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Generalized Brandt Semigroups

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A semigroup S is said to be *regular* if it satisfies the following condition :

(C. 1) For any $a \in S$, there exists an element $a^* \in S$ such that $aa^*a=a$ and $a^*aa^*=a^*$.

In this case, such an element a^* is called an inverse of a. In general, for an element a of a regular semigroup, an inverse of a is not necessarily unique. It is well-known that an inverse of a is unique for every element a of a regular semigroup S if and only if S is an inverse semigroup. For an inverse semigroup S, we shall denote the inverse of $a \in S$ by a^{-1}. Let S be a regular semigroup, and B the set of all idempotents of S. Then B is not necessarily a subsemigroup of S. If B is *a* subsemigroup of S, then S is said to be *strictly regular*. Moreover, in this case B is called *the band of idempotents of* S. The main purpose of this paper is to study the structure of semigroups S satisfying the following (C. 2) :

(C. 2) $\begin{cases} \text{(1)} & S \text{ is a strictly regular semigroup with zero, and} \\ \text{(2)} & \text{every non-zero idempotent of } S \text{ is primitive (see [2]).} \end{cases}$

Now, let S be a semigroup satisfying (C. 2). Since S is a primitive regular semigroup, it follows from Preston [3] that S is a 0-direct union of $\{S_i : i \in I\}$, where each S_i is a completely 0-simple semigroup in which the set of idempotents constitutes a subsemigroup. Hence our problem is reduced to the problem of determining the structure of semigroups S satisfying the following condition :

(C. 3) $\begin{cases} \text{(1)} & S \text{ is completely 0-simple, and} \\ \text{(2)} & \text{the set } B \text{ of idempotents of } S \text{ is a subsemigroup of } S \text{ (i. e.,} \\ & B \text{ forms a band).} \end{cases}$

We shall call a semigroup satisfying (C. 3) a *generalized Brandt semigroup*.

Remark. It is well-known that a semigroup S is a Brandt semigroup if and only if S satisfies (1) of (C. 3) and the following :

(C. 3, 2*) The set B of idempotents of S forms a semilattice.

In the following sections, we shall mainly deal with generalized Brandt semigroups.

§1. Characterizations of generalized Brandt semigroups. At first, we have the following theorem :

Theorem 1. *A semigroup S is a generalized Brandt semigroup if and only if S satisfies*

$$(1.\ 1)\ \begin{cases} (1) & S \text{ is a strictly regular semigroup with zero } 0, \\ (2) & \text{every non-zero idempotent of } S \text{ is primitive, and} \\ (3) & \text{for any non-zero idempotents } e, f,\ eSf \neq \{0\}. \end{cases}$$

Proof. Let S be a generalized Brandt semigroup. Since S satisfies (1) of (C. 3), it is obvious from Clifford and Preston [2] that S satisfies (2) and (3) of (1. 1). Further, it also follows from (C. 3) that S is strictly regular. Hence S satisfies (1) of (1. 1). Conversely, let S be a semigroup satisfying (1. 1). Since S is strictly regular, S satisfies (2) of (C. 3). Further since S is regular and since S satisfies (2) and (3) of (1. 1), S is completely 0-simple (see [2]). Hence, S is a generalized Brandt semigroup.

Next, we shall show another characterization of generalized Brandt semigroups.

Lemma 1. *If S is a generalized Brandt semigroup, then (1) $efe = e$ and $fef = f$ or (2) $ef = fe = 0$ for every non-zero idempotents e, f of S.*

Proof. Since efe is an idempotent and $efee = eefe = efe$, we have $efe = 0$ or $efe = e$. Similarly, $fef = f$ or $fef = 0$.

Case 1 ($efe = 0$). In this case, $efef = 0$ and $fefe = 0$. Hence $ef = (ef)^2 = 0$ and $fe = (fe)^2 = 0$, and hence $ef = fe = 0$.

Case 2 ($efe = e$). If $fef = 0$, then we have $ef = fe = 0$ by an analogous way to Case 1. Hence we have $efe = 0$, which contradicts to our assumption $efe = e$. Therefore $fef \neq 0$, and hence $fef = f$.

