Modified Strichartz estimates with an application to the critical nonlinear Schrödinger equation

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Abstract

The Cauchy problem for the critical nonlinear Schrödinger equation is considered in the Sobolev space of fractional order. Some modified Strichartz estimates are constructed and applied to the problem to obtain small global solutions with less regularity assumption for the nonlinear term.

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1 Introduction

We consider the Cauchy problem for nonlinear Schrödinger equations

$$\begin{cases} \partial_t u(t,x) + i\Delta u(t,x) = f(u)(t,x) \quad \text{for} \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,\cdot) = u_0(\cdot) \in H^s(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $n \geq 1$, $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$ is the Laplacian, $f(u) := \lambda |u|^{p-1}u$ or $f(u) := \lambda |u|^p$ with $\lambda \in \mathbb{C}$ for example, $1 < p < \infty$, u_0 is a given initial datum in the Sobolev space $H^s(\mathbb{R}^n)$ for $0 \leq s < \infty$. Cazenave and Weissler [3] proved the existence of time global solutions of (1.1) for small data under the conditions

$$0 \leq s < \frac{n}{2}, \qquad [s]+1 < p = p(s) := 1 + \frac{4}{n-2s}, \tag{1.2}$$

where $p(s)$ is the critical number for (1.1) by the scaling $u_R(t,x) = R^{2/(p-1)}u(R^2t, Rx)$ for any $R > 0$, and $[s]$ denotes the largest integer less than or equal to s. The condition

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$[s] + 1 < p$ is the required regularity for $f(u)$, and it can be improved to $s < p$ by the method by Ginibre, Ozawa and Velo in [4] (see [12]). The aim of this paper is mainly to improve this condition to

$$\frac{s}{2} < p = p(s), \tag{1.3}$$

and to that end we also aim to refine the modified Strichartz estimate by Pecher [13]. Here, the special case $s = 2$ was proved in [3, Theorem 1.4]. However, the other case has been left open for long time.

To describe the corresponding results, we define $p_0(s)$ by

$$p_0(s) := \begin{cases} 1 & \text{for } \ s \le 2, \\ s - 1 & \text{for } \ 2 < s < 4, \\ s - 2 & \text{for } \ 4 \le s. \end{cases}$$

And we consider the problem (1.1) under the condition

$$0 \le s < \frac{n}{2}, \quad p_0(s) < p < p(s). \tag{1.4}$$

The condition $p_0(s) < p$ for $s \le 2$ and $s \ge 4$ is natural since $1 < p$ and the s-derivative of u in the spatial variables requires the $(s-2)$-derivative of $f(u)$ by the first equation in (1.1). The existence of time local solutions of (1.1) under (1.4) has been shown by Tsutsumi [16] for $s = 0$, Ginibre and Velo [5, Theorem 3.1] for $s = 1$ (see also [6]), Tsutsumi [15] for $s = 2$ for $f(u) = \lambda|u|^{p-1}u$ with $i\lambda \in \mathbb{R}$ mainly by the use of the $L^p - L^q$ estimate and the regularization technique. Kato [9, 10] used the Strichartz estimate and gave alternative proofs for the cases $s = 0, 1, 2$ both for $f(u) = \lambda|u|^{p-1}u$ and $f(u) = \lambda|u|^p$ with $\lambda \in \mathbb{C}$. Pecher [13] used the fractional Besov space for the time variable and proved the result when s is a real number with (1.4) and $s > 1$. He has also shown the existence of time global solutions when the initial data are sufficiently small. The condition $p_0(s) < p$ was improved to $s/2 < p$ for $2 < s < 4$ in [17], which seems to be natural since $p_0(s)$ is discontinuous at $s = 4$ and by the property of the Schrödinger equation (one time derivative corresponds to two spatial derivatives). However, the methods in [13] and [17] are not applicable to time global solutions for the critical case $p = p(s)$ by the technical conditions on the Strichartz estimates there. Especially, the interpolation argument to construct the Strichartz estimates prevent us from treating the critical point $p(s)$ in its application to (1.1). In this paper, we improve the Strichartz estimates in [13] and [17] using the auxiliary space $\ell^\alpha L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ defined by (1.5) below, and we show the time global solutions for $p = p(s)$.

To state the main results in this paper, we prepare several function spaces. Let $\{\varphi_j\}_{j=-\infty}^{\infty}$ be the Littlewood-Paley decomposition of the unity on $\mathbb{R}$. Namely, let φ

be a function whose Fourier transform $\widehat{\varphi}$ is a non-negative function which satisfies $\operatorname{supp}\widehat{\varphi} \subset \{\tau \in \mathbb{R}; 1/2 \le |\tau| \le 2\}$ and $\sum_{j=-\infty}^{\infty} \widehat{\varphi}(\tau/2^j) = 1$ for $\tau \neq 0$. We define ψ and φ_j for $j \in \mathbb{N}$ by $\widehat{\varphi_j}(\cdot) = \widehat{\varphi}(\cdot/2^j)$, $\widehat{\psi} = \sum_{j\le 0} \widehat{\varphi_j}$. We define $\chi_j := \sum_{k=j-1}^{j+1} \varphi_k$ for $j \ge 1$, $\chi_0 := \psi + \varphi_1$. We put $\psi(x) := \mathscr{F}_\xi^{-1}\widehat{\psi}(|\xi|)$ and $\varphi_j(x) := \mathscr{F}_\xi^{-1}\widehat{\varphi_j}(|\xi|)$ for $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. For $s \in \mathbb{R}$ and $1 \le r, \alpha \le \infty$, the Besov space is defined by $B^s_{r,\alpha}(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n); \|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)} < \infty\}$, where $\mathscr{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$,

$$\|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)} := \|\psi *_x u\|_{L^r(\mathbb{R}^n)} + \begin{cases} \left\{ \sum_{j\ge1} \left(2^{sj}\|\varphi_j *_x u\|_{L^r(\mathbb{R}^n)}\right)^\alpha \right\}^{1/\alpha} & \text{if } \alpha < \infty, \\ \sup_{j\ge1} 2^{sj}\|\varphi_j *_x u\|_{L^r(\mathbb{R}^n)} & \text{if } \alpha = \infty, \end{cases}$$

where $*_x$ denotes the convolution in the variables in $\mathbb{R}^n$. We prepare the Besov space of vector-valued functions (see [1], [14]). For functions $u = u(t,x)$ and $v = v(t,x)$, we denote their convolutions in t and x variables by $u *_t v$ and $u *_x v$, respectively. For $1 \le q, \ell \le \infty$ and a Banach space V, we denote the Lebesgue space for functions on $\mathbb{R}$ to V by $L^q(\mathbb{R}, V)$ and the Lorentz space by $L^{q,\ell}(\mathbb{R}, V)$. We define the Sobolev space $W^{1,q}(\mathbb{R}, V) := \{u \in L^q(\mathbb{R}, V); \partial_t u \in L^q(\mathbb{R}, V)\}$ and the Besov space $B^s_{q,\alpha}(\mathbb{R}, V) := \left\{u \in \mathscr{S}'(\mathbb{R}, V); \|u\|_{B^s_{q,\alpha}(\mathbb{R},V)} < \infty\right\}$, where

$$\|u\|_{B^s_{q,\alpha}(\mathbb{R},V)} := \|\psi *_t u\|_{L^q(\mathbb{R},V)} + \left\{ \sum_{j\ge1} \left(2^{sj}\|\varphi_j *_t u\|_{L^q(\mathbb{R},V)}\right)^\alpha \right\}^{1/\alpha}$$

if $\alpha < \infty$ with trivial modification if $\alpha = \infty$. We define the space $\ell^\alpha L^q(\mathbb{R}, L^r(\mathbb{R}^n)) := \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}, L^r(\mathbb{R}^n)); \|u\|_{\ell^\alpha L^q(\mathbb{R},L^r(\mathbb{R}^n))} < \infty\right\}$, where

$$\|u\|_{\ell^\alpha L^q(\mathbb{R},L^r(\mathbb{R}^n))} := \|\psi *_x u\|_{L^q(\mathbb{R},L^r(\mathbb{R}^n))} + \left(\sum_{j\ge1} \|\varphi_j *_x u\|^\alpha_{L^q(\mathbb{R},L^r(\mathbb{R}^n))} \right)^{1/\alpha} \tag{1.5}$$

if $\alpha < \infty$ with trivial modification if $\alpha = \infty$. We also define $\ell^\alpha L^{q,\ell}(\mathbb{R}, L^r(\mathbb{R}^n))$ similarly.

