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On the Γ-distribution of Matrix Argument and Its Related Distributions

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Recently we have many results of the distribution of the quadratic forms of vector variates with multivariate normal distribution after A. T. James and others. These distributions are related to the Wishart distribution. The Wishart distribution play the analogous role to the Chi-square distribution in the univariate case. *C. S.* Herz [*H*] discusses exhaustively properties of the Bessel function of matrix argument. It is well known that we can derive the Γ-distribution, the B-distribution and the F-distribution in the unified manner in the univariate case. In this paper, we treat the unified derivation of the Γ-distribution, the B-distribution and the F-distribution of matrix argument. Some results may be contained in the paper by *I.* Olkin and *H.* Rubin [0].

§1. Distributions in the central case

Let the matrix S be the (p,p) real symmetric matrix in the following; we define the Γ-distribution $\Gamma_p(\alpha;C)$ of matrix argument as follows:

Definition 1.1.

The matrix S is distributed according to the distribution $\Gamma_p(\alpha;C)$ when its distribution has the probability element

$$(1.1)\quad dF(S)=\begin{cases}\dfrac{1}{\det(C)^{\alpha}\Gamma_p(\alpha)}\operatorname{etr}(-C^{-1}S)\det(S)^{\alpha-\frac{p+1}{2}}dS, \\ \qquad\qquad S \text{ positive definite } (S>0),\ \alpha>\dfrac{p-1}{2} \\ 0, \quad \text{otherwise}\end{cases}$$

, where C is any (p,p) real positive definite matrix, and etr($\cdot$) denotes exp (trace), and $\Gamma_p(\alpha)=\pi^{\frac{p(p-1)}{4}}\prod_{i=1}^{p}\Gamma\left(\alpha-\frac{i-1}{2}\right)$. We call the case of $C=I_p$, the unit matrix of order p, as the canonical form of this distribution. It is the generalization of the central Wishart distribution $W_p(\alpha; C)$.

Proposition 1.1.

The canonical case of the distribution $\Gamma_p(\alpha; C)$ has the constant probability element on the sphere passing through S and with center I_p in the space of real positive definite matrix.

[Proof] The probability element in the canonical case is given in the form

$$dF(S) = \frac{1}{\Gamma_p(\alpha)} \operatorname{etr}(-S) \det(S)^{\alpha - \frac{p+1}{2}} dS.$$

Let us make the transformation $S \rightarrow H'SH$, $H \in 0(p)$ the orthogonal group of degree p, then we have the result

$$dF(H'HS) = dF(S).$$

Q. E. D.

Proposition 1.2.

If the (p, p) matrix S is distributed according to $\Gamma_p(\alpha; I_p)$, then eigenvalues and eigenvectors of matrix S are independently distributed. Moreover, the distribution of the eigen vectors is the uniform distribution over the orthogonal group $0(p)$.

[Proof] It is well known that there exists a factorization of the positive definite real matrix S as $S = HD_{\lambda^2}H'$, where $H \in 0(p)$ and D_{λ^2} is a diagonal matrix with diagonal elements $\lambda_1^2 \geqq \lambda_2^2 \geqq \ldots \geqq \lambda_p^2 > 0$, and this factorization is uniquely defined whenever $\lambda_1^2 > \lambda_2^2 > \ldots > \lambda_p^2$. This latter condition holds with probability one. As performing the above factorization, the corresponding volume element dS is expressed as

$$(1.2) \quad dS = \prod_{i<j}^{p} (\lambda_i^2 - \lambda_j^2) \prod_{i=1}^{p} d\lambda_i^2 \frac{1}{2^p} \prod_{i<j}^{p} h_j' \, dh_i,$$

where $H = [h_1, h_2, \ldots, h_p]$.

Then the probability element of the distribution $\Gamma_p(\alpha; I_p)$ is expressed as

$$(1.3) \quad \frac{1}{2^p \Gamma_p(\alpha)} \operatorname{etr}(-D_{\lambda^2}) \det(D_{\lambda^2})^{\alpha - \frac{p+1}{2}} \prod_{i<j}^{p} (\lambda_i^2 - \lambda_j^2) \prod_{i=1}^{p} d\lambda_i^2 \cdot \prod_{i<j}^{p} h_j' dh_i.$$

In this expression, the term $\prod_{i<j}^{p} h_j' dh_i$ is the differential form corresponding to the invariant measure on the group $0(p)$, that is,

$$\int_{0(p)} \prod_{i<j}^{p} h_j' dh_i = \prod_{i=1}^{p} \frac{2\pi^{i/2}}{\Gamma\left(\frac{i}{2}\right)}.$$

Q. E. D.

We call a function of the symmetric real matrix S as the symmetric function of the matrix S whenever it satisfies the expression $f(S)=f(HSH')$ for any $H \in 0(p)$ and distributions with symmetric probability element as the symmetric distribution. We have the following generalizations of Prop. 1 and Prop. 2:

Proposition 1.1.'

For the symmetric distribution, the distribution has the constant probability element on every sphere with center I_p.

Proposition 1.2.'

