,\ \Omega\backslash\Gamma_0\}$ of Γ_0 and $\Omega\backslash\Gamma_0$ is a composite system of C-factors of G relative to N. Conversely, it is easily seen that every composite system of C-factors of G relative to N is obtained by this method if the order of N is 2.

Now, we obtain the following construction theorem for commutative 0-extensions of null semigroups :

Theorem 4. *Every commutative* 0-*extension of* N *by* G *is constructed by the following manner : Let* $\{\Gamma_a : a\in N\}$ *be a composite system of C-factors of* G *relative to* N, *and define a binary operation* $\circ$ *in* $S=G^*\dot{+}N$ *by*

$$(2.9)\quad \begin{cases} A\circ B=AB \ \ if\ AB\neq 0\ \ in\ G,\ A, B\in G^*, \\ A\circ B=a\ \ if\ AB=0\ in\ G,\ A, B\in G^*,\ (A, B)\in\Gamma_a,\ a\in N, \\ A\circ a=a\circ A=0\ \ if\ a\in N,\ A\in G^*, \\ a\circ b=ab=0\ \ if\ a, b\in N. \end{cases}$$

Then $S(\circ)$ *becomes a commutative* 0-*extension of* N *by* G.

Proof. *The first half of the theorem.* Let $S(\circ)$ be the set S with the binary operation $\circ$ defined by (2. 9). To prove $S(\circ)$ to be a commutative 0-extension of N by G, we need only to show that $S(\circ)$ satisfies the associative law $(\alpha\circ\beta)\circ\gamma=\alpha\circ(\beta\circ\gamma)$. If two or all of α, β, γ are elements of N, then each of $\alpha\circ(\beta\circ\gamma)$ and $(\alpha\circ\beta)\circ\gamma$ is the zero element of N. Hence, in this case $(\alpha\circ\beta)\circ\gamma=\alpha\circ(\beta\circ\gamma)$ is satisfied. Therefore, we may consider only the case where at most one of α, β, γ is contained in N and the others are contained in G^*.

Case 1. $\alpha=a$, $\beta=B$, $\gamma=C(a\in N\ ;\ B, C\in G^*)$. It is obvious that $a\circ(B\circ C)=0$ and $(a\circ B)\circ C=0\circ C=0$. Hence $a\circ(B\circ C)=(a\circ B)\circ C$. Both $A\circ(b\circ C)=(A\circ b)\circ C$ and $A\circ(B\circ c)=(A\circ B)\circ c$, where $A, B, C\in G^*$ and $b, c\in N$, are also proved by similar methods.

Case 2. $\alpha=A$, $\beta=B$, $\gamma=C$ $(A, B, C\in G^*)$.

Subcase (i). *The case where* $ABC\neq 0$ *in* G. Since $ABC\neq 0$, $(A\circ B)\circ C=(AB)C=A(BC)=A\circ(B\circ C)$.

4) The symbol $\Box$ means the empty set.

Subcase (ii). *The case where* $ABC=\mathbf{0}$ *in* G. If $AB=\mathbf{0}$ in G, then $(AB, C)=(\mathbf{0}, C)$. Suppose that $(\mathbf{0}, C)$ is contained in Γ_a. Then, $a=0$. Since $(AB, C)=(\mathbf{0}, C)\in\Gamma_0$, $(A, BC)\in\Gamma_0$. Hence $A\circ(B\circ C)=0$ if $BC=\mathbf{0}$, and also $A\circ(B\circ C)=0$ even if $BC\neq\mathbf{0}$. Thus in both of the cases, we have $A\circ(B\circ C)=0$. On the other hand, $(A\circ B)\circ C=u\circ C=0$ for some $u\in N$. Therefore $A\circ(B\circ C)=(A\circ B)\circ C$. If $BC=\mathbf{0}$ in G, then in this case $A\circ(B\circ C)=(A\circ B)\circ C$ can be proved by a similar method. Finally, assume that $AB\neq\mathbf{0}$ and $BC\neq\mathbf{0}$. Then since $(A\circ B)\circ C=AB\circ C=a$ and $A\circ(B\circ C)=A\circ BC=a$, we have $(A\circ B)\circ C=A\circ(B\circ C)$.