We recall the Strichartz estimate. For $1 \le r \le \infty$, we put $\delta(r) := n(1/2 - 1/r)$. We say that the pair (q, r) is admissible if $2 \le q, r \le \infty$ and $2/q = \delta(r)$ with $(q, r, n) \neq (2, \infty, 2)$. For $1 \le r \le \infty$, r' denotes its conjugate number defined by $1/r + 1/r' = 1$.

Lemma 1.1. *(See, e.g., [8], [11] and the references therein.) Let $s \in \mathbb{R}$, and let (q, r) and (γ, ρ) be admissible pairs. Then the solution u of*

$$\begin{cases} \partial_t u + i\Delta u = f \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^n, \\ u(0, \cdot) = u_0(\cdot) \end{cases}$$

satisfies

$$\|u\|_{L^\infty(\mathbb{R},H^s(\mathbb{R}^n))\cap L^q(\mathbb{R},B^s_{r,2}(\mathbb{R}^n))} \le C\|u_0\|_{H^s(\mathbb{R}^n)} + C\|f\|_{L^{\gamma'}(\mathbb{R},B^s_{\rho',2}(\mathbb{R}^n))},$$

where the constant $C > 0$ is independent of u, f and u_0. Moreover, $u \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.

Lemma 1.2. *([13, Proposition 2.5], [17, Lemma 2.3]) Let $s > 0$, and let (q, r) be an admissible pair with $2 < q < \infty$. Then the solution u of the problem*

$$\begin{cases} \partial_t u + i\Delta u = 0 \quad on \quad \mathbb{R}\times\mathbb{R}^n, \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

satisfies

$$\|u\|_{B^{s/2}_{q,2}(\mathbb{R},L^r(\mathbb{R}^n))} \le C\|u_0\|_{H^s(\mathbb{R}^n)},$$

where the constant $C > 0$ is independent of u and u_0.

In this paper, we firstly show the following Strichartz estimate for the inhomogeneous problem.

Theorem 1.3. *Let $n \ge 1$, $0 < \theta < 1$, $2 \le \alpha \le \infty$. Let (q, r) and (γ, ρ) be admissible pairs. Assume $\rho < \infty$ when $\alpha < \infty$. Let $1 \le \bar{q}, \bar{r} \le \infty$ satisfy $2/\bar{q} - \delta(\bar{r}) = 2(1-\theta)$. For any fixed function f, let us consider the problem*

$$\begin{cases} \partial_t u + i\Delta u = f \quad on \quad \mathbb{R}\times\mathbb{R}^n, \\ u(0,\cdot) = 0. \end{cases} \tag{1.6}$$

Then there exists a constant $C > 0$ which is independent of u and f such that

$$(1) \quad \|u\|_{B^\theta_{q,\alpha}(\mathbb{R},L^r(\mathbb{R}^n))} \le C\|f\|_{B^\theta_{\gamma',\alpha}(\mathbb{R},L^{\rho'}(\mathbb{R}^n))} + C\|f\|_{\ell^\alpha L^{\bar{q}}(\mathbb{R},L^{\bar{r}}(\mathbb{R}^n))}. \tag{1.7}$$

Moreover, if $\max\{\alpha, \bar{q}\} \le q$, then

$$(2) \quad \|u\|_{L^q(\mathbb{R},B^{2\theta}_{r,\alpha}(\mathbb{R}^n))} \le C\|f\|_{B^\theta_{\gamma',\alpha}(\mathbb{R},L^{\rho'}(\mathbb{R}^n))} + C\|f\|_{\ell^\alpha L^{\bar{q}}(\mathbb{R},L^{\bar{r}}(\mathbb{R}^n))}. \tag{1.8}$$

In addition, $u \in C(\mathbb{R}, H^s(\mathbb{R}^n))$ if $\alpha = 2$.

Remark 1.4. *Theorem 1.3 is a refinement of the following inequality previously obtained by Pecher [13, Propositions 2.6 and 2.7], Uchizono and Wada [17, Lemma 2.3].*

$$\|u\|_{B^\theta_{q,2}(\mathbb{R},L^r(\mathbb{R}^n))\cap L^q(\mathbb{R},B^{2\theta}_{r,q}(\mathbb{R}^n))} \le C\|f\|_{B^\theta_{q',2}(\mathbb{R},L^{r'}(\mathbb{R}^n))} + C\max_{\pm}\|f\|_{L^{\bar{q}_\pm}(\mathbb{R},L^{\bar{r}_\pm}(\mathbb{R}^n))}, \tag{1.9}$$

where $1/\bar{q}_\pm = (1-\theta\mp\varepsilon)/q' + (\theta\pm\varepsilon)/q$ and $1/\bar{r}_\pm = (1-\theta\mp\varepsilon)/r' + (\theta\pm\varepsilon)/r$ with sufficiently small $\varepsilon > 0$. The most important advantage of our new estimates (1.7) *and*

(1.8) *is that they are scale invariant. Namely, if we consider the scaling* $u_R(t,x) = R^{n/2-2\theta}u(R^2t, Rx)$, $f_R(t,x) = R^{n/2+2-2\theta}f(R^2t, Rx)$ *for any* $R > 0$, (1.6) *is invariant under this transform, and (the homogeneous counterparts of) the both sides of* (1.7) *and* (1.8) *are also invariant under this transform. On the other hand, the last term in the right-hand side of* (1.9) *is not scale-invariant because of the presence of* ε *in the definition of* $\bar{q}_\pm, \bar{r}_\pm$. *This is why we cannot treat the critical nonlinear problem as long as we rely on* (1.9). *Moreover, in our estimates, we can choose the pairs* (q,r), (γ,ρ) *and* $(\bar{q},\bar{r})$ *independently, whereas in* (1.9), *we have to choose* $(\gamma,\rho) = (q,r)$ *and* $(\bar{q}_\pm, \bar{r}_\pm)$ *satisfying the above formulas. This can be an advantage when we consider not only pure power but also more complicated nonlinear terms.*