Lemma 2. *Let B be the band of idempotents of a generalized Brandt semigroup S. Then B is a normal band*[1], *and accordingly S is a generalized inverse semigroup*[2].

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1) A band G (that is, an idempotent semigroup) is said to be normal if it satisfies the polynomial identity $xyzw = xzyw$. It has been proved by [5] that a band G is normal if and only if it satisfies the polynomial identity $xyzx = xzyx$.

2) A regular semigroup G is called a generalized inverse semigroup if the idempotents of G form a normal band (see [6]).

Proof. Let e, f and g be elements of B. If at least one of these e, f and g is the zero element 0 of S, then clearly $efge = egfe = 0$. Suppose that all the elements e, f, g are not 0. By Lemma 1, $efge = 0$ or $efge = e$.

Case 1 ($efge = 0$). If $ef = 0$, then $fe = 0$. Hence $g(fe) = 0$, and hence $egfe = 0$. If $ef \neq 0$, then $e(ef) \neq 0$. Hence $efe \neq 0$, and hence $efe = e$. Therefore, we have $ge = g(efe) = g(ef)e$. Since $e(fg)e = 0$, we have $efg = e(efg) = 0$. Hence $g(ef) = 0$. Accordingly $ge = (gef)e = 0$, which also implies $eg = 0$. Thus we have $egfe = 0$, and hence $egfe = efge = 0$.

Case 2 ($efge \neq 0$). Since $e(fg)e \neq 0$, we have $e(fg)e = e$. If $egfe = 0$, then it follows from Case 1 that $efge = 0$. This contradicts to our assumption. Therefore $egfe \neq 0$, and hence $egfe = e$. Thus we have $efge = egfe = e$. Since S is regular and B is a normal band, S is a generalized inverse semigroup.

By using Lemmas 1, 2, we obtain the following theorem :

Theorem 2. *A semigroup S with zero is a generalized Brandt semigroup if and only if S is a generalized inverse semigroup and completely* 0-*simple.*

Proof. If S is a generalized Brandt semigroup, then by Lemma 2 S is a generalized inverse semigroup. It is obvious from the definition of generalized Brandt semigroups that S is completely 0-simple. Conversely, let S be a generalized inverse semigroup which is also completely 0-simple. Then since the set of idempotents of S is a subsemigroup of S, S is a generalized Brandt semigroup.

Remark. It is well-known that a semigroup S is a Brandt semigroup if and only if S is an inverse semigroup and completely 0-simple (*e. g.*, see [2]).

Theorem 2 is analogous to this result.

§ 2. A structure theorem. If S is a generalized Brandt semigroup, then it follows from Theorem 2 that S is a generalized inverse semigroup. Hence the following result follows from the author [6], [7] :

The relation ρ on S defined by

(2. 1) $a \rho b$ if and only if $\{a^* : a^*$ is an inverse of $a\} = \{b^* : b^*$ is an inverse of $b\}$

is a congruence on S, and

(2. 2) $\left\{\begin{array}{l}\text{(1) the factor semigroup } S/\rho \text{ of } S \text{ mod } \rho \text{ is an inverse semigroup ;} \\ \text{(2) } S/\rho \ni \bar{e} \text{ (where } \bar{e} \text{ is the congruence class containing } e\text{)} \\ \quad \text{is an idempotent if and only if } e \text{ is an idempotent of } S\text{;} \\ \text{(3) } \rho|B \text{ (the restriction of } \rho \text{ to } B\text{) gives the greatest semilattice} \\ \quad \text{decomposition of } B\text{, where } B \text{ is the band of idempotents of } S.\end{array}\right.$

Of course, it is obvious from (2) of (2. 2) that every congruence class of B mod $\rho_B = \rho|B$ constitutes a complete (congruence) class of S mod ρ.

