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FURTHER BOUNDS FOR ČEBYŠEV FUNCTIONAL FOR POWER SERIES IN BANACH ALGEBRAS VIA GRÜSS-LUPAŞ TYPE INEQUALITIES FOR p-NORMS

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Abstract. Some Grüss-Lupaş type inequalities for p-norms of sequences in Banach algebras are obtained. Moreover, if $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$, a Banach algebra, with $xy = yx$, then we also establish some new upper bounds for the norm of the *Čebyšev type difference*

$$f(\lambda) f(\lambda xy) - f(\lambda x) f(\lambda y), \lambda \in D(0,R).$$

These results build upon the earlier results obtained by the authors. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

1. Introduction

In 1935, G. Grüss [11] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x) g(x) dx - \frac{1}{b-a}\int_a^b f(x) dx \cdot \frac{1}{b-a}\int_a^b g(x) dx\right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma) \quad (1)$$

where $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and satisfying the assumption

$$\varphi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a,b]$ where $\varphi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

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For a simple proof of (1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [12] by Mitrinović, Pečarić and Fink.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [1] established the following discrete version of Grüss' inequality, see also [12, Ch. X]:

Theorem 1.1. *Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n)$ be two n-tuples of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, ..., n$. Then one has the inequality:*

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right| \leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)(S - s) \tag{2}$$

when $[x]$ is the integer part of $x, x \in \mathbb{R}$.

In 1981, A. Lupaş [12, Ch. X] proved some similar results for the first difference of a as follows :

Theorem 1.2. *Let a, b two monotonic n-tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned} &\min_{1\leq i\leq n-1} |a_{i+1} - a_i| \min_{1\leq i\leq n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right] \\ &\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \\ &\leq \max_{1\leq i\leq n-1} |a_{i+1} - a_i| \max_{1\leq i\leq n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right]. \end{aligned} \tag{3}$$

If there exists the numbers $\bar{a}, \bar{a}_1, r, r_1, (rr_1 > 0)$ such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then in (3) the equality holds.

For some generalizations of Gruss' inequality for isotonic linear functionals defined on certain spaces of mappings see Chapter X of the book [12] where further references are given .

In order to extend the above results for Banach algebras, we need some preliminary facts as follows:

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in$Inv$\mathcal{B}$;
(ii) $\{a \in \mathcal{B}\colon\ \|1 - a\| < 1\} \subset$Inv$\mathcal{B}$;
(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map Inv$\mathcal{B} \ni a \longmapsto a^{-1} \in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} :\ \lambda - a \in \text{Inv}\mathcal{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to$Inv$\mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < R)$. If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \|a^j\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x + y).$$

Also, if x is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx)\, dt = x^{-1}[\exp(bx) - \exp(ax)].$$

Moreover, if x and y are commuting and $y - x$ is invertible, then

$$\int_0^1 \exp((1 - s)x + sy)\, ds = (y - x)^{-1}[\exp(y) - \exp(x)].$$

Inequalities for functions of operators in Hilbert spaces may be found in the recent monographs [5], [6], [10] and the references therein.

Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \le R \le \infty$, but we consider only the case $0 < R \le \infty$.

Denote by:

$$D(0,R) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0,R) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0,R) = \begin{cases} \{x \in \mathcal{B} : \|x\| < R\}, & \text{if } R < \infty \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map:

$$x \mapsto \widetilde{f}(x) : B(0,R) \to \mathcal{B}, \widetilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

Obviously, $\widetilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n x^n\| \le \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.

In addition, we assume that $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| < \infty$. Let $s_0 := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $s_1 := \sum_{n=0}^{\infty} n |\alpha_n| < \infty$.