As an application of Theorem 1.3, we show the global well-posedness of the problem (1.1) for small initial data under the condition (1.3). For any function f from $\mathbb{C}$ to $\mathbb{C}$, we denote the derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ by f', where $\bar{z}$ is the complex conjugate of z. For $1 < p < \infty$, we say that f satisfies $N(p)$ if $f \in C^1(\mathbb{C},\mathbb{C})$ in the sense of the derivatives by z and $\bar{z}$, $f(0) = f'(0) = 0$, and

$$|f'(z_1) - f'(z_2)| \le \begin{cases} C \max\limits_{w=z_1,z_2} |w|^{p-2}|z_1 - z_2| & \text{if } p \ge 2, \\ C|z_1 - z_2|^{p-1} & \text{if } 1 < p < 2 \end{cases} \tag{1.10}$$

for any z_1, $z_2 \in \mathbb{C}$. We note that $f(z) = \lambda|z|^{p-1}z$ and $f(z) = \lambda|z|^p$ with $\lambda \in \mathbb{C}$ satisfy $N(p)$ (see [7, Remark 2.3′]). For $s \ge 0$, an admissible pair (q,r) and $1 \le \alpha \le \infty$, we define a function space $X^s_{q,r,\alpha}$ by

$$\begin{aligned} X^s_{q,r,\alpha} := {} & C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{s-2}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}, H^s(\mathbb{R}^n)) \cap W^{1,\infty}(\mathbb{R}, H^{s-2}(\mathbb{R}^n)) \\ & \cap B^{s/2}_{q,2}(\mathbb{R}, L^r(\mathbb{R}^n)) \cap L^q(\mathbb{R}, B^s_{r,\alpha}(\mathbb{R}^n)) \cap W^{1,q}(\mathbb{R}, B^{s-2}_{r,\alpha}(\mathbb{R}^n)) \end{aligned}$$

with the metric in $L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)) \cap L^q(\mathbb{R}, L^r(\mathbb{R}^n))$, where we remove $C^1(\mathbb{R}, H^{s-2}(\mathbb{R}^n))$, $W^{1,\infty}(\mathbb{R}, H^{s-2}(\mathbb{R}^n))$ and $W^{1,q}(\mathbb{R}, B^{s-2}_{r,\alpha}(\mathbb{R}^n))$ when $s < 2$.

Theorem 1.5. *Let* $n \ge 8$. *Let* $1 < s < 4$ *with* $s \ne 2$, *and let* $s/2 < p = p(s) \le s$. *If* $s \ge 3$ *with* $p < 2$, *or equivalently if* $3 \le s < (n-4)/2$, *we further assume either of the following:*

$$\text{(i) } n = 11; \text{ or (ii) } n = 12 \text{ and } 7 - \sqrt{15}\,(= 3.12\cdots) \le s < 5 - \sqrt{3}\,(= 3.26\cdots). \tag{1.11}$$

Let f *satisfy* $N(p)$. *Then, there exists an admissible pair* (q,r) *such that if* $u_0 \in H^s(\mathbb{R}^n)$ *is sufficiently small, then the Cauchy problem* (1.1) *has a unique global solution* u *in* $X^s_{q,r,\alpha}$. *Moreover, the solutions depend on the initial data continuously, namely, the* *flow mapping* $u_0 \mapsto u$ *is continuous from* $H^s(\mathbb{R}^n)$ *to* $X^s_{q,r,\alpha}$. *Here, we put* $\alpha := 2$ *for* $s < 3$, *and* $\alpha := q$ *for* $3 \le s$.

Remark 1.6. *The assumption* $n \ge 8$ *is natural since the condition* $p(s) \le s$ *holds if and only if*

$$s_1 := \frac{n+2-\sqrt{(n-2)^2-32}}{4} \le s \le s_2 := \frac{n+2+\sqrt{(n-2)^2-32}}{4}, \tag{1.12}$$

and $n \ge 8$ *is required for* $(n-2)^2 - 32 \ge 0$. *We see that* $s_1 \le 2$ *when* $n \ge 8$, $s_2 = 3$ *when* $n = 8$, $s_2 = (11 + \sqrt{17})/4 = 3.78\cdots$ *when* $n = 9$, *and* $s_2 > 4$ *when* $n \ge 10$.

Remark 1.7. *Let* $s/2 < p(s) < 2$ *with* $3 \le s < 4$. *The inequality* $p(s) < 2$ *implies* $s < (n-4)/2$. *If* $n \le 11$, *then* $s/2 < p(s)$ *holds; on the other hand, if* $n \ge 12$, *the inequality* $s/2 < p(s)$ *implies* $s < s_3 := (n + 4 - \sqrt{(n+4)(n-12)})/4$. *(If* $n \le 11$, *we put* $s_3 = \infty$.*) Therefore, we have* $3 \le s < \min\{(n-4)/2, s_3\}$, *so that* $11 \le n \le 13$ *and*

$$3 \le s < \begin{cases} 7/2 & \text{when} \quad n = 11, \\ 4 & \text{when} \quad n = 12, \\ (17-\sqrt{17})/4\,(= 3.21\cdots) & \text{when} \quad n = 13. \end{cases} \tag{1.13}$$

However, in Theorem 1.5, we further assume (1.11). *Namely, we can treat all the range of* s *when* $n = 11$, *while there are open ranges of* s *to be filled when* $n \ge 12$, *possibly by the technical reason (see §4.3).*

Combining the known results with Theorem 1.5, we obtain the following corollary.

Corollary 1.8. *Let* $n \ge 1$, $0 \le s < \min\{n/2, 4\}$, $s/2 < p = p(s)$. *When* $s \ge 3$ *with* $p < 2$, *assume* (1.11). *Let* f *satisfy* $N(p)$. *The Cauchy problem* (1.1) *is well-posed time-globally in* $C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap Y$ *for small initial data in* $H^s(\mathbb{R}^n)$, *where* Y *is some auxiliary function space.*

Proof. The case $s = 2$ has been shown in [3, Theorem 1.4]. The case $n \ge 1$, $0 \le s < n/2$, $s < p = p(s)$ has been shown in [4, 12]. The remaining case is proved by Theorem 1.5. □

Throughout the paper, we denote by $A \lesssim B$ the inequality $A \le CB$ for some constant $C > 0$ which is not essential in our argument. For any function $f = f(t)$ or $f = f(x)$, its Fourier transform is denoted by $\widehat{f}$. For any function $f = f(t, x)$, $\widehat{f}$ and $\widetilde{f}$ denote its Fourier transform by x and (t, x) variables, respectively. We abbreviate $L^r(\mathbb{R}^n)$ by L^r, $L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ by L^qL^r, $\ell^\alpha L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ by $\ell^\alpha L^qL^r$ as long as no fear of confusion. We use the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ and Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ (see [2] for the definitions and properties).

2 Preliminaries

We prepare several lemmas which are required in this paper.

Lemma 2.1. *Let $1 < p < \infty$, $0 < s < \min\{2, p\}$, $1 \leq r_0, r_1, \rho, \alpha \leq \infty$. Let f satisfy $N(p)$. Let $1/r_0 = (p-1)/\rho + 1/r_1$. Then we have*

$$\|f(u)\|_{\dot{B}^s_{r_0,\alpha}(\mathbb{R}^n)} \lesssim \|u\|^{p-1}_{L^\rho(\mathbb{R}^n)} \|u\|_{\dot{B}^s_{r_1,\alpha}(\mathbb{R}^n)}$$

for any $u \in L^\rho(\mathbb{R}^n) \cap \dot{B}^s_{r_1,\alpha}(\mathbb{R}^n)$.

Proof. The case $\alpha = 2$ has been proved in [4, Lemma 3.4]. The case $\alpha \neq 2$ follows from its straightforward modification. □

Lemma 2.2. *Let $s \in \mathbb{R}$, $0 < \theta < 1$, $1 \leq q_0, q_1 \leq \infty$, $1 \leq \alpha \leq \infty$. Put $1/q := (1-\theta)/q_0 + \theta/q_1$. Let V, V_0, V_1 be Banach spaces which satisfy $V_0 \cap V_1 \subset V$ and $\|u\|_V \lesssim \|u\|^{1-\theta}_{V_0} \|u\|^\theta_{V_1}$ for any $u \in V_0 \cap V_1$. Then the inequality*

$$\|u\|_{B^{\theta s}_{q,\alpha/\theta}(\mathbb{R},V)} \lesssim \|u\|^{1-\theta}_{B^0_{q_0,\infty}(\mathbb{R},V_0)} \|u\|^\theta_{B^s_{q_1,\alpha}(\mathbb{R},V_1)}$$

holds for any $u \in B^0_{q_0,\infty}(\mathbb{R}, V_0) \cap B^s_{q_1,\alpha}(\mathbb{R}, V_1)$.