The second half of the theorem. Let $S(\circledcirc)$ be a commutative 0-extension of N by G, with a binary operation $\circledcirc$. Let $\Omega=\{(A, B) : AB=\mathbf{0}$ in G, $A, B\in G\}$, $\Gamma_a=\{(A, B) : A\circledcirc B=a,\ A, B\in G^*\}$ for each $a\neq 0$, $a\in N$, and $\Gamma_0=\{(A, B) : A\circledcirc B=0,\ A, B\in G^*\}\cup\{(A, \mathbf{0}) : A\in G\}\cup\{(\mathbf{0}, A) : A\in G\}$. Then it is obvious that $\Omega=\cup\{\Gamma_a : a\in N\}$ and $\Gamma_a\cap\Gamma_b=\square$ for $a\neq b$, $a, b\in N$. Let a be a non-zero element of N. Suppose that (A, B) is an element of Γ_a. Then $A\circledcirc B=a$, and hence $B\circledcirc A=a$. Hence $(B, A)\in\Gamma_a$. If $(AB, C)\in\Gamma_a$, then $AB\circledcirc C=a$. Since $AB\circledcirc C=a$, we have $A\circledcirc(B\circledcirc C)=(A\circledcirc B)\circledcirc C=AB\circledcirc C=a$. Hence, $BC\neq\mathbf{0}$ and $A\circledcirc BC=a$. Therefore, $(A, BC)\in\Gamma_a$. Consequently, Γ_a is a C-factor of G. Next we prove that Γ_0 is a principal C-factor of G. If $A\neq\mathbf{0}$, $B\neq\mathbf{0}$ and $(A, B)\in\Gamma_0$, then $A\circledcirc B=0$. Since $0=A\circledcirc B=B\circledcirc A$, (B, A) is an element of Γ_0. If $A=\mathbf{0}$ or $B=\mathbf{0}$ and if $(A, B)\in\Gamma_0$, then clearly $(B, A)\in\Gamma_0$. Next, let (AB, C) be an element of Γ_0. If $C=\mathbf{0}$, then $(A, BC)=(A, \mathbf{0})\in\Gamma_0$. If $C\neq\mathbf{0}$ and $BC=\mathbf{0}$, then $(A, BC)=(A, \mathbf{0})\in\Gamma_0$. If $BC\neq\mathbf{0}$ and $A=\mathbf{0}$, then $(A, BC)=(\mathbf{0}, BC)\in\Gamma_0$. Finally, assume that $BC\neq\mathbf{0}$, $A\neq\mathbf{0}$ and $C\neq\mathbf{0}$. If $AB=\mathbf{0}$, then $(A\circledcirc B)\circledcirc C=u\circledcirc C$ for some $u\in N$. Since $u\circledcirc C=0$, we have $(A\circledcirc B)\circledcirc C=0$. Hence $0=(A\circledcirc B)\circledcirc C=A\circledcirc(B\circledcirc C)=A\circledcirc BC$, and hence $(A, BC)\in\Gamma_0$. If $AB\neq\mathbf{0}$, then $(A\circledcirc B)\circledcirc C=AB\circledcirc C$. Since $(AB, C)\in\Gamma_0$, $AB\circledcirc C=0$. Hence $0=AB\circledcirc C=(A\circledcirc B)\circledcirc C=A\circledcirc(B\circledcirc C)=A\circledcirc BC$. This means that (A, BC) is an element of Γ_0. Thus, $\Gamma_0\ni(AB, C)$ implies $(A, BC)\in\Gamma_0$. It is obvious that $\Gamma_0\ni(A, \mathbf{0}), (\mathbf{0}, A)$ for all $A\in G$. Therefore, Γ_0 is a principal C-factor of G. From the results above, it is easy to see that the collection $\{\Gamma_a : a\in N\}$ is a composite system of C-factors of G relative to N. Let $S(\circ)$ be the commutative 0-extension of N by G determined by the system $\{\Gamma_a : a\in N\}$ and the binary operation $\circ$ given by (2. 9). Then, $A\circ B=AB=A\circledcirc B$ if $AB\neq\mathbf{0}$, $A, B\in G^*$; $A\circ B=a=A\circledcirc B$ if $AB=\mathbf{0}$, $A, B\in G^*$, $(A, \mathrm{B})\in\Gamma_a$; $A\circ b=0=A\circledcirc b$ and $b\circ A=0=b\circledcirc A$ if $A\in G^*$, $b\in N$; $a\circ b=0=a\circledcirc b$ if $a, b\in N$. Hence, $S(\circ)=S(\circledcirc)$.