The distributions of eigenvalues and eigenvectors of the matrix are indepen-dendently distributed and the distributions of the latter is the uniform distribution on the orthogonal group $0(p)$ if and only if the distribution is symmetric. These two propositions are shown in the same way as Prop. 1 and Prop. 2.

Propotition 1.3.

The characteristic function of the distribution $\Gamma_p(\alpha; C)$ is given by the expression

$$(1.4) \quad \varphi(\theta)=\det(I_p-i\theta C)^{-\alpha},$$

where the symmetric matrix θ is of the form

$$\theta=(\eta_{ij}\theta_{ij}), \qquad \eta_{ii}=1,\ \eta_{ij}=\frac{1}{2}\ (i\neq j).$$

[Proof] We calculate $\varphi(\theta)=E\ \mathrm{etr}(i\,\theta\,S)$:

$$\begin{aligned} E\ \mathrm{etr}(i\,\theta\,S) &= \frac{1}{\det(C)^{\alpha}\,\Gamma_p(\alpha)}\int_{S>0}\mathrm{etr}(i\,\theta\,S)\,\mathrm{etr}(-C^{-1}S)\det(S)^{\alpha-\frac{p+1}{2}}\,dS \\ &= \frac{1}{\det(C)^{\alpha}\,\Gamma_p(\alpha)}\int_{S>0}\mathrm{etr}\left[-(C^{-1}-i\,\theta)S\right]\det(S)^{\alpha-\frac{p+1}{2}}\,dS \\ &= \det(I_p-i\,\theta\,C)^{-\alpha} \end{aligned}$$

Q. E. D.

Corollary

Let matrices S_1 and S_2 be independently distributed, each according to $\Gamma_p(\alpha_1; C)$ and $\Gamma_p(\alpha_2; C)$, then the matrix S_1+S_2 is distributed according to $\Gamma_p(\alpha_1+\alpha_2; C)$.

Definition 1.2.

Let matrices $S_1, S_2, \ldots, S_k$ be (p, p) real symmetric matrices; whenever their joint distribution has the probability element

$$(1.5)\quad dF(S_1, S_2, \ldots, S_k) = \begin{cases} \dfrac{1}{B_p(\alpha_1, \ldots, \alpha_k ; \alpha_{k+1})} \det(S_1)^{\alpha_1 - \frac{p+1}{2}} \det(S_2)^{\alpha_2 - \frac{p+1}{2}} \cdots \\ \det(S_k)^{\alpha_k - \frac{p+1}{2}} \det(\mathrm{I}_p - S_1 - \ldots - S_k)^{\alpha_{k+1} - \frac{p+1}{2}} dS_1 \\ \ldots dS_k,\ \alpha_1, \ldots, \alpha_{k+1} > \dfrac{p-1}{2},\ S_1, \ldots, S_k > 0 \text{ and} \\ S_1 + \ldots + S_k < \mathrm{I}_p, \\ 0, \qquad \text{otherwise} \end{cases}$$

we say that the set of matrices $(S_1, \ldots, S_k)$ are distributed according to the Dirichlet distribution of matrix argument $Di_p(\alpha_1, \ldots, \alpha_k ; \alpha_{k+1})$. In particular, the case of $k=1$, it is the B-distribution of matrix argument $Be_p(\alpha_1 ; \alpha_2)$. In the above expression

$$B_p(\alpha_1, \ldots, \alpha_k ; \alpha_{k+1}) = \int_{\substack{S_1, \ldots, S_k > 0 \\ S_1 + \ldots + S_k < \mathrm{I}_p}} \det(S_1)^{\alpha_1 - \frac{p+1}{2}} \ldots \det(S_k)^{\alpha_k - \frac{p+1}{2}}$$

$$\det(\mathrm{I}_p - S_1 - \ldots - S_k)^{\alpha_{k+1} - \frac{p+1}{2}} dS_1 \ldots dS_k.$$

$$= \left\{ \frac{\Gamma_p(\alpha_1 + \ldots + \alpha_{k+1})}{\Gamma_p(\alpha_1) \ldots \Gamma_p(\alpha_{k+1})} \right\}^{-1}.$$

The distribution $Be_p(\alpha_1 ; \alpha_2)$ is a symmetric distribution. At present, we might consider some connections between Γ-distribution $\Gamma_p(\alpha ; C)$ and the distribution $Be_p(\alpha ; \beta)$, and connections between distributions $Be_p(\alpha ; \beta)$ and $Di_p(\alpha_1, \ldots, \alpha_k ; \alpha_{k+1})$. These results are summarized in

Proposition 1.4.

Let matrices S_1 and S_2 be independently distributed, each according to $\Gamma_p(\alpha ; C)$ and $\Gamma_p(\beta ; C)$, then the matrix

$$S = (S_1 + S_2)^{-\frac{1}{2}} \quad S_1 (S_1 + S_2)^{-\frac{1}{2}}$$

is distributed according to $Be_p(\alpha ; \beta)$, where $S^{-\frac{1}{2}}$ is the symmetric root of the matrix S^{-1}, that is, $(S^{-\frac{1}{2}})' = S^{-\frac{1}{2}}$.