Proof. By the inequality $\|\zeta *_t u\|_V \lesssim \|\zeta *_t u\|^{1-\theta}_{V_0} \|\zeta *_t u\|^\theta_{V_1}$ for $\zeta = \psi, \varphi_j$, we have $\|\psi *_t u\|_{L^q(\mathbb{R},V)} \lesssim \|\psi *_t u\|^{1-\theta}_{L^{q_0}(\mathbb{R},V_0)} \|\psi *_t u\|^\theta_{L^{q_1}(\mathbb{R},V_1)}$, and

$$\left\{ \sum_{j \geq 1} \left(2^{\theta s j} \|\varphi_j *_t u\|_{L^q(\mathbb{R},V)} \right)^{\alpha/\theta} \right\}^{\theta/\alpha}$$
$$\lesssim \sup_{k \geq 1} \|\varphi_k *_t u\|^{1-\theta}_{L^{q_0}(\mathbb{R},V_0)} \left\{ \sum_{j \geq 1} \left(2^{sj} \|\varphi_j *_t u\|_{L^{q_1}(\mathbb{R},V_1)} \right)^\alpha \right\}^{\theta/\alpha}$$

when $\alpha \neq \infty$. So that, we have the required estimate when $\alpha \neq \infty$. The case $\alpha = \infty$ also follows by trivial modification. □

Lemma 2.3. *Let $s \in \mathbb{R}$, $1 \leq r, \alpha \leq \infty$. The norm defined by*

$$\|u\|_{\widetilde{B}^s_{r,\alpha}(\mathbb{R}^n)} := \left\| \left(\mathscr{F}^{-1}_\xi \left(\widehat{\psi}(|\xi|^2) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} + \begin{cases} \left\{ \sum_{j \geq 1} \left(2^{sj/2} \left\| \left(\mathscr{F}^{-1}_\xi \left(\widehat{\varphi}(|\xi|^2/2^j) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} \right)^\alpha \right\}^{1/\alpha} & \text{if } \alpha < \infty, \\ \sup_{j \geq 1} 2^{sj/2} \left\| \left(\mathscr{F}^{-1}_\xi \left(\widehat{\varphi}(|\xi|^2/2^j) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} & \text{if } \alpha = \infty \end{cases}$$

for any function u is equivalent to the norm $\|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)}$.

Proof. First, we show $\|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)} \lesssim \|u\|_{\widetilde{B}^s_{r,\alpha}(\mathbb{R}^n)}$. We use the identity

$$\sum_{k=-2}^{2} \widehat{\varphi}(|\xi|^2/2^{2j+k}) = 1$$

on the support of $\widehat{\varphi}(|\xi|/2^j)$, and the Young inequality to have

$$\begin{aligned}\|\varphi_j *_x u\|_{L^r(\mathbb{R}^n)} &= \left\| \left(\mathscr{F}_\xi^{-1} \widehat{\varphi}(|\xi|/2^j) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} \\ &\lesssim \sum_{k=-2}^{2} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi}(|\xi|^2/2^{2j+k}) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)}.\end{aligned}$$

Then we have

$$\begin{aligned}\left\{ \sum_{j\geq 1} \left(2^{sj} \|\varphi_j *_x u\|_{L^r(\mathbb{R}^n)} \right)^\alpha \right\}^{1/\alpha} &\lesssim \left\{ \sum_{j\geq 1} \left(2^{sj/2} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi}(|\xi|^2/2^j) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} \right)^\alpha \right\}^{1/\alpha} \\ &\quad + \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi}(|\xi|^2) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)}.\end{aligned}$$

The last term in the right hand side is bounded by

$$\left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\psi}(|\xi|^2) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} + \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi}(|\xi|^2/2) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)}$$

since $\widehat{\psi}(|\xi|^2) + \widehat{\varphi}(|\xi|^2/2) = 1$ on the support of $\widehat{\varphi}(|\xi|^2)$. Similarly, by

$$\widehat{\psi}(|\xi|^2) + \sum_{j=1}^{2} \widehat{\varphi}(|\xi|^2/2^j) = 1$$

on the support of $\widehat{\psi}(|\xi|)$, we have

$$\|\psi *_x u\|_{L^r(\mathbb{R}^n)} \lesssim \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\psi}(|\xi|^2) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} + \sum_{j=1}^{2} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi}(|\xi|^2/2^j) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)}.$$

So that, we obtain $\|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)} \lesssim \|u\|_{\widetilde{B}^s_{r,\alpha}(\mathbb{R}^n)}$. Next, we show the opposite inequality. Since $\sum_{k=-1}^{1} \widehat{\varphi}(|\xi|/2^{[j/2]+k}) = 1$ on the support of $\widehat{\varphi}(|\xi|^2/2^j)$ for $j \geq 1$, we have

$$\left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi}(|\xi|^2/2^j) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)} \lesssim \sum_{k=-1}^{1} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi}(|\xi|/2^{[j/2]+k}) \right) \right) *_x u \right\|_{L^r(\mathbb{R}^n)}$$

for $j \geq 1$ by the Young inequality. Then we have

$$\left\{\sum_{j\geq 1}\left(2^{sj/2}\left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi}(|\xi|^2/2^j)\right)\right)*_x u\right\|_{L^r(\mathbb{R}^n)}\right)^\alpha\right\}^{1/\alpha}$$
$$\lesssim \sum_{k=-1}^{1}\left\{\sum_{j\geq 1}\left(2^{sj/2}\left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi}(|\xi|/2^{[j/2]+k})\right)\right)*_x u\right\|_{L^r(\mathbb{R}^n)}\right)^\alpha\right\}^{1/\alpha}$$
$$\lesssim \|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)} + \sum_{k=-1}^{0}\left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi}(|\xi|/2^k)\right)\right)*_x u\right\|_{L^r(\mathbb{R}^n)}, \quad (2.1)$$

where we have used

$$2^{sj/2}\widehat{\varphi}(|\xi|/2^{[j/2]+k}) = \begin{cases} 2^{sm}\widehat{\varphi}(|\xi|/2^{m+k}) & \text{for } j=2m, \quad m=1,2,\cdots, \\ 2^{s/2}\cdot 2^{sm}\widehat{\varphi}(|\xi|/2^{m+k}) & \text{for } j=2m+1, \quad m=0,1,2,\cdots \end{cases}$$

for the last inequality. The last term in (2.1) is bounded by $\|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)}$ since $\widehat{\psi}(|\xi|)+\widehat{\varphi}(|\xi|/2)=1$ on the supports of $\widehat{\varphi}(|\xi|/2^{-1})$ and $\widehat{\varphi}(|\xi|)$. Similarly, since $\widehat{\psi}(|\xi|)+\widehat{\varphi}(|\xi|/2)=1$ on the support of $\widehat{\psi}(|\xi|^2)$, we have

$$\left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\psi}(|\xi|^2)\right)\right)*_x u\right\| \lesssim \|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)}.$$

Therefore, we have obtained $\|u\|_{\widetilde{B}^s_{r,\alpha}(\mathbb{R}^n)} \lesssim \|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)}$. □

Lemma 2.4. *(1) Let V be a Banach space. Let $1 \leq q < p < \infty$, and put $\theta := 1/q - 1/p$. Let $1 \leq \alpha \leq \infty$. Then the embedding $B^\theta_{q,\alpha}(\mathbb{R},V) \hookrightarrow L^{p,\alpha}(\mathbb{R},V)$ holds, namely, $\|u\|_{L^{p,\alpha}(\mathbb{R},V)} \lesssim \|u\|_{B^\theta_{q,\alpha}(\mathbb{R},V)}$ holds for any $u \in B^\theta_{q,\alpha}(\mathbb{R},V)$.*

(2) Let $n \geq 1$, $1 \leq r < r_0 < \infty$, and put $s/n := 1/r - 1/r_0$. Let $1 \leq \alpha \leq \infty$. Then the embedding $B^s_{r,\alpha}(\mathbb{R}^n) \hookrightarrow L^{r_0,\alpha}(\mathbb{R}^n)$ holds, namely, $\|u\|_{L^{r_0,\alpha}(\mathbb{R}^n)} \lesssim \|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)}$ holds for any $u \in B^s_{r,\alpha}(\mathbb{R}^n)$. Especially, $B^s_{r,\alpha}(\mathbb{R}^n) \hookrightarrow L^{r_0}(\mathbb{R}^n)$ holds if $\alpha \leq r_0$.