Further, we have

Lemma 3. *S/ρ is a Brandt semigroup.*

Proof. It is obvious that S/ρ is regular. For $x \in S$, let $\bar{x}$ be the congruence class containing x mod ρ. At first, we shall show that $\bar{0} = \{0\}$. Let $\bar{0} = \bar{e}$. Then since 0 is an inverse of 0, $0 = e0e = e$. Hence $\bar{0} = \{0\}$. Now, let $\bar{e}$, $\bar{f}$ be idempotents of S/ρ such that $\bar{e}\,\bar{f} = \bar{f}\,\bar{e} = \bar{e} \neq \bar{0}$. Since $\overline{ef} = \overline{fe} \neq \bar{0}$ and e, f are idempotents of S, we have $0 \neq e = efe$ and $0 \neq f = fef$. Therefore, $e\ \rho_B f$ since ρ_B gives the greatest semilattice decomposition of B. Hence $e \rho f$, which implies $\bar{e} = \bar{f}$. Thus, every non-zero idempotent of S/ρ is primitive. Next, let $\bar{e}$, $\bar{f}$ be non-zero idempotents of S/ρ. Since e, f are non-zero idempotents of S and since S is completely 0-simple, there exists x of S such that $exf \neq 0$. If $\bar{e}\,\bar{x}\,\bar{f} = \bar{0}$ then $\overline{exf} = \bar{0}$, which implies $exf = 0$. This contradicts to $exf \neq 0$. Hence $\bar{e}\,\bar{x}\,\bar{f} \neq \bar{0}$, and hence $\bar{e}\ S/\rho\ \bar{f} \neq \{\bar{0}\}$. Therefore, S/ρ is completely 0-simple. Since S/ρ is completely 0-simple and an inverse semigroup, S/ρ is a generalized Brandt semigroup.

From Lemma 3 and the author [6], it follows that S is isomorphic to a semigroup S^* constructed in the following way :

Let $\mathscr{B}$ be a Brandt semigroup, $\mathscr{L}$ the basic semilattice of $\mathscr{B}$ (that is, the semilattice of idempotents of $\mathscr{B}$). In this case, it is obvious that $ab = ba = 0$ for any $a, b \in \mathscr{L}$ (see [2]). Let L, R be a left normal band and a right normal band [3] respectively having $L \sim \Sigma\{L_k : k \in \mathscr{L}\}$ and $R \sim \Sigma\{R_k : k \in \mathscr{L}\}$ as their structure decompositions (for the definition of structure decompositions, see [6]). Of course, it is obvious that the zero element 0 of $\mathscr{B}$ is contained in $\mathscr{L}$ and each of R_0 and L_0 consists of a single element : Put $R_0 = \{0_r\}$ and $L_0 = \{0_l\}$. Let S^* be the quasi-direct product (abbrev.

3) A band G is said to be left [right] normal if it satisfies the polynomial identity $xyz = xzy$ [$xyz = yxz$] (see [8]).

Q-product) $Q(L\otimes \mathscr{B}\otimes R;\ \mathscr{L})$ of L, $\mathscr{B}$, R with respect to $\mathscr{L}$ (see [6]). That is,

$$(2.\ 3)\begin{cases} Q(L\otimes \mathscr{B}\otimes R;\ \mathscr{L})=\{(e,a,f)\ :\ e\in L_{aa^{-1}},\ \ f\in R_{a^{-1}a},\ \ a\in \\ \mathscr{B}\},\ \text{where } a^{-1} \text{ means the inverse of } a \text{ in } \mathscr{B}; \\ \text{the multiplication in } Q(L\otimes \mathscr{B}\otimes R;\ \mathscr{L}): \\ (e,a,f)(g,b,h)=(eu,ab,vh),\ \text{where } u\in L_{ab(ab)^{-1}} \text{ and } v\in R_{(ab)^{-1}ab}\text{[4]}. \end{cases}$$

Now, we obtain the following result as to S^*:

Lemma 4. (1) $(e,a,f)\in S^*$ *implies* (i) $a=0$, $e=0_l$, $f=0_r$ *or* (ii) $a\neq 0$, $e\neq 0_l$, $f\neq 0_r$.