With the above assumptions we have that [7]:

Theorem 1.3. *Let $\lambda \in \mathbb{C}$ such that $\max\{|\lambda|, |\lambda|^2\} < R < \infty$ and let $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \le 1$ and $xy = yx$. Then:*

(i) *We have*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \le \sqrt{2} \psi \min\{\|x-1\|, \|y-1\|\} f_A\left(|\lambda|^2\right) \tag{4}$$

where:

$$\psi^2 := s_0 s_2 - s_1^2. \tag{5}$$

(ii) *We also have*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \le \sqrt{2} \min\{\|x-1\|, \|y-1\|\} f_A(|\lambda|) \times \left\{ f_A(|\lambda|) \left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|) \right] - \left[|\lambda| f_A'(|\lambda|) \right]^2 \right\}^{1/2}. \tag{6}$$

For other similar results, see [7], [8] and [9]

Motivated by the above results we establish in this paper other similar inequalities for the norm of the *Čebyšev difference*

$$\widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y)$$

via some Grüss'-Lupaş type inequality for p-norms with $p \geq 1$, where λ is a complex number and the vectors x, y belong to the Banach algebra $\mathcal{B}$. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

2. A Discrete Inequality of Grüss Type for 1-Norm

The following inequality of Grüss type holds.

Theorem 2.1. *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \leq \frac{1}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right] \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| , \tag{7}$$

where $\Delta a_i := a_{i+1} - a_i$ $(i = 1, ..., n-1)$ and $\Delta b_i := b_{i+1} - b_i$ $(i = 1, ..., n-1)$ are the usual forward differences.
The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Let us start with the following identity in Banach algebras which can be proved by direct computation

$$\sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i = \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j (a_j - a_i)(b_j - b_i) = \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j (a_j - a_i)(b_j - b_i) .$$

As $i < j$, we can write

$$a_j - a_i = \sum_{k=i}^{j-1} (a_{k+1} - a_k) = \sum_{k=i}^{j-1} \Delta a_k \text{ and } b_j - b_i = \sum_{l=i}^{j-1} (b_{l+1} - b_l) = \sum_{l=i}^{j-1} \Delta b_l.$$

Using the generalized triangle inequality, we have successively:

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| &= \left\| \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \sum_{k=i}^{j-1} \Delta a_k \sum_{l=i}^{j-1} \Delta b_l \right\| \\ &\leq \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \left\| \sum_{k=i}^{j-1} \Delta a_k \right\| \left\| \sum_{l=i}^{j-1} \Delta b_l \right\| \\ &\leq \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \sum_{k=i}^{j-1} \|\Delta a_k\| \sum_{l=i}^{j-1} \|\Delta b_l\| =: A. \end{aligned} \tag{8}$$

It is obvious for all $1 \leq i < j \leq n-1$, we have that

$$\sum_{k=i}^{j-1} \|\Delta a_k\| \leq \sum_{k=1}^{n-1} \|\Delta a_k\|$$

and

$$\sum_{l=i}^{j-1} \|\Delta b_l\| \leq \sum_{l=1}^{n-1} \|\Delta b_l\|$$

and then

$$A \leq \sum_{k=1}^{n-1} \|\Delta a_k\| \sum_{l=1}^{n-1} \|\Delta b_l\| \sum_{1\leq i<j\leq n} |\alpha_i| |\alpha_j| . \tag{9}$$

Now, let us observe that

$$\begin{aligned} \sum_{1\leq i<j\leq n} |\alpha_i| |\alpha_j| &= \frac{1}{2} \left[\sum_{i,j=1}^{n} |\alpha_i| |\alpha_j| - \sum_{i=j} |\alpha_i| |\alpha_j| \right] \\ &= \frac{1}{2} \left[\sum_{i=1}^{n} |\alpha_i| \sum_{j=1}^{n} |\alpha_j| - \sum_{i=1}^{n} |\alpha_i|^2 \right] \\ &= \frac{1}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right] . \end{aligned} \tag{10}$$

Using (8)-(10) , we deduce the desired inequality (7) .
To prove the sharpness of the constant $\frac{1}{2}$, let us assume that (7) holds with a constant $c > 0$. That is,