In particular, for the case where G is point-wise nilpotent and the order of N is 2, we can paraphrase Theorem 3 in the following form by using Corollary to Theorem 3 and Remark of p. 14 :

Theorem 3*. *If G is point-wise nilpotent and if the order of N is 2* ($N=\{u, 0\}$ *such that* $u^2=0$, $u\neq 0$), *then every commutative* (0-) *extension of N by G is constructed by the following manner : Let* $\Omega=\{(A, B) : AB=\mathbf{0}$ *in* G, $A, B\in G\}$. *Let Γ_0 be any principal C-factor of G, and define a binary operation $\circ$ in* $S=G^*\dot{+}N$ *by*

$$(2.10)\quad \begin{cases} A\circ B=AB & \text{if } AB\neq\mathbf{0},\ A, B\in G^*, \\ A\circ B=u & \text{if } AB=\mathbf{0},\ A, B\in G^*,\ (A, B)\in\Omega\backslash\Gamma_0, \\ A\circ B=0 & \text{if } AB=\mathbf{0},\ A, B\in G^*,\ (A, B)\in\Gamma_0, \\ A\circ a=a\circ A=0 & \text{if } A\in G^*,\ a\in N, \\ a\circ b=ab=0 & \text{if } a, b\in N. \end{cases}$$

Then, the resulting system $S(\circ)$ is a commutative (0-) *extension of N by G.*[5)]

Proof. Obvious.

Remark. Suppose that G is point-wise nilpotent and N is finite. In this case, every commutative extension of N by G is constructed by the following manner : Let $N=\{a_1, a_2, \ldots, a_n, 0\}$, where 0 is the zero element of N, $a_i\neq a_j$ for $i\neq j$, $a_i\neq 0$ for all i. For every a_i, let $N_i=\{a_i, 0_i\}$ be a null semigroup of order 2 such that $a_i^2=0_i^2=a_i 0_i=0_i a_i=0_i$. Put $G=G_0$, $G_0\backslash\mathbf{0}=G_0^*$, where $\mathbf{0}$ is the zero element of G, and let G_i ($i=1\sim n$) be a commutative extension of N_i by G_{i-1} such that $ab=0_i$ for all a, $b\in G^*_{i-1}\backslash G_0^*$, where $G^*_{i-1}=G_{i-1}\backslash 0_{i-1}$ if $i\geq 2$. Since each G_{i-1} is of course point-wise nilpotent, each G_i can be obtained from G_{i-1} and N_i by slightly modifying the method of Theorem 3* (that is, by adding the following restriction to the method of Theorem 3* : Restriction. A principal C-factor Γ_0 of G_{i-1} contains all pairs (a, b) of elements a, b of $G^*_{i-1}\backslash G_0^*$). Now, it is obvious that G_n is a commutative extension of N by G. Conversely, it is also easy to see that every commutative extension of N by G is obtained by this method.

By Theorem 4, the problem of constructing all commutative 0-extensions of a null semigroup N by a commutative semigroup G with zero is reduced to the problem of determining all composite systems of C-factors of G relative to N. Hence, we shall discuss this problem in the next paragraph.

§ 3. C-factors and principal C-factors.