Proof. (1) For sufficiently small $\varepsilon > 0$, we define $p_\pm$ by $1/p_\pm := 1/p \mp \varepsilon$. We have

$$\|u\|_{L^{p_\pm}(\mathbb{R},V)} \leq \|\psi * u\|_{L^{p_\pm}(\mathbb{R},V)} + \sum_{j=1}^{\infty}\|\varphi_j * u\|_{L^{p_\pm}(\mathbb{R},V)},$$
$$\|\psi * u\|_{L^{p_\pm}(\mathbb{R},V)} \lesssim \|\psi * u\|_{L^q(\mathbb{R},V)}, \qquad \|\varphi_j * u\|_{L^{p_\pm}(\mathbb{R},V)} \lesssim 2^{(\theta\pm\varepsilon)j}\|\varphi_j * u\|_{L^q(\mathbb{R},V)}$$

by $\psi = \psi * (\psi + \varphi_1)$, $\varphi_j = \varphi_j * \sum_{k=-1}^{1}\varphi_{j+k}$ and the Young inequality. So that, we obtain

$$\|u\|_{L^{p_\pm}(\mathbb{R},V)} \lesssim \|u\|_{B^{\theta\pm\varepsilon}_{q,1}(\mathbb{R},V)}.$$

By the real interpolation $(L^{p_+}, L^{p_-})_{1/2,\alpha} = L^{p,\alpha}$ and $(B^{\theta+\varepsilon}_{q,1}, B^{\theta-\varepsilon}_{q,1})_{1/2,\alpha} = B^{\theta}_{q,\alpha}$, we obtain the required result. The proof of (2) follows analogously. We put $1/r_\pm := 1/r-(s\pm\varepsilon)/n = 1/r_0 \mp \varepsilon/n$ for sufficiently small $\varepsilon > 0$. We have $B^{s\pm\varepsilon}_{r,1}(\mathbb{R}^n) \hookrightarrow L^{r_\pm}(\mathbb{R}^n)$. By the interpolation $(L^{r_+}, L^{r_-})_{1/2,\alpha} = L^{r_0,\alpha}$ and $(B^{s+\varepsilon}_{r,1}, B^{s-\varepsilon}_{r,1})_{1/2,\alpha} = B^s_{r,\alpha}$, we have $B^s_{r,\alpha}(\mathbb{R}^n) \hookrightarrow L^{r_0,\alpha}(\mathbb{R}^n)$. By the identity $L^{r_0,r_0} = L^{r_0}$ and the embedding $B^s_{r,\alpha} \hookrightarrow B^s_{r,r_0}$ for $\alpha \le r_0$, we obtain the last result. □

We note the embedding $\|u\|_{\ell^\alpha(L^q(\mathbb{R},L^r(\mathbb{R}^n)))} \lesssim \|u\|_{L^q(\mathbb{R},B^0_{r,\alpha}(\mathbb{R}^n))}$ if $q \le \alpha$ by the Minkowski inequality, so that, $\|u\|_{\ell^\alpha(L^q(\mathbb{R},L^r(\mathbb{R}^n)))} \lesssim \|u\|_{L^q(\mathbb{R},L^r(\mathbb{R}^n))}$ if $q \le \alpha$ and $1 < r \le 2 \le \alpha$ by $\|u\|_{B^0_{r,\alpha}} \lesssim \|u\|_{B^0_{r,2}} \lesssim \|u\|_{L^r}$ (see [2, Theorem 6.4.4]).

Lemma 2.5. *Let $n \ge 1$, $0 < \theta < 1$, $1 \le r_0, \bar{r}, \bar{q}, \rho, \gamma \le \infty$, $1 < q_0 < \infty$, and let $2/\bar{q} - \delta(\bar{r}) = 2/q_0 - \delta(r_0) = 2(1-\theta)$, $2/\gamma = \delta(\rho)$. Let $\max\{2, \rho', \gamma'\} \le \alpha \le \infty$. Let $\rho < \infty$ if $\alpha < \infty$. Let r_0 satisfy $\rho' \le r_0 < \bar{r}$ or $\bar{r} < r_0 \le \rho'$. Then the inequality*

$$\|u\|_{\ell^\alpha L^{q_0,\infty}(\mathbb{R},L^{r_0}(\mathbb{R}^n))} \lesssim \|u\|^{1-k}_{\ell^\alpha L^{\bar{q}}(\mathbb{R},L^{\bar{r}}(\mathbb{R}^n))} \|u\|^k_{B^\theta_{\gamma',\alpha}(\mathbb{R},L^{\rho'}(\mathbb{R}^n))}$$

holds for any u, where $0 < k \le 1$ is the number defined by $1/r_0 = (1-k)/\bar{r} + k/\rho'$.

Proof. Since q_1 defined by $1/q_1 := 1/q_0 + k\theta$ satisfies $1/q_1 = (1-k)/\bar{q} + k/\gamma'$, we have $1 \le q_1 < q_0 < \infty$. We consider only the case $\alpha < \infty$ since the case $\alpha = \infty$ follows by its straightforward modification. By Lemmas 2.2 and 2.4, we have

$$\begin{aligned} \|\varphi_j *_x u\|_{L^{q_0,\infty}L^{r_0}} &\lesssim \|\varphi_j *_x u\|_{B^{k\theta}_{q_1,\infty}L^{r_0}} \\ &\lesssim \|\varphi_j *_x u\|^{1-k}_{B^0_{\bar{q},\infty}L^{\bar{r}}} \|\varphi_j *_x u\|^k_{B^\theta_{\gamma',\infty}L^{\rho'}}, \end{aligned}$$

which yields

$$\left(\sum_{j\ge1} \|\varphi_j *_x u\|^\alpha_{L^{q_0,\infty}L^{r_0}}\right)^{1/\alpha} \lesssim \left(\sum_{j\ge1} \|\varphi_j *_x u\|^\alpha_{B^0_{\bar{q},\infty}L^{\bar{r}}}\right)^{(1-k)/\alpha} \left(\sum_{j\ge1} \|\varphi_j *_x u\|^\alpha_{B^\theta_{\gamma',\infty}L^{\rho'}}\right)^{k/\alpha}.$$