Corollary

Let matrices S_1 and S_2 be independently distributed, each according to the

Wishart distribution $W_p(m; C)$ and $W_p(n; C)$, then the matrix $S=(S_1+S_2)^{-\frac{1}{2}} S_1(S_1+S_2)^{-\frac{1}{2}}$ is distributed according to the distribution $Be_p\left(\frac{m}{2}; \frac{n}{2}\right)$.

Proposition 1.5.

Let matrices $S_1, S_2, \ldots, S_k$ be independently distributed, each according to $Be_p(\alpha_i, \alpha-\alpha_1-\ldots-\alpha_i)$ $(i=1, 2, \ldots, k)$, then the set of matrices $\theta_i=(\mathrm{I}_p-\theta_1-\ldots-\theta_{i-1})^{\frac{1}{2}} S_i (\mathrm{I}_p-\theta_1-\ldots-\theta_{i-1})^{\frac{1}{2}}$ $(i=1, 2, \ldots, k)$, $\theta_0=0$, is distributed according to the distribution $Di_p(\alpha_1, \ldots, \alpha_k; \alpha-\alpha_1-\ldots-\alpha_k)$. The Prop. 4 is generalized into

Proposition 1.4.'

Let matrices $S_1, \ldots, S_k, S_{k+1}$ be independently distributed, each according to $\Gamma_p(\alpha_i; C)$, then the set of matrices

$$\theta_i=(S_1+\ldots+S_{k+1})^{-\frac{1}{2}} S_i (S_1+\ldots+S_{k+1})^{-\frac{1}{2}} \quad (i=1, \ldots, k)$$

is distributed according to $Di_p(\alpha_1, \ldots, \alpha_k; \alpha_{k+1})$.

[Proof of Prop. 1.4.'] We have the joint distribution for the set of matrices $(S_1, \ldots, S_k; S_{k+1})$,

$$dF(S_1, \ldots, S_k; S_{k+1})=\frac{1}{\det(C)^{\alpha}\Gamma_p(\alpha_1)\ldots\Gamma_p(\alpha_{k+1})}\mathrm{etr}\left[-C^{-1}(S_1+\ldots+S_{k+1})\right]$$

$$\det(S_1)^{\alpha_1-\frac{p+1}{2}}\ldots\det(S_{k+1})^{\alpha_{k+1}-\frac{p+1}{2}}\, dS_1\ldots dS_{k+1},$$

where $\alpha=\alpha_1+\ldots+\alpha_{k+1}$.
Make the transformation

$$S=S_1+\ldots+S_{k+1}, \qquad \theta_i=S^{-\frac{1}{2}} S_i S^{-\frac{1}{2}} \quad (i=1, \ldots, k),$$

then since $S_1, \ldots, S_{k+1}>0$, θ_i $(i=1, \ldots, k)>0$ and $\mathrm{I}_p-\theta_i=S^{\frac{1}{2}}(S-S_i)S^{-\frac{1}{2}}>0$, $\mathrm{II}_p-(\theta_1+\ldots+\theta_k)=S^{-\frac{1}{2}}S^{k+1}S^{-\frac{1}{2}}>0$. Thus the joint distribution of the set of matrices $(\theta_1, \ldots, \theta_k)$ has the support $\theta_i>0$ $(i=1, \ldots, k)$, $\mathrm{I}_p>\theta_1+\ldots+\theta_k$. Under the above transformation the volume element $dS_1\ldots dS_{k+1}$ is transformed into $dS\, d\theta_1\ldots d\theta_k$, which is equivalent to $\det(S)^{-\frac{p+1}{2}k}\, dS_1\ldots dS_k\, dS_{k+1}$, and we have the joint distribution for matrices $\theta_1, \ldots, \theta_k, S$ as

$$dF(\theta_1, \ldots, \theta_k; S)=\frac{1}{\Gamma_p(\alpha_1)\ldots\Gamma_p(\alpha_{k+1})}\det(\theta_1)^{\alpha_1-\frac{p+1}{2}}\ldots\det(\theta_k)^{\alpha_k-\frac{p+1}{2}}$$

$$\det(\mathrm{I}_p-\theta_1-\ldots-\theta_k)^{\alpha_{k+1}-\frac{p+1}{2}}\, d\theta_1\ldots d\theta_k \frac{1}{\det(C)^{\alpha}}\mathrm{etr}(-C^{-1}S)\det(S)^{\alpha-\frac{p+1}{2}}dS,$$

and integrating out with respect to the matrix S, we have the joint distribution

$$dF(\theta_1, \ldots, \theta_k) = \frac{1}{B_p(\alpha_1, \ldots, \alpha_k ; \alpha_{k+1})} \det(\theta_1)^{\alpha_1 - \frac{p+1}{2}}$$

$$\ldots \det(\theta_k)^{\alpha_k - \frac{p+1}{2}} \det(\mathrm{I}_p - \theta_1 - \ldots - \theta_k)^{\alpha_{k+1} - \frac{p+1}{2}} \, d\theta_1 \ldots d\theta_k,$$

the probability element for the distribution $Di_p(\alpha_1, \ldots, \alpha_k ; \alpha_{k+1})$.