Let M be a commutative semigroup with zero, and 0 the zero element of

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5) This theorem is a generalization of Theorem 3 given by the author [6] (see also Theorem 5 of [7]).

M. Put $M\backslash 0=M^*$, and $\Omega=\{(a, b) : ab=0, a, b\in M\}$. Hereafter, through this paragraph, we denote elements of M by small letters a, b, c etc. At first, we determine all C-factors and principal C-factors of M. For this purpose, we introduce some necessary concepts.

(I) Two elements (u, v) and (w, t) of Ω is said to be *chainable* if the following (3. 1) holds :

(3. 1)
(a) $(u, v)=(w, t)$ or
(b) there exist elements $t_0, t_1, t_2, \ldots, t_r$ (where r is an even integer$\geqq 2$) of M such that $t_0=v$, $u=t_1t_2$, $t_0t_1=t_3t_4, \ldots, t_{r-6}t_{r-5}=t_{r-3}t_{r-2}$, $t_{r-4}t_{r-3}=t_{r-1}t_r$, $t_{r-2}t_{r-1}=w$, $t_r=t$ ($t_0=v$, $u=t_1t_2$, $vt_1=w$, $t_2=t$ if $r=2$) in M.

(II) Define a relation ρ on Ω as follows : $(u, v)\ \rho\ (w, t)$ if and only if $(u, v), (w, t)$ are chainable ; $(u, v), (t, w)$ are chainable ; $(v, u), (w, t)$ are chainable ; or $(v, u), (t, w)$ are chainable.

Lemma 1. *ρ is an equivalence relation on Ω.*

Proof. It is obvious that $(u, v)\ \rho\ (u, v)$ for all $(u, v)\in\Omega$. Next, we prove that $(u, v)\ \rho\ (w, t)$ implies $(w, t)\ \rho\ (u, v)$. Suppose that $(u, v)\ \rho\ (w, t)$. Then, (1) $(u, v), (w, t)$ are chainable, (2) $(u, v), (t, w)$ are chainable, (3) $(v, u), (w, t)$ are chainable or (4) $(v, u), (t, w)$ are chainable. If $(v, u), (w, t)$ are chainable, then $(v, u)=(w, t)$ or there exist elements $t_0, t_1, \ldots, t_r\in M$ such that $u=t_0$, $v=t_1t_2$, $t_0t_1=t_3t_4, \ldots, t_{r-4}t_{r-3}=t_{r-1}t_r$, $t_{r-2}t_{r-1}=w$, $t_r=t$. If $(v, u)=(w, t)$, then $(w, t), (v, u)$ are chainable. Hence $(w, t)\ \rho\ (u, v)$. If $(v, u)\neq(w, t)$, then $t=s_0$, $w=s_1s_2$, $s_0s_1=s_3s_4, \ldots, s_{r-2}s_{r-1}=v$, $s_r=u$ where $s_i=t_{r-i}$ ($i=0\sim r$). Hence $(w, t), (v, u)$ are chainable, and hence $(w, t)\ \rho\ (u, v)$. Thus in the case (3), we proved $(w, t)\ \rho\ (u, v)$. In the other cases (1), (2) and (4), we can also prove $(w, t)\ \rho\ (u, v)$ by similar methods. Finally. we prove that $(u, v)\ \rho\ (w, t)$, $(w, t)\ \rho\ (s, x)$ imply $(u, v)\ \rho\ (s, x)$. Suppose that $(u, v)\ \rho\ (w, t)$ and $(w, t)\ \rho\ (s, x)$. Then (1) $(u, v), (w, t)$ are chainable, (2) $(u, v), (t, w)$ are chainable, (3) $(v, u), (w, t)$ are chainable or (4) $(v, u), (t, w)$ are chainable, and (1′) $(w, t)\ (s, x)$ are chainable, (2′) $(w, t), (x, s)$ are chainable, (3′) $(t, w), (s, x)$ are chainable or (4′) $(t, w), (x, s)$ are chainable. For each case $\{(i), (j')\}$ $(i, j=1, 2, 3, 4)$, we should prove that $(u, v)\ \rho\ (s, x)$.