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\ &\leq c \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right] \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| \end{aligned} \tag{11}$$

for all a_i, b_i, α_i $(i = 1, ..., n)$ as above and $n \geq 2$.
Choose in (7) $n = 2$ and compute

$$\begin{aligned} \sum_{i=1}^{2} \alpha_i \sum_{i=1}^{2} \alpha_i a_i b_i - \sum_{i=1}^{2} \alpha_i a_i \sum_{i=1}^{2} \alpha_i b_i &= \frac{1}{2} \sum_{i,j=1}^{2} \alpha_i \alpha_j (a_i - a_j)(b_i - b_j) \\ &= \sum_{1\leq i<j\leq 2} \alpha_i \alpha_j (a_i - a_j)(b_i - b_j) \\ &= \alpha_1 \alpha_2 (a_1 - a_2)(b_1 - b_2) . \end{aligned}$$

Also,

$$\sum_{1\le i<j\le 2}|\alpha_i|\,|\alpha_j|\sum_{i=1}^{1}\|\Delta a_i\|\sum_{i=1}^{1}\|\Delta b_i\| = |\alpha_1|\,|\alpha_2|\,\|a_1-a_2\|\,\|b_1-b_2\|\,.$$

Substituting in (11), we obtain

$$|\alpha_1|\,|\alpha_2|\,\|a_1-a_2\|\,\|b_1-b_2\| \le 2c\,|\alpha_1|\,|\alpha_2|\,\|a_1-a_2\|\,\|b_1-b_2\|\,.$$

If we assume that $\alpha_1,\alpha_2>0,\ a_1\ne a_2,\ b_1\ne b_2$, then we obtain $c\ge\frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. □

Remark 2.2. Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ (=$\mathbb{R}$,$\mathbb{C}$), $a_i\in\mathcal{B}$ and $\alpha_i\in\mathbb{K}$ $(i=1,...,n)$. Then we have the inequality:

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i^2-\left(\sum_{i=1}^{n}\alpha_i a_i\right)^2\right\| \le \frac{1}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]\left(\sum_{i=1}^{n-1}\|\Delta a_i\|\right)^2. \tag{12}$$

The constant $\frac{1}{2}$ is best possible.

The following corollary holds.

Corollary 2.3. *Under the above assumptions for a_i,b_i $(i=1,...,n)$, we have the inequality*

$$\left\|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right\| \le \frac{1}{2}\left(1-\frac{1}{n}\right)\sum_{i=1}^{n-1}\|\Delta a_i\|\sum_{i=1}^{n-1}\|\Delta b_i\|, \tag{13}$$

and the constant $\frac{1}{2}$ is sharp.

In particular, we have

$$\left\|\frac{1}{n}\sum_{i=1}^{n}a_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^2\right\| \le \frac{1}{2}\left(1-\frac{1}{n}\right)\left(\sum_{i=1}^{n-1}\|\Delta a_i\|\right)^2.$$

3. An Inequality of Grüss Type for p-Norm

The following result that provides a version for the p-norm, $p>1$ of the forward difference also holds.

Theorem 3.1. *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \leq \sum_{1\leq j<i\leq n} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}, \tag{14}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $C = 1$ in the right hand side of (7) *is sharp in the sense that it cannot be replaced by a smaller one.*

Proof. From the proof of Theorem 2.1 we have

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \leq \sum_{1\leq j<i\leq n} |\alpha_i| |\alpha_j| \sum_{k=j}^{i-1} \|\Delta a_k\| \sum_{l=j}^{i-1} \|\Delta b_l\| =: A. \tag{15}$$

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1} \|\Delta a_k\| \leq (i-j)^{\frac{1}{q}} \left(\sum_{k=j}^{i-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}}$$

and

$$\sum_{k=j}^{i-1} \|\Delta b_k\| \leq (i-j)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and then, by multiplication, we have

$$A \leq \sum_{1\leq j<i\leq n} |\alpha_i| |\alpha_j| (i-j) \left(\sum_{k=j}^{i-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}. \tag{16}$$

As

$$\sum_{k=j}^{i-1} \|\Delta a_k\|^p \leq \sum_{k=1}^{n-1} \|\Delta a_k\|^p$$

and

$$\sum_{k=j}^{i-1} \|\Delta b_k\|^q \leq \sum_{k=1}^{n-1} \|\Delta b_k\|^q,$$

for all $1 \leq j < i \leq n$, then by (15) and (16) , we get the desired inequality (14) .