For the last term, by the definition of the Besov space, we have

$$\begin{aligned} \left(\sum_{j\ge1} \|\varphi_j *_x u\|^\alpha_{B^\theta_{\gamma',\infty}L^{\rho'}}\right)^{1/\alpha} &\lesssim \left(\sum_{j\ge1} \|\psi *_t \varphi_j *_x u\|^\alpha_{L^{\gamma'}L^{\rho'}}\right)^{1/\alpha} \\ &\quad + \left(\sum_{j,m\ge1} \left(2^{\theta m}\|\varphi_m *_t \varphi_j *_x u\|_{L^{\gamma'}L^{\rho'}}\right)^\alpha\right)^{1/\alpha}, \end{aligned}$$

where we have used $B^{\theta}_{\gamma',\alpha} \hookrightarrow B^{\theta}_{\gamma',\infty}$. By the assumption $\gamma' \le \alpha$, the first term in the right hand side is bounded by $\|\psi *_t u\|_{L^{\gamma'} B^0_{\rho',\alpha}}$, and the second term is bounded by $\left(\sum_{m\ge 1} \left(2^{\theta m} \|\varphi_m *_t u\|_{L^{\gamma'} B^0_{\rho',\alpha}} \right)^{\alpha} \right)^{1/\alpha}$. So that, we have

$$\left(\sum_{j\ge 1} \|\varphi_j *_x u\|^{\alpha}_{B^{\theta}_{\gamma',\infty} L^{\rho'}} \right)^{1/\alpha} \lesssim \|u\|_{B^{\theta}_{\gamma',\alpha} B^0_{\rho',\alpha}}.$$

Therefore, we obtained

$$\left(\sum_{j\ge 1} \|\varphi_j *_x u\|^{\alpha}_{L^{q_0,\infty} L^{r_0}} \right)^{1/\alpha} \lesssim \|u\|^{1-k}_{\ell^{\alpha} B^0_{\bar{q},\infty} L^{\bar{r}}} \|u\|^{k}_{B^{\theta}_{\gamma',\alpha} B^0_{\rho',\alpha}}. \tag{2.2}$$

Since we have

$$\|\psi *_x u\|_{L^{q_0,\infty} L^{r_0}} \lesssim \|u\|^{1-k}_{\ell^{\alpha} B^0_{\bar{q},\infty} L^{\bar{r}}} \|u\|^{k}_{B^{\theta}_{\gamma',\alpha} B^0_{\rho',\alpha}} \tag{2.3}$$

by the similar argument, we have obtained

$$\|u\|_{\ell^{\alpha} L^{q_0,\infty} L^{r_0}} \lesssim \|u\|^{1-k}_{\ell^{\alpha} B^0_{\bar{q},\infty} L^{\bar{r}}} \|u\|^{k}_{B^{\theta}_{\gamma',\alpha} B^0_{\rho',\alpha}}.$$

By the embeddings $L^{\bar{q}} \hookrightarrow B^0_{\bar{q},\infty}$ and $L^{\rho'} \hookrightarrow B^0_{\rho',\alpha}$, we obtain the required result, where we have used $2 \le \rho \le \infty$ by $2/\gamma = \delta(\rho)$, and $\rho < \infty$ if $\alpha < \infty$ for $L^{\rho'} \hookrightarrow B^0_{\rho',\alpha}$. □

3 Proof of Theorem 1.3

Throughout this section we put $U(t) = \exp(-it\Delta)$. The solution u to (1.6) is written as

$$u(t) = \int_0^t U(t-t') f(t') dt'.$$

We prepare several claims to prove the theorem. We define a function v_0 by

$$\widehat{v_0}(\xi) := \text{p.v.} \int_{-\infty}^{\infty} \frac{\widetilde{f}(\tau,\xi)}{2\pi i(\tau - |\xi|^2)} d\tau.$$

Claim 3.1. $\|u\|_{B^{\theta}_{q,\alpha} L^r} \lesssim \|f\|_{B^{\theta}_{\gamma',\alpha} L^{\rho'}} + \|v_0\|_{B^{2\theta}_{2,\alpha}}.$

Proof. Since u is written as $\widehat{u}(t,\xi) = \int_0^t e^{i(t-t')|\xi|^2} \widehat{f}(t',\xi) dt'$, we have

$$\widehat{u}(t,\xi) = \text{p.v.} \int_{-\infty}^{\infty} \widetilde{f}(\tau,\xi) \frac{e^{it\tau} - e^{it|\xi|^2}}{2\pi i(\tau - |\xi|^2)} d\tau. \tag{3.1}$$

So that, we have

$$
\begin{aligned}
\varphi_j *_t \widehat{u}(t,\xi) &= \text{p.v.} \int_{-\infty}^{\infty} \widetilde{f}(\tau,\xi) \frac{e^{it\tau}\widehat{\varphi_j}(\tau) - e^{it|\xi|^2}\widehat{\varphi_j}(|\xi|^2)}{2\pi i(\tau - |\xi|^2)} d\tau \\
&= \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{2\pi i(\tau - |\xi|^2)} \widehat{\varphi_j}(\tau)\widetilde{f}(\tau,\xi) d\tau - e^{it|\xi|^2}\widehat{\varphi_j}(|\xi|^2)\widehat{v_0}(\xi),
\end{aligned}
$$

where we have used $\varphi_j *_t e^{it\tau} = e^{it\tau}\widehat{\varphi_j}(\tau)$ and $\varphi_j *_t e^{it|\xi|^2} = e^{it|\xi|^2}\widehat{\varphi_j}(|\xi|^2)$. By the fact

$$
\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{2\pi i(\tau - |\xi|^2)} d\tau = e^{it|\xi|^2} \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{it\nu}}{2\pi i\nu} d\nu = \frac{1}{2}\text{sign}(t) e^{it|\xi|^2}, \tag{3.2}
$$

we have

$$
\begin{aligned}
\varphi_j *_t u(t,x) = \frac{1}{2} \int_{-\infty}^{\infty} & \text{sign}(t-t')U(t-t')(\varphi_j *_t f)(t',x) dt' \\
& - U(t) \left(\left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi_j}(|\xi|^2) \right) \right) *_x v_0 \right)(x).
\end{aligned}
$$

By Lemma 1.1, we obtain

$$
\|\varphi_j *_t u\|_{L^q L^r} \lesssim \|\varphi_j *_t f\|_{L^{\gamma'} L^{\rho'}} + \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi_j}(|\xi|^2) \right) \right) *_x v_0 \right\|_{L^2}.
$$

Similarly, we also have

$$
\|\psi *_t u\|_{L^q L^r} \lesssim \|\psi *_t f\|_{L^{\gamma'} L^{\rho'}} + \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\psi}(|\xi|^2) \right) \right) *_x v_0 \right\|_{L^2}.
$$

So that, we obtain the required estimate by the definition of the Besov space and Lemma 2.3. □

Claim 3.2. *For any real numbers q_0 and r_0 with*

$$
1 \le q_0 \le \infty, \quad 1 \le r_0 \le 2, \quad 2/q_0 - \delta(r_0) = 2(1-\theta), \tag{3.3}
$$

the following estimate holds;

$$
\|v_0\|_{B^{2\theta}_{2,\alpha}} \lesssim \|f\|_{B^\theta_{\gamma',\alpha} L^{\rho'}} + \|f\|_{\ell^\alpha L^{q_0,\infty} L^{r_0}}. \tag{3.4}
$$