However, we omit its complete proof and prove it only in the case $\{(1), (4')\}$ since we can prove it by similar methods in the other cases. Suppose that $(u, v,)\ (w, t)$ are chainable and $(t, w), (x, s)$ are chainable. If $(u, v)=(w, t)$, then $(v, u)=(t, w)$. Hence $(v, u), (x, s)$ are chainable, and hence $(u, v)\ \rho\ (s, x)$. If $(t, w)=(x, s)$, then $(w, t)=(s, x)$. Hence $(u, v), (s, x)$ are chainable, and hence $(u, v)\ \rho\ (s, x)$. Now, assume that $(u, v)\neq(w, t)$ and $(t, w)\neq(x, s)$. In this case, there exist elements $t_0, t_1, \ldots, t_r$ of M and elements

$u_0, u_1, \ldots, u_k$ of M such that $v=t_0$, $u=t_1t_2$, $t_0t_1=t_3t_4, \ldots$, $t_{r-4}t_{r-3}=t_{r-1}t_r$, $t_{r-2}t_{r-1}=w$, $t_r=t$ and $w=u_0$, $t=u_1u_2$, $u_0u_1=u_3u_4, \ldots$, $u_{k-4}u_{k-3}=u_{k-1}u_k$, $u_{k-2}u_{k-1}=x$, $u_k=s$. Since $v=t_0$, $u=t_1t_2$, $t_0t_1=t_3t_4, \ldots$, $t_{r-4}t_{r-3}(=t_{r-1}t_r)=(t_{r-1}u_1)u_2$, $t_{r-2}(t_{r-1}u_1)=u_3u_4$, $u_2u_3=u_5u_6, \ldots$, $u_{k-2}u_{k-1}=x$, $u_k=s$, (u, v) and (x, s) are chainable. Hence $(u, v)\ \rho\ (s, x)$. Thus, ρ is an equivalence relation on Ω.

Hereafter, $E(u, v)$ will denote the equivalence class $(\in \Omega/\rho)$ containing (u, v).

Lemma 2. (1) *If* $\{\Gamma_\alpha : \alpha \in I\}$ *is a collection of C-factors* Γ_α *of* M, *then* $\bigcap \{\Gamma_\alpha : \alpha \in I\}$ *is also a C-factor of* M.

(2) *If* $\{\Lambda_\xi : \xi \in I^*\}$ *is a collection of principal C-factors* Λ_ξ *of* M, *then* $\bigcap \{\Lambda_\xi : \xi \in I^*\}$ *is also a principal C-factor of* M.

Proof. Obvious.

For $(u, v) \in \Omega$, let $\{\Gamma_\alpha : \alpha \in I\}$ be the collection of all C-factors Γ_α containing (u, v). Let $\{\Lambda_\xi : \xi \in I^*\}$ be the collection of all principal C-factors Λ_ξ. Then $\Gamma(u, v)=\bigcap \{\Gamma_\alpha : \alpha \in I\}$ is the least C-factor containing (u, v), and $\Lambda_0=\bigcap \{\Lambda_\xi : \xi \in I^*\}$ is the least principal C-factor of M.

Theorem 5.

(1) $\Lambda_0=\{(w, t)\in \Omega : (w, t)\ \rho\ (v, 0)$ *(hence also* $(w, t)\ \rho\ (0, v)$*) for some* $v \in M\}$.

(2) $\Gamma(u, v)=\{(w, t)\in \Omega : (w, t)\ \rho\ (u, v)$ *(hence also* $(w, t)\ \rho\ (v, u)$*)*$\}=E(u, v)$.

Proof. This can be proved by slightly modifying the proofs of Theorems 7 and 8 of the author [7] (see also [6]).

Remark. "C-factor" in the author [6], [7] means "principal C-factor" of this paper.