To prove the sharpness of the constant, let us assume that (14) holds with a constant $C > 0$. That is,

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \tag{17}$$
$$\leq C \sum_{1 \leq j < i \leq n} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}.$$

Choose $n = 2$. Then

$$\left\| \sum_{i=1}^{2} \alpha_i \sum_{i=1}^{2} \alpha_i a_i b_i - \sum_{i=1}^{2} \alpha_i a_i \sum_{i=1}^{2} \alpha_i b_i \right\| = |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\|$$

and

$$\sum_{1 \leq j < i \leq 2} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}$$
$$= |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\|.$$

Therefore, from (17), we obtain

$$|\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\| \leq C |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\|$$

for all $a_1 \neq a_2$, $b_1 \neq b_2$, and then $C \geq 1$, which proves the sharpness of the constant. □

Remark 3.2. A coarser upper bound, which can be more useful may be obtained by applying Cauchy-Schwartz's inequality:

$$\sum_{1 \leq j < i \leq n} (i-j) |\alpha_i| |\alpha_j| \leq \left(\sum_{1 \leq j < i \leq n} |\alpha_i| |\alpha_j| \right)^{\frac{1}{2}} \left(\sum_{1 \leq j < i \leq n} |\alpha_i| |\alpha_j| (i-j)^2 \right)^{\frac{1}{2}}$$

and taking into account that

$$\sum_{1 \leq j < i \leq n} |\alpha_i| |\alpha_j| = \frac{1}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right]$$

and

$$\sum_{1 \leq j < i \leq n} |\alpha_i| |\alpha_j| (i-j)^2 = \frac{1}{2} \sum_{i,j=1}^{n} |\alpha_i| |\alpha_j| (i-j)^2$$
$$= \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right].$$

Thus, from (14), we can state the inequality

$$\begin{aligned}(18)\quad &\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i b_i-\sum_{i=1}^{n}\alpha_i a_i\sum_{i=1}^{n}\alpha_i b_i\right\| \\ &\leq \frac{\sqrt{2}}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]^{\frac{1}{2}} \\ &\times\left(\sum_{k=1}^{n-1}\|\Delta a_k\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta b_k\|^q\right)^{\frac{1}{q}},\end{aligned}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

The following corollary holds.

Corollary 3.3. *With the above assumptions, we have*

$$\begin{aligned}(19)\quad &\left\|\frac{1}{n}\sum_{i=1}^{n}a_i b_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right\| \\ &\leq \frac{n^2-1}{6n}\left(\sum_{k=1}^{n-1}\|\Delta a_k\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta b_k\|^q\right)^{\frac{1}{q}}\end{aligned}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

The constant $\frac{1}{6}$ *is the best possible.*

Proof. The proof follows by (7), putting $\alpha_i=\frac{1}{n}$ and taking into account that

$$\begin{aligned}&\sum_{1\leq j<i\leq n}(i-j) \\ &=\sum_{1\leq j\leq 2}(2-j)+\sum_{1\leq j\leq 3}(3-j)+...+\sum_{1\leq j\leq n}(n-j) \\ &=2\cdot 2-(1+2)+3\cdot 3-(1+2+3)+...+n\cdot n-(1+2+...+n) \\ &=1^2+2^2+...+n^2-1-(1+2)-(1+2+3)-...-(1+2+...+n) \\ &=\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}\frac{k(k+1)}{2}=\frac{1}{2}\left(\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k\right)=\frac{n(n^2-1)}{6},\end{aligned}$$

and the corollary is thus proved. □

Remark 3.4. If in (14) and (19) we assume that $p = q = 2$, then we get the inequalities:

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \le \sum_{1 \le j < i \le n} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^2 \right)^{\frac{1}{2}} \tag{20}$$

and

$$\left\| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right\| \le \frac{n^2-1}{6n} \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^2 \right)^{\frac{1}{2}}, \tag{21}$$

respectively.