Proof. The definition of v_0 gives the equation

$$
\begin{aligned}
\widehat{\varphi_j}(|\xi|^2)\widehat{v_0}(\xi) &= \text{p.v.} \int_{-\infty}^{\infty} \frac{\widehat{\varphi_j}(|\xi|^2)\widehat{\chi_j}(\tau)\widetilde{f}(\tau,\xi)}{2\pi i(\tau - |\xi|^2)} d\tau \\
&\quad + \int_{-\infty}^{\infty} \frac{\widehat{\varphi_j}(|\xi|^2)\left(1 - \widehat{\chi_j}(\tau)\right)\widehat{\chi_j}(|\xi|^2)\widetilde{f}(\tau,\xi)}{2\pi i(\tau - |\xi|^2)} d\tau \\
&=: V_{1j} + V_{2j},
\end{aligned}
$$

where we have put the two terms in the right hand side as V_{1j} and V_{2j}. The term V_{1j} is estimated by

$$\begin{aligned} V_{1j} &= \widehat{\varphi_j}(|\xi|^2) \text{ p.v. } \mathscr{F}_t^{-1}\left(\frac{\widehat{\chi_j}(\tau)\widetilde{f}(\tau,\xi)}{i(\tau-|\xi|^2)}\right)\Big|_{t=0} \\ &= \frac{1}{2}\,\widehat{\varphi_j}(|\xi|^2)\mathscr{F}_x\int_{-\infty}^{\infty}\text{sign}(-t')U(-t')\left(\chi_j *_t f\right)(t',\xi)dt', \end{aligned}$$

where we have used (3.2) for the second equation. The term V_{2j} is estimated by

$$\begin{aligned} V_{2j} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{K_j}(\tau,\xi)\widehat{\chi_j}(|\xi|^2)\widetilde{f}(\tau,\xi)d\tau, \\ &= \mathscr{F}_t^{-1}\left(\widetilde{K_j}(\tau,\xi)\widehat{\chi_j}(|\xi|^2)\widetilde{f}(\tau,\xi)\right)\Big|_{t=0} \\ &= \mathscr{F}_x\left(K_j *_{t,x}\left(\mathscr{F}_\xi^{-1}\left(\widehat{\chi_j}(|\xi|^2)\right)\right) *_x f\right)\Big|_{t=0}, \end{aligned}$$

where we have put

$$\begin{aligned} K_0(t,x) &:= \frac{1}{(2\pi)^{n+1}}\iint_{\mathbb{R}^{1+n}}\frac{e^{it\tau+ix\xi}\widehat{\varphi}(|\xi|^2)\left(1-\widehat{\chi}(\tau)\right)}{i(\tau-|\xi|^2)}d\tau d\xi, \\ K_j(t,x) &:= 2^{nj/2}K_0(2^j t, 2^{j/2}x) \end{aligned}$$

for $j \geq 1$. By the dual of Lemma 1.1, we have $\|V_{1j}\|_{L^2} \lesssim \|\chi_j *_t f\|_{L^{\gamma'}L^{\rho'}}$. Let $1 \leq \widetilde{q}_0, \widetilde{r}_0 \leq \infty$ be the numbers defined by $1 = 1/\widetilde{q}_0 + 1/q_0$, $3/2 = 1/\widetilde{r}_0 + 1/r_0$, where such $\widetilde{r}_0$ exists by $1 \leq r_0 \leq 2$. We have

$$\|V_{2j}\|_{L^2} \leq \|K_j\|_{L^{\widetilde{q}_0,1}L^{\widetilde{r}_0}}\left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\chi_j}(|\xi|^2)\right)\right) *_x f\right\|_{L^{q_0,\infty}L^{r_0}}$$

by the Hölder and Young inequalities. By scaling, we have

$$\|K_j\|_{L^{\widetilde{q}_0,1}L^{\widetilde{r}_0}} = 2^{-\theta j}\|K_0\|_{L^{\widetilde{q}_0,1}L^{\widetilde{r}_0}}.$$

Since K_0 satisfies $|K_0(t,x)| \lesssim (1+|t|+|x|)^{-\ell}$ for any $\ell \geq 1$ by the integration by parts, the term $\|K_0\|_{L^{\widetilde{q}_0,1}L^{\widetilde{r}_0}}$ is finite. So that, we have obtained

$$\|\widehat{\varphi_j}(|\xi|^2)\widehat{v_0}(\xi)\|_{L^2_\xi} \lesssim \|\chi_j *_t f\|_{L^{\gamma'}L^{\rho'}} + 2^{-\theta j}\left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\chi_j}(|\xi|^2)\right)\right) *_x f\right\|_{L^{q_0,\infty}L^{r_0}}.$$

Similarly, we also have

$$\|\widehat{\psi}(|\xi|^2)\widehat{v_0}(\xi)\|_{L^2_\xi} \lesssim \|\chi_0 *_t f\|_{L^{\gamma'}L^{\rho'}} + \left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\chi_0}(|\xi|^2)\right)\right) *_x f\right\|_{L^{q_0,\infty}L^{r_0}}.$$

Therefore we obtain the required result under the condition (3.3). □

Claim 3.3. *Let $\alpha \le q$. For any real numbers q_1 and r_1 with*

$$1 \le q_1 \le q, \quad 1 \le r_1 \le r, \quad 2/q_1 - \delta(r_1) = 2(1-\theta), \tag{3.5}$$

the following estimate holds;

$$\|u\|_{L^q B^{2\theta}_{r,\alpha}} \lesssim \|f\|_{B^{\theta}_{\gamma',\alpha} L^{\rho'}} + \|f\|_{\ell^\alpha L^{q_1,\infty} L^{r_1}} + \|v_0\|_{B^{2\theta}_{2,\alpha}}. \tag{3.6}$$

Proof. We use (3.1) to have

$$\begin{aligned}\widehat{\varphi_j}(|\xi|^2)\widehat{u}(t,\xi) &= \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{it\tau}\widehat{\varphi_j}(|\xi|^2)}{2\pi i(\tau - |\xi|^2)}\widetilde{f}(\tau,\xi)d\tau - e^{it|\xi|^2}\widehat{\varphi_j}(|\xi|^2)\widehat{v_0}(\xi) \\ &=: U_{1j} - U_{2j},\end{aligned}$$

where we have put the terms in the right hand side as U_{1j} and U_{2j}. The term U_{1j} is estimated by

$$\begin{aligned}U_{1j} &= \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{it\tau}\widehat{\varphi_j}(|\xi|^2)\widehat{\chi_j}(\tau)}{2\pi i(\tau - |\xi|^2)}\widetilde{f}(\tau,\xi)d\tau \\ &\quad + \int_{-\infty}^{\infty} \frac{e^{it\tau}\widehat{\varphi_j}(|\xi|^2)(1-\widehat{\chi_j}(\tau))\widehat{\chi_j}(|\xi|^2)}{2\pi i(\tau - |\xi|^2)}\widetilde{f}(\tau,\xi)d\tau \\ &= \mathscr{F}_t^{-1}\left(\frac{1}{i(\tau-|\xi|^2)}\mathscr{F}_t\mathscr{F}_x\left(\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi_j}(|\xi|^2)\right)\right) *_x \chi_j *_t f\right)\right) \\ &\quad + \mathscr{F}_x\left(K_j *_{t,x}\left(\mathscr{F}_\xi^{-1}\left(\widehat{\chi_j}(|\xi|^2)\right)\right) *_x f\right).\end{aligned}$$

Then we have

$$\begin{aligned}&\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi_j}(|\xi|^2)\right)\right) *_x u(t,x) \\ &\quad = \frac{1}{2}\int_{-\infty}^{\infty} \text{sign}(t-t')U(t-t')\left(\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi_j}(|\xi|^2)\right)\right) *_x \chi_j *_t f\right)(t',x)dt' \\ &\qquad + K_j *_{t,x}\left(\mathscr{F}_\xi^{-1}\left(\widehat{\chi_j}(|\xi|^2)\right)\right) *_x f - U(t)\left(\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi_j}(|\xi|^2)\right)\right) *_x v_0\right)(x),\end{aligned}$$

where we have used (3.2) for the first term in the right hand side. By Lemma 1.1, we have

$$\begin{aligned}&\left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi_j}(|\xi|^2)\right)\right) *_x u\right\|_{L^q L^r} \lesssim \left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi_j}(|\xi|^2)\right)\right) *_x \chi_j *_t f\right\|_{L^{\gamma'} L^{\rho'}} \\ &\quad + \left\|K_j *_{t,x}\left(\mathscr{F}_\xi^{-1}\left(\widehat{\chi_j}(|\xi|^2)\right)\right) *_x f\right\|_{L^q L^r} + \left\|\left(\mathscr{F}_\xi^{-1}\left(\widehat{\varphi_j}(|\xi|^2)\right)\right) *_x v_0\right\|_{L^2} \\ &\qquad =: I_j + II_j + III_j.\end{aligned}$$