Q. E. D.

The Prop. 5 is shown in the same way.

Definition 1.3.

Let a matrax S be a (p, p) real symmetric matrix. Whenever its probability element has the expression

$$(1.6) \quad dF(S) = \begin{cases} \dfrac{1}{B_p\left(\dfrac{\alpha}{2}, \dfrac{\beta}{2}\right)} \dfrac{\det(S)^{\frac{\alpha}{2} - \frac{p+1}{2}}}{\det(\mathrm{I}_p + S)^{\frac{\alpha+\beta}{2}}} \, dS, & S > 0 \text{ and } \alpha, \beta > p-1 \\ 0, & \text{otherwise} \end{cases}$$

we say that matrix S is distributed according to the F-distribution of matrix argument $F_p(\alpha ; \beta)$.

The distribution $F_p(\alpha ; \beta)$ is also the symmetric distribution. Now in the univariate case, we have the following proposition; "Let random variables s_1 and s_2 be independently distributed, each according to $\Gamma_1(\alpha ; C)$ and $\Gamma_1(\beta ; C)$. Then the random variable $S = s_1/s_2$ is distributed according to $F_1(2\alpha ; 2\beta)$." In the case of matrix argument, the corresponding problem is the following; "Let matrices S_1 and S_2 be independently distributed, each according to $\Gamma_p(\alpha ; C)$ and $\Gamma_p(\beta ; C)$. Then how is the distribution of the matrix $S = \theta' S_1 \theta$, where $\theta\theta' = S_2^{-1}$?" I. Olkin and H. Rubin [0] solved this problem under the triangular decomposition of matrix S_2 and they also mention the result for the symmetric root decomposition of matrix S_2 for the case of $C = \mathrm{I}_p$.

At present, the problem for the symmetric root decomposition of matrix S_2 for the general case of matrix C is unsolved, and in the following we see the sufficient facts to solve this problem.

The joint distribution of matrices S_1 and S_2 is given by the expression

$$(1.7) \quad dF(S_1, S_2) = \frac{1}{\Gamma_p(\alpha)\Gamma_p(\beta)\det(C)^{\alpha+\beta}} \operatorname{etr}[-C^{-1}(S_1 + S_2)]$$

$$\det(S)^{\alpha - \frac{p+1}{2}} \det(S_2)^{\beta - \frac{p+1}{2}} \, dS_1 dS_2.$$

Put $S = S_2^{-\frac{1}{2}} S_1 S_2^{-\frac{1}{2}}$ in the expression (1.7), then we have the transformation of volume element as

$$dS_1\, dS_2 = \det(S_2)^{\frac{p+1}{2}}\, dS\, dS_2 \text{ and}$$

$$dF(S, S_2) = \frac{\det(S)^{\alpha-\frac{p+1}{2}}}{\Gamma_p(\alpha)\Gamma_p(\beta)\det(C)^{\alpha+\beta}} \operatorname{etr}[-S_2^{\frac{1}{2}} C^{-1} S_2^{\frac{1}{2}}(\mathrm{I}_p+S)]$$
$$\det(S_2)^{\alpha+\beta-\frac{p+1}{2}} dS_2 dS,$$

thus the distribution of the matrix S in given by the formula

$$(1.8)\quad dF(S) = \frac{1}{B_p(\alpha,\beta)} \det(S)^{\alpha-\frac{p+1}{2}} \left\{ \frac{1}{\Gamma_p(\alpha+\beta)\det(C)^{\alpha+\beta}} \right.$$
$$\left. \int_{S_2>0} \operatorname{etr}[-S_2^{\frac{1}{2}} C^{-1} S_2^{\frac{1}{2}}(\mathrm{I}_p+S)] \det(S_2)^{\alpha+\beta-\frac{p+1}{2}} dS_2 \right\} dS.$$

Thus it in sufficient to have the evaluation of the integral

$$(1.9)\quad \mathrm{I}(\alpha;\, C,\, S) = \frac{\det(C)^{\alpha}}{\Gamma_p(\alpha)} \int_{R>0} \operatorname{etr}[-R^{\frac{1}{2}} C R^{\frac{1}{2}} S] \det(R)^{\alpha-\frac{p+1}{2}} dR$$

for matrices C and S are positive definite and $Re(\alpha) > p - \frac{1}{2}$.

For the case of a scalar matrix C, we have the evaluation of the integral (1.9) as $\det(S)^{-\alpha}$, but in the general case of a matrix C, it may depend on the matrix C. Incidentally it has the property of $\mathrm{I}(\alpha;C,S) = \mathrm{I}(\alpha;H'CH, H'SH)$ for any matrix $H \in O(p)$, and moreover, eigenvalues of the matrix S has the same distribution as it for the distribution $F_p(2\alpha;2\beta)$, which holds for any decomposition of the matrix S_2.

§ 2. Distributions in the noncentral case

In this section we consider the noncentral distribution corresponding to central distributions defined in § 1.

Definition 2.1.