We also have the inequality

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\ &\le \frac{\sqrt{2}}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right]^{\frac{1}{2}} \\ &\times \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{22}$$

In the case when $b_i = a_i$, $i \in \{1, ..., n\}$ we get from (20)

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i^2 - \left(\sum_{i=1}^{n} \alpha_i a_i \right)^2 \right\| \le \sum_{1 \le j < i \le n} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2 \right) \tag{23}$$

and from (22)

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i^2 - \left(\sum_{i=1}^{n} \alpha_i a_i \right)^2 \right\| \\ &\le \frac{\sqrt{2}}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right]^{\frac{1}{2}} \\ &\times \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2 \right). \end{aligned} \tag{24}$$

4. Inequalities for Power Series

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned}
f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \\
g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C}; \\
h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C}; \\
l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);
\end{aligned} \tag{25}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned}
f_A(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \\
g_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C}; \\
h_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C}; \\
l_A(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).
\end{aligned} \tag{26}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{27}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma>0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function.*

The following new result holds:

Theorem 4.1. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R>0$. If $x,y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$\begin{aligned} &\left\| \widetilde{f}(\lambda \cdot 1)\widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x)\widetilde{f}(\lambda y) \right\| \\ &\le \frac{1}{2}\frac{\|x-1\|\|y-1\|}{(1-\|x\|)(1-\|y\|)}\left[f_A^2(|\lambda|) - f_{A^2}\left(|\lambda|^2\right)\right], \end{aligned} \tag{28}$$

where

$$f_{A^2}(\lambda) := \sum_{n=0}^{\infty} |\alpha_n|^2 \lambda^n \tag{29}$$

has the radius of convergence R^2.

Proof. From the inequality (7) we have

$$\begin{aligned} &\left\| \sum_{i=0}^{n} \alpha_i \lambda^i \sum_{i=0}^{n} \alpha_i \lambda^i (xy)^i - \sum_{i=0}^{n} \alpha_i \lambda^i x^i \sum_{i=0}^{n} \alpha_i \lambda^i y^i \right\| \\ &\le \frac{1}{2}\left[\left(\sum_{i=0}^{n} |\alpha_i| |\lambda|^i\right)^2 - \sum_{i=0}^{n} |\alpha_i|^2 |\lambda|^{2i}\right] \sum_{j=0}^{n-1} \left\|x^{j+1} - x^j\right\| \sum_{j=0}^{n-1} \left\|y^{j+1} - y^j\right\|, \\ &= \frac{1}{2}\left[\left(\sum_{i=0}^{n} |\alpha_i| |\lambda|^i\right)^2 - \sum_{i=0}^{n} |\alpha_i|^2 |\lambda|^{2i}\right] \sum_{j=0}^{n-1} \left\|x^j(x-1)\right\| \sum_{j=0}^{n-1} \left\|y^j(y-1)\right\| \end{aligned} \tag{30}$$

$$\le \frac{1}{2}\|x-1\|\|y-1\|\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]\sum_{j=0}^{n-1}\|x\|^j\sum_{j=0}^{n-1}\|y\|^j$$
$$=\frac{1}{2}\|x-1\|\|y-1\|\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]$$
$$\times\frac{1-\|x\|^n}{1-\|x\|}\frac{1-\|y\|^n}{1-\|y\|}$$

for any $n \ge 1$.

Since all the series whose partial sums are involved in (30) are convergent, then by letting $n \to \infty$ in (30) we deduce the desired inequality (28). □

Corollary 4.2. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R>0$. If $x \in \mathcal{B}$ and $\|x\|<1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda|<R$ the inequality:*

$$\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}\left(\lambda x^2\right)-\left[\widetilde{f}(\lambda x)\right]^2\right\| \le \frac{1}{2}\frac{\|x-1\|^2}{(1-\|x\|)^2}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]. \tag{31}$$

We have the following result as well:

Theorem 4.3. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R>0$. If $x,y \in \mathcal{B}$ with $xy=yx$ and $\|x\|,\|y\|<1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda|<R$ the inequality:*

$$\begin{aligned}(32)\quad &\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}(\lambda xy)-\widetilde{f}(\lambda x)\widetilde{f}(\lambda y)\right\| \\ &\le \frac{\sqrt{2}}{2}\frac{\|x-1\|\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\ &\times\left\{f_A(|\lambda|)\left[|\lambda|f_A'(|\lambda|)+|\lambda|^2f_A''(|\lambda|)\right]-\left[|\lambda|f_A'(|\lambda|)\right]^2\right\}^{1/2}\end{aligned}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have

$$\begin{aligned}(33)\quad &\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}\left(\lambda x^2\right)-\left[\widetilde{f}(\lambda x)\right]^2\right\| \\ &\le \frac{\sqrt{2}}{2}\frac{\|x-1\|^2}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|x\|^q)^{\frac{1}{q}}}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\ &\times\left\{f_A(|\lambda|)\left[|\lambda|f_A'(|\lambda|)+|\lambda|^2f_A''(|\lambda|)\right]-\left[|\lambda|f_A'(|\lambda|)\right]^2\right\}^{1/2}.\end{aligned}$$

Proof. Using the inequality (18) we have for $n \geq 1$ that

$$\left\| \sum_{i=0}^{n} \alpha_i \lambda^i \sum_{i=0}^{n} \alpha_i \lambda^i (xy)^i - \sum_{i=0}^{n} \alpha_i \lambda^i x^i \sum_{i=0}^{n} \alpha_i \lambda^i y^i \right\| \tag{34}$$
$$\leq \frac{\sqrt{2}}{2} \left[\left(\sum_{i=0}^{n} |\alpha_i| |\lambda|^i \right)^2 - \sum_{i=0}^{n} |\alpha_i|^2 |\lambda|^{2i} \right]^{1/2}$$
$$\times \left[\sum_{i=0}^{n} |\alpha_i| |\lambda|^i \sum_{i=0}^{n} i^2 |\alpha_i| |\lambda|^i - \left(\sum_{i=0}^{n} i |\alpha_i| |\lambda|^i \right)^2 \right]^{\frac{1}{2}}$$
$$\times \left(\sum_{j=0}^{n-1} \left\| x^{j+1} - x^j \right\|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{n-1} \left\| y^{j+1} - y^j \right\|^q \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Observe that

$$\sum_{j=0}^{n-1} \left\| x^{j+1} - x^j \right\|^p \leq \sum_{j=0}^{n-1} \left\| x^j (x-1) \right\|^p \leq \|x-1\|^p \sum_{j=0}^{n-1} \left\| x^j \right\|^p$$
$$\leq \|x-1\|^p \sum_{j=0}^{n-1} \|x\|^{jp} = \|x-1\|^p \frac{1 - \|x\|^{np}}{1 - \|x\|^p},$$

which implies that

$$\left(\sum_{j=0}^{n-1} \left\| x^{j+1} - x^j \right\|^p \right)^{\frac{1}{p}} \leq \|x-1\| \left(\frac{1 - \|x\|^{np}}{1 - \|x\|^p} \right)^{\frac{1}{p}}.$$

Similarly,

$$\left(\sum_{j=0}^{n-1} \left\| y^{j+1} - y^j \right\|^q \right)^{\frac{1}{q}} \leq \|y-1\| \left(\frac{1 - \|y\|^{nq}}{1 - \|y\|^q} \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

From (34) we get

$$\left\|\sum_{i=0}^{n}\alpha_i\lambda^i\sum_{i=0}^{n}\alpha_i\lambda^i(xy)^i-\sum_{i=0}^{n}\alpha_i\lambda^ix^i\sum_{i=0}^{n}\alpha_i\lambda^iy^i\right\| \tag{35}$$
$$\leq\frac{\sqrt{2}}{2}\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]^{1/2}$$
$$\times\left[\sum_{i=0}^{n}|\alpha_i||\lambda|^i\sum_{i=0}^{n}i^2|\alpha_i||\lambda|^i-\left(\sum_{i=0}^{n}i|\alpha_i||\lambda|^i\right)^2\right]^{\frac{1}{2}}$$
$$\times\|x-1\|\,\|y-1\|\left(\frac{1-\|x\|^{np}}{1-\|x\|^p}\right)^{\frac{1}{p}}\left(\frac{1-\|y\|^{nq}}{1-\|y\|^q}\right)^{\frac{1}{q}},$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