(2) (e,a,f), $(g,b,h)\in S^*$ *and* $ab=0$ *imply* $(e,a,f)(g,b,h)=(0_l,0,0_r)$.

(3) (e,a,f), $(g,b,h)\in S^*$ *and* $ab\neq 0$ *imply* $(e,a,f)(g,b,h)=(e,ab,h)$.

Proof. The part (1) is obvious from the fact that $L_0=\{0_l\}$ and $R_0=\{0_r\}$. Further the part (2) is also obvious from the part (1). Hence, we shall next prove the part (3). Suppose that (e,a,f), $(g,b,h)\in S^*$ and $ab\neq 0$. If $aa^{-1}\neq ab(ab)^{-1}$, then $0=(aa^{-1})(ab(ab)^{-1})=ab(ab)^{-1}$. Hence $ab=(ab(ab)^{-1})ab=0$, which contradicts to $ab\neq 0$. Therefore, $aa^{-1}=ab(ab)^{-1}$. Similarly, we have $b^{-1}b=(ab)^{-1}ab$. Now $(e,a,f)(g,b,h)=(eu,ab,vh)$, where $u\in L_{ab(ab)^{-1}}=L_{aa^{-1}}$ and $v\in R_{(ab)^{-1}ab}=R_{b^{-1}b}$. Since $u\in L_{aa^{-1}}$, $e\in L_{aa^{-1}}$, $v\in R_{b^{-1}b}$, $h\in R_{b^{-1}b}$ and since $L_{aa^{-1}}$, $R_{b^{-1}b}$ are a left zero semigroup, a right zero semigroup respectively, we have $eu=e$ and $vh=h$. Hence $(e,a,f)(g,b,h)=(e,ab,h)$.

From the above-mentioned lemmas, we have the following theorem:

Theorem 3. *For any generalized Brandt semigroup S, there exist a Brandt semigroup $\mathscr{B}$, having $\mathscr{L}$ as its basic semilattice, and families of sets $\{I_a : a\in\mathscr{L}\}$, $\{J_a : a\in\mathscr{L}\}$, where each of I_0, J_0 consists of a single element, such that S is isomorphic to the semigroup S^* defined by*

$$(2.\ 4)\begin{cases} S^*=\{(e,a,f)\ :\ e\in I_{aa^{-1}},\ f\in J_{a^{-1}a},\ a\in\mathscr{B}\};\ \textit{and} \\ \textit{the multiplication in } S^*: \\ (e,a,f)(g,b,h)=\begin{cases}(e,ab,h) & \textit{if } ab\neq 0,\\ (0_l,0,0_r) & \textit{if } ab=0,\end{cases} \\ \textit{where } 0_l,\ 0_r \textit{ are the elements of } I_0,\ J_0 \textit{ respectively.}\end{cases}$$

Conversely, any semigroup S^* constructed as in (2.4) is a generalized Brandt semigroup.

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4) By [6], it has been shown that (eu,ab,vh) does not depend on the selection of u, v. Hence the product of (e,a,f) and (g,b,h) is well-defined.

Proof. The first half of the assertion follows from the above-mentioned lemmas. Let S^* be a semigroup constructed as in (2. 4). Then it is easy to see that S^* satisfies (1), (2) and (3) of (1. 1). Hence, S^* is a generalized Brandt semigroup.

§ 3. **Matrix representations.** By Clifford and Preston [2] (see also Rees [4]), a semigroup S is a Brandt semigroup if and only if S is isomorphic to a Rees $\Lambda\times\Lambda$ matrix semigroup over a group with zero and with the $\Lambda\times\Lambda$ identity matrix as sandwich matrix. Hence, it follows from Theorem 3 that : For any generalized Brandt semigroup S, there exist a Rees $\Lambda\times\Lambda$ matrix semigroup $\mathscr{B}$ over a group with zero G^0 and with the $\Lambda\times\Lambda$ identity matrix P as sandwich matrix and families of sets $\{I_k : k\in\mathscr{L}\}$, $\{J_k : k\in\mathscr{L}\}$, where $\mathscr{L}$ is the set of idempotents of $\mathscr{B}$ and each of I_0, J_0 consists of a single element, such that