Let a matrix S be (p, p) real symmetric matrix. It is distributed according to the noncentral Γ-distribution of matrix argument $\Gamma_p(\alpha;\, C,\, \Omega)$ whenever it has the probability element

$$(2.1)\quad dF(S)=\begin{cases}\dfrac{\mathrm{etr}(-C^{-1}\Omega)\det(S)^{\alpha-\frac{p+1}{2}}}{\Gamma_p(\alpha)\det(C)^{\alpha}}\,\mathrm{etr}(-C^{-1}S)\,{}_0F_1(\alpha\,;\,C^{-1}SC^{-1}\Omega)\,dS, \\ \qquad\qquad S>0,\ C>0 \text{ and } \alpha>\dfrac{p-1}{2} \\ 0, \qquad \text{otherwise,}\end{cases}$$

where ${}_0F_1(\alpha\,;\,S)$ is the confluent hypergeometric function of matrix argument (*C. S.* Herz [*H*] ; *A. G.* Constantine [*C*]). We define the canonical form of the distribution as the distribution with probability element

$$(2.1')\quad dF(S)=\begin{cases}\dfrac{1}{\Gamma_p(\alpha)}\,\mathrm{etr}(-S)\det(S)^{\alpha-\frac{p+1}{2}}\,\mathrm{etr}(-\Omega)\,{}_0F_1(\alpha\,;\,S\Omega)\,dS, \\ \qquad S>0 \text{ and } \alpha>\dfrac{p-1}{2}, \\ 0, \qquad \text{otherwise}\end{cases}$$

Proposition 2.1.

The characteristic function of the distribution $\Gamma_p(\alpha\,;\,C,\,\Omega)$ is given by the expression

$$(2.2)\quad \varphi(\theta)=\mathrm{etr}\{-[\mathrm{I}_p-(\mathrm{I}_p-i\,\theta\,C)^{-1}]\,C^{-1}\Omega\}\det(\mathrm{I}_p-i\,\theta\,C)^{-\alpha}$$

[Proof]

$$\varphi(\theta)=E\ \mathrm{etr}(i\,\theta\,S)=\frac{\mathrm{etr}(-C^{-1}\Omega)}{\Gamma_p(\alpha)\det(C)^{\alpha}}\int_{S>0}\mathrm{etr}(i\,\theta\,S)\,\mathrm{etr}(-C^{-1}S)\det(S)^{\alpha-\frac{p+1}{2}}$$
$${}_0F_1(\alpha\,;\,C^{-1}SC^{-1}\Omega)\,dS$$
$$=\frac{\mathrm{etr}(-C^{-1}\Omega)}{\det(C)^{\alpha}}\det(C^{-1}-i\,\theta)^{-\alpha}\mathrm{etr}\ [(\mathrm{I}_p-i\,\theta\,C)^{-1}C^{-1}\Omega]$$
$$=\mathrm{etr}\{-[\mathrm{I}_p-(\mathrm{I}_p-i\,\theta\,C)^{-1}]\,C^{-1}\Omega\}\det(\mathrm{I}_p-i\,\theta\,C)^{-\alpha}.$$

Q. E. D.

Corollary

Let matrices S_1 and S_2 be independently distributed, each according to $\Gamma_p(\alpha_1\,;\,C,\,\Omega_1)$ and $\Gamma_p(\alpha_2\,;\,C,\,\Omega_2)$, respectively. Then the matrix $S=S_1+S_2$ is distributed according to $\Gamma_p(\alpha_1+\alpha_2\,;\,C,\ \Omega_1+\Omega_2)$.

Proposition 2.2.

Let a matrix S be distributed according to $\Gamma_p(\alpha\,;\,\mathrm{I}_p,\,\Omega)$, then its average over the orthogonal group $0(p)$ in the following sense has the probability element

$$(2.3)\quad d\tilde{F}(S)=\int_{0(p)} dF((H'SH)dV(H)=\frac{1}{\Gamma_p(\alpha)}\operatorname{etr}(-S)\det(S)^{\alpha-\frac{p+1}{2}} \cdot \operatorname{etr}(-\Omega)\,{}_0F_1(\alpha\,;S,\Omega)\,dS,$$

where the function ${}_0F_1(\alpha\,;S,\Omega)$ is defined by the formula

$${}_0F_1(\alpha\,;S,\Omega)=\sum_{k=0}^{\infty}\sum_{(k)}\frac{1}{k!\,(\alpha)_{(k)}}\frac{C_{(k)}(S)C_{(k)}(\Omega)}{C_{(k)}(\mathrm{I}_p)}$$

and $C_{(k)}(S)$ is the zonal polynomial corresponding to partition (k) of k (A. T. James [J]), and $dV(H)$ is the normalized invariant measure over the orthogonal group $0(p)$.

Before proving this proposition, we note that the distribution $\Gamma_p(\alpha\,;\mathrm{I}_p,\Omega)$ is the symmetric distribution only when the matrix Ω is a scalar matrix, on the other hand the distribution $d\tilde{F}(S)$ is always the symmetric distribution.