If we denote $g(u):=\sum_{n=0}^{\infty}\alpha_nu^n$, then for $|u|<R$ we have

$$\sum_{n=0}^{\infty}n\alpha_nu^n=ug'(u)$$

and

$$\sum_{n=0}^{\infty}n^2\alpha_nu^n=u(ug'(u))'.$$

However

$$u(ug'(u))'=ug'(u)+u^2g''(u)$$

and then

$$\sum_{n=0}^{\infty}n^2\alpha_nu^n=ug'(u)+u^2g''(u).$$

Therefore

$$\sum_{n=0}^{\infty}n^2|\alpha_n||\lambda|^n=|\lambda|f_A'(|\lambda|)+|\lambda|^2f_A''(|\lambda|)$$

and

$$\sum_{n=0}^{\infty}n|\alpha_n||\lambda|^n=|\lambda|f_A'(|\lambda|)$$

for $|\lambda|<R$.

Since all the series whose partial sums are involved in (35) are convergent, then by letting $n\to\infty$ in (35) we deduce the desired inequality (32). □

Corollary 4.4. *With the assumptions of Theorem 4.3, we have*

$$\begin{aligned}(36)\qquad &\left\|\widetilde{f}(\lambda\cdot 1)\,\widetilde{f}(\lambda xy)-\widetilde{f}(\lambda x)\,\widetilde{f}(\lambda y)\right\| \\ &\leq \frac{\sqrt{2}}{2}\frac{\|x-1\|\,\|y-1\|}{\left(1-\|x\|^2\right)^{\frac{1}{2}}\left(1-\|y\|^2\right)^{\frac{1}{2}}}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\ &\times\left\{f_A(|\lambda|)\left[|\lambda|\,f_A'(|\lambda|)+|\lambda|^2 f_A''(|\lambda|)\right]-\left[|\lambda|\,f_A'(|\lambda|)\right]^2\right\}^{1/2}\end{aligned}$$

and, in particular

$$\begin{aligned}(37)\qquad &\left\|\widetilde{f}(\lambda\cdot 1)\,\widetilde{f}\left(\lambda x^2\right)-\left[\widetilde{f}(\lambda x)\right]^2\right\| \\ &\leq \frac{\sqrt{2}}{2}\frac{\|x-1\|^2}{1-\|x\|^2}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\ &\times\left\{f_A(|\lambda|)\left[|\lambda|\,f_A'(|\lambda|)+|\lambda|^2 f_A''(|\lambda|)\right]-\left[|\lambda|\,f_A'(|\lambda|)\right]^2\right\}^{1/2}.\end{aligned}$$

5. Some Particular Cases of Interest

Consider the function $f : D(0,1) \to \mathbb{C}$ defined by

$$f(\lambda) = (1-\lambda)^{-1} = \sum_{k=0}^{\infty}\lambda^k = f_A(\lambda).$$

Then

$$f_{A^2}(\lambda) := \sum_{n=0}^{\infty}\lambda^n = (1-\lambda)^{-1},$$

which implies that

$$f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right) = \frac{2|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)},\ \ |\lambda|<1$$

and by (28), we have for $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ that

$$\begin{aligned}(38)\qquad &\left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1}-(1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \\ &\leq \frac{|\lambda|\,\|x-1\|\,\|y-1\|}{(1-|\lambda|)^2(1+|\lambda|)(1-\|x\|)(1-\|y\|)}.\end{aligned}$$

We also have for $|\lambda|, \|x\| < 1$ that

$$(39)\qquad \left\|(1-\lambda)^{-1}\left(1-\lambda x^2\right)^{-1}-(1-\lambda x)^{-2}\right\| \leq \frac{|\lambda|\,\|x-1\|^2}{(1-|\lambda|)^2(1+|\lambda|)(1-\|x\|)^2}.$$