$$(3.\ 1)\begin{cases} S^*=\{(e,\ a,\ f) : e\in I_{aa^{-1}},\ f\in J_{aa^{-1}},\ a\in\mathscr{B}\}\ ; \\ \text{the multiplication in } S^* : \\ (e,\ a,\ f)\ (g,\ b,\ h)=\begin{cases}(e,\ ab,\ h) & \text{if } ab\neq 0, \\ (0_l,\ 0,\ 0_r) & \text{if } ab=0,\end{cases} \\ \text{where } 0_l,\ 0_r \text{ are the elements of } I_0,\ J_0 \text{ respectively.}\end{cases}$$

In this case,

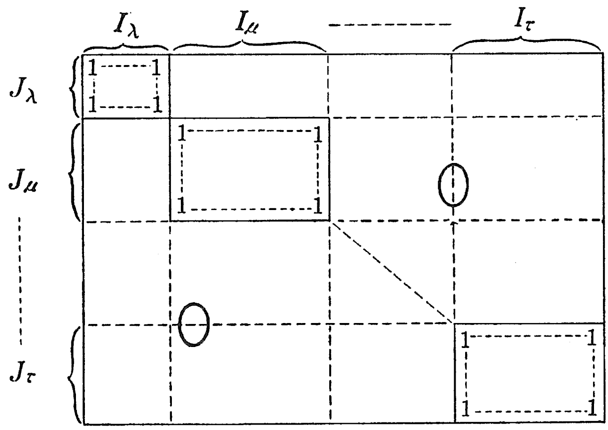
$$(3.\ 2)\begin{cases} \mathscr{B}=\{(x)_{\lambda\mu} : (\lambda,\ \mu)\in\Lambda\times\Lambda,\ x\in G^0\}^{5)}, \text{ where all } (0)_{\lambda\mu},\ (\lambda,\ \mu)\in \\ \Lambda\times\Lambda, \text{ should be considered as the same element denoted by } 0 \text{ (in} \\ \text{fact, this element } 0 \text{ becomes a zero element with respect to the} \\ \text{multiplication in } \mathscr{B})\ ; \\ \text{the multiplication in } \mathscr{B} : \\ (x)_{\lambda\mu}\ (y)_{\tau\eta}=(x\,\delta_{\mu\tau}\,y)_{\lambda\eta}, \text{ where } \delta_{\mu\tau} \text{ means the Kronecker } \delta.\end{cases}$$

Definition. Let G^0 be a group with zero 0, and P a $J\times I$ matrix over G^0. If there exist families $\{I_\lambda : \lambda\in\Lambda\}$ and $\{J_\lambda : \lambda\in\Lambda\}$ of subsets I_λ, J_λ of I, J such that

(1) $I=\Sigma\{I_\lambda : \lambda\in\Lambda\}$, $J=\Sigma\{J_\lambda : \lambda\in\Lambda\}$ (Σ means disjoint sum), and

(2) all elements in $J_\lambda\times I_\mu$ submatrix of P are (i) 0 if $\lambda\neq\mu$ and (ii) 1 if $\lambda=\mu$, then at present we shall call P *the $J\times I$ semi-identity matrix of type* $\{J_\lambda\times I_\lambda : \lambda\in\Lambda\}$.

5) $(x)_{\lambda\mu}$ denotes the $\Lambda\times\Lambda$ matrix which has x in the $(\lambda,\ \mu)$-position and has 0 in all the other positions.