[Proof]

$$\begin{aligned} d\tilde{F}(S) &= \frac{\operatorname{etr}(-S)\det(S)^{\alpha-\frac{p+1}{2}}}{\Gamma_p(\alpha)}\cdot\operatorname{etr}(-\Omega)\int_{0(p)}{}_0F_1(\alpha\,;H'SH\Omega)\,dV(H)\cdot dS \\ &= \frac{\operatorname{etr}(-S)\det(S)^{\alpha-\frac{p+1}{2}}}{\Gamma_p(\alpha)}\cdot\operatorname{etr}(-\Omega)\sum_{k=0}^{\infty}\sum_{(k)}\frac{1}{k!\,(\alpha)_{(k)}} \\ &\quad \int_{0(p)} C_{(k)}(H'SH\Omega)\,dV(H)\cdot dS \\ &= \frac{\operatorname{etr}(-S)\det(S)^{\alpha-\frac{p+1}{2}}}{\Gamma_p(\alpha)}\operatorname{etr}(-\Omega)\,{}_0F_1(\alpha\,;S,\Omega)\,dS. \end{aligned}$$

Q. E. D.

Corollary

Let a matrix S be distributed according to $\Gamma_p(\alpha\,;\mathrm{I}_p,\Omega)$, then the joint distribution of eigenvalues of matrix S has the probability element

$$(2.4)\quad dF(D_{\lambda^2})=\frac{\pi^{\frac{p}{2}}}{\prod_{i=1}^{p}\Gamma\left(\frac{i}{2}\right)\Gamma\left(\alpha-\frac{i-1}{2}\right)}\operatorname{etr}(-D_{\lambda^2})\det(D_{\lambda^2})^{\alpha-\frac{p+1}{2}} \operatorname{etr}(-\Omega)\,{}_0F_1(\alpha\,;D_{\lambda^2},\Omega)\cdot\prod_{i<j}^{p}(\lambda_i^2-\lambda_j^2)\prod_{i=1}^{p}d\lambda_i^2,$$

where the matrix D_{λ^2} is the diagonal matrix $(\lambda_1^2,\lambda_2^2,\ldots,\lambda_p^2)$, and eigenvectors are independently distributed of eigenvalues when the matrix Ω is a scalar matrix.

[Proof] The second part is the consequence of Prop. 1.2′. The first part may be calculated as the following; the joint distribution of eigenvalues and eigenvectors under the distribution (2.3) has the expression

$$dF(D_{\lambda^2}, H) = \frac{1}{\Gamma_p(\alpha)} \operatorname{etr}(-D_{\lambda^2}) \det(D_{\lambda^2})^{\alpha-\frac{p+1}{2}} \operatorname{etr}(-\Omega)\, {}_0F_1(\alpha\,;\, D_{\lambda^2}, \Omega)$$

$$\cdot \prod_{i<j}^{p} (\lambda_i^2-\lambda_j^2) \prod_{i=1}^{p} d\lambda_i^2 \cdot \frac{1}{2^p} \prod_{i<j}^{p} h_j' dh_i,$$

and integrating out of eigenvectors, we have the joint distribution of eigenvalues

$$dF(D_{\lambda^2}) = \frac{\pi^{\frac{p(p+1)}{4}}}{\Gamma_p(\alpha) \prod_{i=1}^{p} \Gamma\left(\frac{i}{2}\right)} \operatorname{etr}(-D_{\lambda^2}) \det(D_{\lambda^2})^{\alpha-\frac{p+1}{2}} \operatorname{etr}(-\Omega)\, {}_0F_1(\alpha\,;\, D_{\lambda^2}, \Omega)$$

$$\cdot \prod_{i>j}^{p} (\lambda_i^2-\lambda_j^2) \prod_{i=1}^{p} d\lambda_i^2.$$

Q. E. D.

Definition 2.2.

Let a matrix S be a (p, p) real symmetric matrix. If its distribution has the probability element

$$(2.5)\quad dF(S) = \begin{cases} \dfrac{1}{B_p(\alpha, \beta)} \det(S)^{\alpha-\frac{p+1}{2}} \det(\mathrm{I}_p - S)^{\beta-\frac{p+1}{2}} \\ \qquad \operatorname{etr}(-\Omega)\, {}_1F_1(\alpha+\beta\,;\, \alpha\,;\, S\Omega)\, dS, \\ \qquad 0 < S < \mathrm{I}_p,\ \alpha, \beta > \dfrac{p-1}{2} \\ 0, \qquad \text{otherwise} \end{cases}$$

we say that the matrix S is distributed according to the noncentral B-distribution of matrix argument $Be_p(\alpha, \beta\,;\, \Omega)$.

Propotisition 2.3.