For the function $f(\lambda) = (1-\lambda)^{-1}$ we have

$$\begin{aligned} & f_A(|\lambda|)\left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|)\right] - \left[|\lambda| f_A'(|\lambda|)\right]^2 \\ & = \frac{1}{1-|\lambda|}\left[\frac{|\lambda|}{(1-|\lambda|)^2} + \frac{2|\lambda|^2}{(1-|\lambda|)^3}\right] - \frac{|\lambda|^2}{(1-|\lambda|)^4} \\ & = \frac{|\lambda|}{(1-|\lambda|)^4}. \end{aligned}$$

From the inequality (32) we then have for $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ that

$$\begin{aligned} & \left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \\ & \leq \frac{\sqrt{2}}{2} \frac{\|x-1\|\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}}\left[\frac{2|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)},\right]^{1/2} \\ & \times\left\{\frac{|\lambda|}{(1-|\lambda|)^4}\right\}^{1/2}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \\ & \leq \frac{|\lambda|\|x-1\|\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}(1-|\lambda|)^3(1+|\lambda|)^{1/2}}, \end{aligned} \tag{40}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

If we consider the function

$$f(\lambda) = (1+\lambda)^{-1} = \sum_{k=0}^{\infty}(-1)^k \lambda^k,$$

then the inequalities (38)-(40) also holds with "$+$" instead of "$-$" in the left hand side expressions such as $(1-\lambda x)^{-1}$ etc.

We consider the *modified Bessel function functions of the first kind*

$$I_\nu(\lambda) := \left(\frac{1}{2}\lambda\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}\lambda^2\right)^k}{k!\Gamma(\nu+k+1)}, \ \lambda \in C$$

where Γ is the *Gamma function* and ν is a real number. An integral formula to represent I_ν is

$$I_\nu(\lambda) = \frac{1}{\pi}\int_0^\pi e^{\lambda\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\lambda\cosh t - \nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(\lambda) = \frac{1}{\pi}\int_0^\pi e^{\lambda\cos\theta}\cos(n\theta)\,d\theta.$$

For $n = 0$ we have

$$I_0(\lambda) = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos\theta} d\theta = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}\lambda^2\right)^k}{(k!)^2},\ \lambda \in C.$$

Now, if we consider the exponential function

$$f(\lambda) = \exp(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k,$$

then for $\rho > 0$ we have

$$f_{A^2}(\rho) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\rho^k = I_0\left(2\sqrt{\rho}\right),$$

which implies that

$$f_A^2(|\lambda|) - f_{A^2}\left(|\lambda|^2\right) = \exp(2|\lambda|) - I_0(2|\lambda|),\ \lambda \in C.$$

Making use of the inequality (28), we have for $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$ that

$$\begin{aligned} &\|\exp(\lambda(xy+1)) - \exp(\lambda(x+y))\| \\ &\leq \frac{1}{2}\frac{\|x-1\|\,\|y-1\|}{(1-\|x\|)(1-\|y\|)}\left[\exp(2|\lambda|) - I_0(2|\lambda|)\right], \end{aligned} \tag{41}$$

In particular, we have for $\|x\| < 1$

$$\left\|\exp\left(\lambda\left(x^2+1\right)\right) - \exp(2\lambda x)\right\| \leq \frac{1}{2}\frac{\|x-1\|^2}{(1-\|x\|)^2}\left[\exp(2|\lambda|) - I_0(2|\lambda|)\right] \tag{42}$$

for any $\lambda \in \mathbb{C}$.

For $f(\lambda) = \exp(\lambda)$ we have

$$f_A(|\lambda|)\left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|)\right] - \left[|\lambda| f_A'(|\lambda|)\right]^2 = |\lambda|\exp(2|\lambda|).$$

If $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < 1$, then from (32) we have for $\lambda \in \mathbb{C}$ the inequality:

$$\begin{aligned} &\|\exp(\lambda(xy+1)) - \exp(\lambda(x+y))\| \\ &\leq \frac{\sqrt{2}}{2}\frac{\|x-1\|\,\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}}|\lambda|^{1/2}\exp(|\lambda|)\left[\exp(2|\lambda|) - I_0(2|\lambda|)\right]^{1/2}, \end{aligned} \tag{43}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

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