Theorem 6. $\Gamma(u, v) \cap \Gamma(w, t)=\square$ *or* $\Gamma(u, v)=\Gamma(w, t)$.

Proof. By Theorom 5, $\Gamma(u, v)=E(u, v)$ and $\Gamma(w, t)=E(w, t)$. Since $E(u, v)$ and $E(w, t)$ are equivalence classes modulo ρ, $E(u, v) \cap E(w, t)=\square$ or $E(u, v)=E(w, t)$.

Theorem 7. *If* Λ *is a principal C-factor of* M, *then*

(1) $\Lambda \supset \Lambda_0$, *and*

(2) Λ *is a disjoint sum of C-factors of* M.

Conversely, a subset Γ *of* Ω *is a principal C-factor of* M *if* $\Gamma \supset \Lambda_0$ *and if* Γ *is a disjoint sum of C-factors of* M.

Proof. It is obvious that a principal C-factor Λ of M contains Λ_0. Let (u, v) be an element of Λ. Since Λ is a C-factor of M, it follows that $\Lambda \supset \Gamma(u, v)$.

Hence, $\Lambda \supset \cup\{\Gamma(u, v) : (u, v)\in\Lambda\}$. Since $\Lambda\subset\cup\{\Gamma(u, v) : (u, v)\in\Lambda\}$ is obvious, $\Lambda=\cup\{\Gamma(u, v) : (u, v)\in\Lambda\}$ holds. It is also obvious that any two distinct C-factors $\Gamma(u, v)$ and $\Gamma(w, s)$, where $(u, v), (w, s)\in\Lambda$, are disjoint. The second half of the theorem is obvious.

Theorem 8. *Every C-factor* X *of* M *is a union of C-factors* $\Gamma(u, v)$. *Conversely, a union of C-factors* $\Gamma(u, v)$ *is a C-factor of* M.

Proof. This can be proved by a similar method to the proof of Theorem 7.

Remark. By Theorem 6, each $\Gamma(u, v)$ is a minimal C-factor of M. Conversely, it is obvious that if Γ is a minimal C-factor of M and if $\Gamma\ni(u, v)$ then $\Gamma=\Gamma(u, v)$. Hence by Theorems 7 and 8, it should be also noted that any principal C-factor of M [any C-factor of M] is a disjoint sum of minimal C-factors of M.

Now, let G be a commutative semigroup with zero, and N a null semigroup. Let $\mathbf{0}$ and 0 be the zero elements of G and N respectively. Put $G\backslash\mathbf{0}=G^*$. Let us denote elements of G^* by capital letters A, B, C etc., and those of N by small letters a, b, c etc. Put $\Omega=\{(A, B) : A, B\in G, AB=\mathbf{0}$ in $G\}$. Let $\mathfrak{M} = \{\Gamma_\alpha : \alpha\in I\}$ be the set of all minimal C-factors of G. By Theorem 6 and Remark mentioned above, it is obvious that $\Gamma_\alpha\cap\Gamma_\beta=\square$ for $\alpha\neq\beta$. Now, consider any mapping $\varphi : \mathfrak{M}\longrightarrow N$ satisfying the following (3. 1) :

(3. 1) $\varphi(\Gamma_\alpha)=0$ if Γ_α contains some element $(A, \mathbf{0})\in\Omega$, $A\in G$.

For each $a\in N$, put $\mathfrak{M}_a=\{\Gamma_\beta : \varphi(\Gamma_\beta)=a, \Gamma_\beta\in\mathfrak{M}\}$ and $\Gamma_a=\cup\{\Gamma_\beta : \Gamma_\beta \in\mathfrak{M}_a\}$. (If $\mathfrak{M}_a$ is empty, Γ_a means the empty subset of Ω).
Then, the following (3. 2) follows from the results stated above :

(3. 2)
- (1) Γ_0 is a principal C-factor of G,
- (2) each Γ_a is a C-factor of G or the empty subset of Ω,
- (3) $\Omega=\cup\{\Gamma_a : a\in N\}$,
- (4) $\Gamma_a\cap\Gamma_b=\square$ if $a\neq b$, $a, b\in N$.