Let a matrix S be distributed according to $Be_p(\alpha, \beta\,;\, \Omega)$, then the distribution $d\widetilde{F}(S)$ has the probability element

$$(2.6)\quad \frac{1}{B_p(\alpha, \beta)} \det(S)^{\alpha-\frac{p+1}{2}} \det(\mathrm{I}_p - S)^{\beta-\frac{p+1}{2}} \operatorname{etr}(-\Omega)\, {}_1F_1(\alpha+\beta\,;\, \alpha\,;\, S, \Omega)\, dS,$$

and eigenvalues of the matrix S are distributed with probability element

$$(2.7)\quad dF(D_{\lambda^2})=\frac{\pi^{\frac{p(p+1)}{4}}}{B_p(\alpha,\beta)\prod_{i=1}^{p}\Gamma\left(\frac{i}{2}\right)}\det(D_{\lambda^2})^{\alpha-\frac{p+1}{2}}\det(I_p-D_{\lambda^2})^{\beta-\frac{p+1}{2}}$$

$$\cdot\ \mathrm{etr}(-\Omega)\,{}_1F_1(\alpha+\beta;\alpha;D_{\lambda^2},\Omega)\prod_{i<j}^{p}(\lambda_i^2-\lambda_j^2)\prod_{i=1}^{p}d\lambda_i^2.$$

Now, we consider a relationship between the distribution $\Gamma_p(\alpha;C,\Omega)$ and the distribution $Be_p(\alpha,\beta;\Omega)$. The problem is the following; let matrices S_1 and S_2 be independently distributed, each according to $\Gamma_p(\alpha;C,\Omega)$ and $\Gamma_p(\beta;C)$ respectively, and how is the distribution of the matrix $S=[(s_1+s_2)^{-\frac{1}{2}}]'\,s_1(s_1+s_2)^{-\frac{1}{2}}$?

At first, we consider the case of $(s_1+s_2)^{-\frac{1}{2}\prime}=(s_1+s_2)^{-\frac{1}{2}}$. The joint distribution of matrices s_1, s_2 has the probability element

$$(2.8)\quad dF(S_1,S_2)=\frac{1}{\Gamma_p(\alpha)\Gamma_p(\beta)\det(C)^{\alpha+\beta}}\mathrm{etr}[-C^{-1}(S_1+S_2)]\det(S_1)^{\alpha-\frac{p+1}{2}}$$

$$\det(S_2)^{\beta-\frac{p+1}{2}}\cdot\mathrm{etr}(-C^{-1}\Omega)\,{}_0F_1(\alpha;C^{-1}S_1C^{-1}\Omega)\,dS_1dS_2,$$

then the joint distribution of matrices S and $T=S_1+S_2$ has the probability element

$$dF(S,T)=\frac{1}{\Gamma_p(\alpha)\Gamma_p(\beta)\det(C)^{\alpha+\beta}}\det(S)^{\alpha-\frac{p+1}{2}}\det(I_p-S)^{\beta-\frac{p+1}{2}}\mathrm{etr}(-C^{-1}T)$$

$$\cdot\ \det(T)^{\alpha+\beta-\frac{p+1}{2}}\mathrm{etr}(-C^{-1}\Omega)\,{}_0F_1(\alpha;C^{-1}T^{\frac{1}{2}}ST^{\frac{1}{2}}C^{-1}\Omega)\,dSdT,$$

for the region $0<S<I_p$ and $T>0$.

To get the distribution of the matrix S, we need the evaluation of the integral

$$(2.9)\quad \frac{\mathrm{etr}(-C^{-1}\Omega)}{\det(C)^{\alpha+\beta}}\int_{T>0}\mathrm{etr}(-C^{-1}T)\det(T)^{\alpha+\beta-\frac{p+1}{2}}{}_0F_1(\alpha;C^{-1}T^{\frac{1}{2}}ST^{\frac{1}{2}}C^{-1}\Omega)\,dT.$$

At present this integral is explicitly evaluated only for the case of matrices C and Ω being scalar matrices; in this case, our integral becomes

$$\frac{\mathrm{etr}(-C^{-1}\Omega)}{\det(C)^{\alpha+\beta}}\int_{T>0}\mathrm{etr}(-C^{-1}T)\det(T)^{\alpha+\beta-\frac{p+1}{2}}{}_0F_1(\alpha;C^{-1}\Omega C^{-1}ST)\,dT$$

$$=\frac{\mathrm{etr}(-C^{-1}\Omega)}{\det(C)^{\alpha+\beta}}\int_{T>0}\mathrm{etr}(-C^{-1}T)\det(T)^{\alpha+\beta-\frac{p+1}{2}}{}_0F_1(\alpha;TC^{-1}S\Omega C^{-1})\,dT$$

$$=\mathrm{etr}(-C^{-1}\Omega)\Gamma_p(\alpha+\beta)\,{}_1F_1(\alpha+\beta;\alpha;C^{-1}\Omega S),$$

and the matrix S is distributed according to the distribution $Be_p(\alpha, \beta; C^{-1}\Omega)$.