Lemma 5. *Any Rees matrix semigroup over a group with zero and with a semi-identity matrix as sandwich matrix is a generalized Brandt semigroup.*

Proof. Let $M^0(G\ ;\ I, J\ ;\ P)$ be a Rees $I \times J$ matrix semigroup over a group with zero G^0 and with a $J \times I$ semi-identity matrix P as sandwich matrix. Let the semi-identity matrix P be of type $\{J_\lambda \times I_\lambda : \lambda \in \Lambda\}$. This $M^0(G\ ;\ I, J\ ;\ P)$ is clearly a completely 0-simple semigroup (see [2]). Now, let $(x)_{ef} \neq 0$, $(e, f) \in I \times J$, be an idempotent of $M^0(G\ ;\ I, J\ ;\ P)$. Then $(x)_{ef} = (x)_{ef}(x)_{ef} = (xp_{fe}x)_{ef}$, where p_{fe} is the element in the (f, e)-position of P. Hence $f \in J_\lambda$, $e \in I_\lambda$ for some $\lambda \in \Lambda$ and $x = 1$. Therefore, any idempotent of $M^0(G\ ;\ I, J\ ;\ P)$ has the type $(1)_{ef}$, $e \in I_\lambda$, $f \in J_\lambda$, $\lambda \in \Lambda$. Now, let $(1)_{ef}$, $(1)_{gh}$ be any two idempotents of $M^0(G\ ;\ I, J\ ;\ P)$. Then $(1)_{ef}(1)_{gh} = (1\, p_{fg} 1)_{eh}$. If $p_{fg} \neq 0$, then there exists $\mu \in \Lambda$ such that $f \in J_\mu$, $g \in I_\mu$. Hence $e \in I_\mu$ and $h \in J_\mu$, and hence $(1\ p_{fg}\ 1)_{eh} = (1)_{eh}$, $e \in I_\mu$, $h \in J_\mu$. Thus $(1)_{ef}(1)_{gh}$ is an idempotent. If $p_{fg} = 0$, then $(1\ p_{fg}\ 1)_{eh} = 0$. Hence of course in this case $(1\ p_{fg}\ 1)_{eh}$ is an idempotent. Therefore, the product of two idempotents of $M^0(G\ ;\ I, J\ ;\ P)$ is an idempotent. Since $M^0(G\ ;\ I, J\ ;\ P)$ is completely 0-simple and since the set of idempotents of $M^0(G\ ;\ I, J\ ;\ P)$ is a subsemigroup, this $M^0(G\ ;\ I, J\ ;\ P)$ is a generalized Brandt semigroup.

Now, let us consider about the semigroup S^* given by (3. 1) and (3. 2). Since $\mathscr{B} = \{(x)_{\lambda\mu} : (\lambda, \mu) \in \Lambda \times \Lambda,\ x \in G^0\}$, $\mathscr{L}$ of (3. 1) is the set $\{(1)_{\lambda\lambda} : \lambda \in \Lambda\} \cup \{0\}$. Put $I_{(1)_{\lambda\lambda}} = I_\lambda$, $J_{(1)_{\lambda\lambda}} = J_\lambda$ for every non-zero idempotent $(1)_{\lambda\lambda} \in \mathscr{L}$, and let $I = \cup\{I_\lambda : \lambda \in \Lambda\}$ and $J = \cup\{J_\lambda : \lambda \in \Lambda\}$. Now, let P be the $J \times I$ semi-identity matrix of type $\{J_\lambda \times I_\lambda : \lambda \in \Lambda\}$ over G^0 and let $M^0(G\ ;\ I, J\ ;\ P)$ be the Rees $I \times J$ matrix semigroup over G^0 and with P as sandwich matrix. Then, the mapping $\varphi : S^* \longrightarrow M^0(G\ ;\ I, J\ ;\ P)$ defined by

$$(3.3)\begin{cases}\varphi(e,\ (a)_{\lambda\mu},\ f)=(a)_{ef} \text{ if } a\neq 0,\\ \varphi(0_l,\ 0,\ 0_r)=0\end{cases}$$

is an isomorphism of S^* onto $M^0(G\ ;\ I, J\ ;\ P)$.

Therefore, we obtain the following theorem by using Lemma 5 and the above-mentioned result :

Theorem 4. *A semigroup S is a generalized Brandt semigroup if and only if S is isomorphic to a Rees matrix semigroup over a group with zero and with a semi-identity matrix as sandwich matrix.*

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