Hence, $\{\Gamma_a : a\in N\}$ is a composite system of C-factors of G relative to N. Conversely, it is easy to see that every composite system of C-factors of G relative to N is obtained by this method.

§ 4. Applications.

In this paragraph, we shall give some applications of Theorems 1—4 obtained in § 2. As a preparation for this purpose, at first we introduce here the following well-known result given by *A. H.* Cliffod [2] :

Theorem. (Clifford) *Let S be a commutative semigroup with* 1, *and T a commutative semigroup with zero. Let* **0** *be the zero element of T, and put $T\backslash 0=T^*$. Let $\eta : T^* \longrightarrow S$ be a partial homomorphism of the commutative halfgroupoid T^* into S. Then, $G=S\dot{+}T^*$ is a commutative extension of S by T with respect to the binary operation $\circ$ defined as follows :*

$$(4.1)\quad \begin{cases} (1)\ A\circ B=AB & \text{if } A, B\in T^*,\ AB\neq 0, \\ (2)\ A\circ B=\overline{AB} & \text{if } A, B\in T^*,\ AB=0, \\ (3)\ A\circ s=s\circ A=\bar{A}s & \text{if } A\in T^*,\ s\in S, \\ (4)\ s\circ t=st & \text{if } s, t\in S, \end{cases}$$

where $\bar{A}=\eta(A)$ for $A\in T^*$.

Further, every commutative extension of S by T is found in this fashion.

Next, we shall give some necessary definitions. By a *semigroup with chain conditions,* we mean a semigroup which satisfies the ascending chain condition and the descending chain condition for ideals. *A* semigroup S with zero is said to be *nilpotent* if it satisfies the following (4. 2) : [6)]

(4. 2) $S\supset S^2\supset S^3\supset\cdots\cdots\supset S^{n-1}\supset S^n=\{0\}$ for some positive integer n, where 0 is the zero element of S.

A commutative nilpotent semigroup is of course point-wise nilpotent, but a commutative point-wise nilpotent semigroup is not necessarily nilpotent. However, it is easily verified that, for finite commutative semigroups S with zero, the properties "point-wise nilpotent" and "nilpotent" are equivalent. Further, for S, each of these properties is equivalent to the property "having no idempotent except zero" (in [6], a semigroup with zero which satisfies this property has been called a z-semigroup).

(I) *Commutative semigroups with chain conditions.*

Let S be a commutative semigroup with chain conditions. Then, S has a principal series $S=S_1\supsetneqq S_2\supsetneqq\cdots\supsetneqq S_m\supsetneqq S_{m+1}=\square$ and each principal factor S_i/S_{i+1} $(i=1\sim m\text{-}1)$ is a commutative group with zero or a null semigroup. Further, S_m is a commutative group, a commutative group with zero or a null semigroup. Hence, S^0 (the adjunction of a zero element to S) is contained in the class $\mathfrak{S}$ of commutative semigroups constructed as follows : Let T_0, $T_1,\ldots,$ T_n (n is an arbitrary non-negative integer) be arbitrary sequence of semigroups such that each T_i is a commutative group with zero or a null semigroup.

6) The meaning of "nilpotent" is somewhat different from that of the author [7]. In [7], a semigroup S with kernel K has been called nilpotent if $S\supset S^2\supset S^3\supset\cdots\cdots\supset S^n=K$ for some positive integer n. However, both meanings of "nilpotent" are equivalent for semigroups with zero.

Put $T_0 = G_0$, and for $i \geqq 1$ let G_i be a commutative extension of T_i by G_{i-1}. Then, at last we can get G_n. Let $\mathfrak{S}$ be the set of all G_n obtained by this-method. Every commutative extension of a commutative group with zero by a commutative semigroup with zero is obtained by the method of Theorem of Clifford given at the first part of this paragraph. On the other hand, every commutative extension of a null semigroup by a commutative semigroup with zero is obtained by the method of Theorem 1.