On the other hand, the averaging process over the orthogonal group $0(p)$ gives us the distribution $d\widetilde{F}(S)$, and hence the joint distribution of eigenvalues of the matrix S:

$$(2.10) \quad d\widetilde{F}(S)=\frac{1}{B_p(\alpha,\beta)} \det(S)^{\alpha-\frac{p+1}{2}} \det(I_p-S)^{\beta-\frac{p+1}{2}}$$

$$\operatorname{etr}(-C^{-1}\Omega)\,{}_1F_1(\alpha+\beta\,;\alpha\,;,C^{-1}\Omega)\,dS,$$

$$(2.11) \quad dF(D_{\lambda^2})=\frac{\pi^{\frac{p(p+1)}{4}}}{B_p(\alpha,\beta)\prod_{i=1}^{p}\Gamma\left(\frac{i}{2}\right)} \det(D_{\lambda^2})^{\alpha-\frac{p+1}{2}} \det(I_p-D_{\lambda^2})^{\beta-\frac{p+1}{2}}$$

$$\cdot \operatorname{etr}(-C^{-1}\Omega)\,{}_1F_1(\alpha+\beta\,;\alpha\,;D_{\lambda^2},C^{-1}\Omega)\prod_{i<j}^{p}(\lambda_i^2-\lambda_j^2)\prod_{i=1}^{p}d\lambda^2.$$

Thus, the eigenvalues of the matrix S are always distributed with the same probability element as it of the distribution $Be_p(\alpha, \beta; C^{-1}\Omega)$.

Next, let $s_1+s_2=TT'$ with lower triangular matrix and consider the distribution of the matrix

$$(2.12) \quad S=T^{-1}\ S_1(T^{-1})'.$$

The probability element of this distribution is given by the expression

$$(2.13) \quad dF(S)=\frac{1}{B_p(\alpha,\beta)} \det(S)^{\alpha-\frac{p+1}{2}} \det(I_p-S)^{\beta-\frac{p+1}{2}}$$

$$\cdot\frac{\operatorname{etr}(-C^{-1}\Omega)}{\Gamma_p(\alpha+\beta)}\int_T \operatorname{etr}(-T'T)\det(T'T)^{\alpha+\beta}$$

$${}_0F_1(\alpha\,;ST'C^{-1}\Omega T)\,2^p\prod_{i=1}^{p}t_{ii}^{-i}\ dT\ \ dS,$$

and in the case of the matrix $C^{-1}\ \Omega$ having the form $\begin{pmatrix} a^2 & 0 & 0 \\ 0 & \ddots & \\ 0 & & 0 \end{pmatrix}$,

the matrix S is distributed according to the distribution $Be_p(\alpha, \beta; C^{-1}\Omega)$.

Definition 2.3.

Let a matrix S be (p, p) real symmetric matrix; we call its distribution $F_p(\alpha, \beta; \Omega)$, when its distribution has the probability element

$$(2.14)\quad dF(S)=\begin{cases}\dfrac{1}{B_p\left(\frac{\alpha}{2},\frac{\beta}{2}\right)}\det(S)^{\frac{\alpha-p-1}{2}}\det(\mathrm{I}_p+S)^{-\frac{\alpha+\beta}{2}} \\ \qquad\cdot\operatorname{etr}(-\Omega)\,{}_1F_1\left(\frac{\alpha+\beta}{2};\frac{\alpha}{2};\ (\mathrm{I}_p+S^{-1})^{-1}\Omega\right)dS, \\ \qquad S>0 \text{ and } \alpha,\ \beta>p-1, \\ 0, \qquad \text{otherwise.}\end{cases}$$

Proposition 2.4.

Let a matrix S be distributed according to the distribution $F_p(\alpha, \beta; \Omega)$. Then the distribution $dF(S)$ and the distribution of eigenvalues have the following probability element;

$$(2.15)\quad d\widetilde{F}(S)=\frac{1}{B_p\left(\frac{\alpha}{2},\frac{\beta}{2}\right)}\det(S)^{\frac{\alpha-p-1}{2}}\det(\mathrm{I}_p+S)^{-\frac{\alpha+\beta}{2}}\operatorname{etr}(-\Omega)$$

$$\cdot\,{}_1F_1\left(\frac{\alpha+\beta}{2}\right);\ \frac{\alpha}{2};\ (\mathrm{I}_p+S^{-1})^{-1},\ \Omega)\,dS,$$

$$(2.16)\quad dF(D_{\lambda^2})=\frac{\pi^{\frac{p(p+1)}{4}}}{B_p\left(\frac{\alpha}{2},\frac{\beta}{2}\right)\prod_{i=1}^{p}\Gamma\left(\frac{i}{2}\right)}\det(D_{\lambda^2})^{\frac{\alpha-p-1}{2}}\det(\mathrm{I}_p+D_{\lambda^2})^{-\frac{\alpha+\beta}{2}}$$

$$\cdot\operatorname{etr}(-\Omega)\,{}_1F_1\left(\frac{\alpha+\beta}{2}\right);\ \frac{\alpha}{2};\ (\mathrm{I}_p+D_{\lambda^2}^{-1})^{-1},\ \Omega\prod_{i<j}^{p}(\lambda_i^2-\lambda_j^2)\prod_{i=1}^{p}d\lambda_i^2.$$

Proposition 2.5.

Let a matrix S be distributed according to $Be_p(\alpha, \beta; \Omega)$, then the matrix $T(\mathrm{I}_p-S)^{-1}T'$, where $S=TT'$, is distributed according to $F_p(2\alpha, 2\beta; \Omega)$.

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