(II) *Commutative nilpotent semigroups.*

Let S be a commutative nilpotent semigroup. Then, there exists a positive integer m such that $S \supsetneqq S^2 \supsetneqq \cdots\cdots \supsetneqq S^m = \{0\}$. Each of the Rees factor semigroups $S^n/S^{n+1} = N_n$ ($n=1 \sim m$-1) is a null semigroup. Put $G_n = S/S^n$ ($n=1 \sim m$). Then, G_{n+1} is a commutative 0-extension of N_n by G_n (where G_2 is regarded as N_1). Since $G_m = S$, S is contained in the class $\mathscr{N}$ of commutative semigroups constructed as follows : Let $T_0, T_1, \ldots, T_n$ (n is an arbitrary non-negative integer) be arbitrary null semigroups. Put $T_0 = G_0$, and for $i \geqq 0$ let G_{i+1} be a commutative 0-extension of T_{i+1} by G_i. Then, at last we can get G_n. Let $\mathscr{N}$ be the set of all G_n obtained by this method.

Conversely, it is easy to see that any semigroup contained in $\mathscr{N}$ is a commutative nilpotent semigroup. It is also obvious that every commutative 0-extension of a null semigroup by a commutative semigroup with zero is obtained by the method of Theorem 4.

(III) *Finite commuative nilpotent semigroups.*

Let S be a finite commutative nilpotent semigroup. Then there exists a composition series of S such that $S = S_0 \supsetneqq S_1 \supsetneqq \cdots \supsetneqq S_{n+1} = \{0\}$, where each $S_i/S_{i+1} = N_i$ ($i = 0 \sim n$-1) and $S_n = N_n$ are null semigroups of order 2 (see [6], [7] and [3]). Put $S/S_j = G_j$. Then G_j is a finite commutative nilpotent semigroup (hence, G_j is also point-wise nilpotent), and $G_{n+1} = S$. Hence S is contained in the class $\mathscr{F}$ of commutative semigroups constructed as follows : Let T_0, $T_1, \ldots, T_n$ (n is an arbitrary non-negative integer) be a sequence of null semigroups of order 2. Put $T_0 = G_0$, and let G_{i+1} ($i = 0 \sim n$-1) be a commutative (0-) extension of T_{i+1} by G_i. In this case, each G_i is obviously a finite commutative nilpotent semigroup and hence is a commutative point-wise nilpotent semigroup. Then, at last we can get G_n. Let $\mathscr{F}$ be the set of all G_n obtained by this method. It is obvious that every commutative (0-) extension of a null semigroup of order 2 by a commutative point-wise nilpotent semigroup is obtained by the method of Theorems 3 and 3*. (This result has been also obtained by the author [6], [7]). Conversely, it is also easy to see that any semigroup contained in $\mathscr{F}$ is a finite commutative nilpotent semigroup.

(IV) *Finite commutative semigroups containing null semigroups as their maximal ideals.*

Let S be a finite commutative semigroup containing a null semigroup I as its maximal ideal. Then the Rees factor semigroup S/I is a 0-simple semigroup or a null semigroup of order 2. Since S/I is commutative, S/I must be a commutative group with zero or a null semigroup of order 2. Hence S is a commutative extension of a finite null semigroup by a finite commutative group with zero or by a null semigroup of order 2, and hence S is contained in the class $\mathscr{G}$ of commutative semigroups constructed as follows : Let T_0 be an arbitrary finite null semigroup, and T_1 an arbitrary finite commutative group with zero or an arbitrary null semigroup of order 2. Let T be a commutative extension of T_0 by T_1. Then T is a finite commutative semigroup containing T_0 as its maximal ideal. Now, let $\mathscr{G}$ be the set of all T obtained by this method. Every commutative extension of a finite null semigroup by a null semigroup of order 2 is obtained by the method of the author and T. Tamura [8] (see also Remark of p. 16) ; and every commutative extension of a finite null semigroup by a finite commutative group with zero is obtained by the method of Theorem 2.

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