Let $1 \le \widetilde{q}_1, \widetilde{r}_1 \le \infty$ be the numbers defined by $1/q = 1/\widetilde{q}_1 + 1/q_1 - 1$, $1/r = 1/\widetilde{r}_1 + 1/r_1 - 1$, where such $\widetilde{q}_1$ and $\widetilde{r}_1$ exist by $1 \le q_1 \le q$ and $1 \le r_1 \le r$. By the Young and generalized Young inequalities, we have

$$I_j \lesssim \|\chi_j *_t f\|_{L^{\gamma'} L^{\rho'}},$$

$$II_j \lesssim \|K_j\|_{L^{\tilde{q}_1,1}L^{\tilde{r}_1}} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\chi_j}(|\xi|^2) \right) \right) *_x f \right\|_{L^{q_1,\infty}L^{r_1}}.$$

Since $\|K_j\|_{L^{\tilde{q}_1,1}L^{\tilde{r}_1}} = 2^{-\theta j}\|K_0\|_{L^{\tilde{q}_1,1}L^{\tilde{r}_1}}$ by scaling, and $\|K_0\|_{L^{\tilde{q}_1,1}L^{\tilde{r}_1}} < \infty$, we have

$$\begin{aligned} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi_j}(|\xi|^2) \right) \right) *_x u \right\|_{L^qL^r} \lesssim \|\chi_j *_t f\|_{L^{\gamma'}L^{\rho'}} + 2^{-\theta j} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\chi_j}(|\xi|^2) \right) \right) *_x f \right\|_{L^{q_1,\infty}L^{r_1}} \\ + \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi_j}(|\xi|^2) \right) \right) *_x v_0 \right\|_{L^2}. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\psi}(|\xi|^2) \right) \right) *_x u \right\|_{L^qL^r} \lesssim \|\chi_0 *_t f\|_{L^{\gamma'}L^{\rho'}} + \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\chi_0}(|\xi|^2) \right) \right) *_x f \right\|_{L^{q_1,\infty}L^{r_1}} \\ + \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\psi}(|\xi|^2) \right) \right) *_x v_0 \right\|_{L^2}. \end{aligned}$$

So that, by Lemma 2.3, we obtain

$$\begin{aligned} \|u\|_{L^qB^{2\theta}_{r,\alpha}} &\lesssim \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\psi}(|\xi|^2) \right) \right) *_x u \right\|_{L^qL^r} + \left\| 2^{\theta j} \left\| \left(\mathscr{F}_\xi^{-1} \left(\widehat{\varphi_j}(|\xi|^2) \right) \right) *_x u \right\|_{L^qL^r} \right\|_{\ell^\alpha_{j\ge 1}} \\ &\lesssim \|f\|_{B^\theta_{\gamma',\alpha}L^{\rho'}} + \|f\|_{\ell^\alpha L^{q_1,\infty}L^{r_1}} + \|v_0\|_{B^{2\theta}_{2,\alpha}}, \end{aligned}$$

where we have used $\alpha \le q$ for the first inequality. □

Claim 3.4. *Let $2 \le \alpha \le \infty$. Let $\rho < \infty$ when $\alpha < \infty$. Then the following estimate holds;*

$$\|v_0\|_{B^{2\theta}_{2,\alpha}} \lesssim \|f\|_{B^\theta_{\gamma',\alpha}L^{\rho'}} + \|f\|_{\ell^\alpha L^{\bar{q}}L^{\bar{r}}}.$$

Proof. The required estimate follows from Claim 3.2 with $(q_0, r_0) := (\bar{q}, \bar{r})$ if $\bar{r} \le 2$. When $\bar{r} > 2$, we put $r_0 := 2$ and $1/q_0 := 1 - \theta$. Then we have $\rho' \le r_0 < \bar{r}$, and we apply Lemma 2.5 to the last term in (3.4). Then we obtain the required result. □

We prove the theorem. The result (1) follows from Claim 3.1 and Claim 3.4. The result (2) follows from Claim 3.3 with $(q_1, r_1) = (\bar{q}, \bar{r})$ and Claim 3.4 if $\bar{r} \le r$. When $\bar{r} > r$, we put $r_1 := 2$ and $1/q_1 := 1 - \theta$. Then we have $\rho' \le r_1 < \bar{r}$, and we apply Lemma 2.5 to the second term in (3.6). And we obtain the required result.

Finally, we prove that $u \in C(\mathbb{R}, H^s(\mathbb{R}^n))$ if $\alpha = 2$. We put $u_N = \widehat{\psi} *_x u + \sum_{j \le N} \widehat{\varphi_j} *_x u$, which satisfies (1.6) with f replaced by $f_N = \widehat{\psi} *_x f + \sum_{j \le N} \widehat{\varphi_j} *_x f$. By Lemma 1.1, we see that $u_N \in C(\mathbb{R}, H^s(\mathbb{R}^n))$. Moreover, by the estimate (2) in the present theorem, u_N tends to u in $L^\infty(\mathbb{R}, H^s(\mathbb{R}^n))$ since f_N tends to f in $B^\theta_{\gamma',\alpha}(\mathbb{R}, L^{\rho'}(\mathbb{R}^n)) \cap \ell^\alpha L^{\bar{q}}(\mathbb{R}, L^{\bar{r}}(\mathbb{R}^n))$. Therefore $u \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.

4 Proof of Theorem 1.5

We regard the solution of the Cauchy problem (1.1) as the fixed point of the integral equation given by

$$u(t) = \Phi(u)(t) := U(t)u_0 + \int_0^t U(t-t')f(u(t'))dt'$$

for $t \in \mathbb{R}$, where $u(t) := u(t, \cdot)$. Let n, s, p satisfy the assumption in the theorem. For any given $2 \le \gamma \le \infty$, we define ρ, q and r by

$$q := p\gamma', \qquad \frac{2}{\gamma} - \delta(\rho) = \frac{2}{q} - \delta(r) = 0.$$

We note that (γ, ρ) and (q, r) form admissible pairs if $2 \le q \le \infty$. We put

$$\frac{1}{m(r,s)} := \frac{1}{r} - \frac{s}{n} > 0, \tag{4.1}$$

where the last inequality holds since $1/m(r,s) = 2(p/(p-1) - 1/\gamma')/np > 0$ by the assumption $p = 1 + 4/(n-2s)$. Moreover, $m(r,s)$ satisfies

$$\frac{1}{\rho'} = \frac{p-1}{m(r,s)} + \frac{1}{r}. \tag{4.2}$$

For any $2 \le \alpha \le \infty$, we put $X := X^s_{q,r,\alpha}$ and $X(R) := \{u \in X \,;\, u(0) = u_0,\ \|u\|_X \le R\}$ for $R > 0$. We show that Φ is a contraction mapping on $X(R)$ for some $R > 0$. We separate the proof of the theorem into three cases $1 < s < 2$, $2 < s < 3$ and $3 \le s < 4$. We prove the continuous dependence of the solutions to the initial data only for the case $1 < s < 2$ since the case $2 < s < 4$ follows analogously. In the following, after we put $\alpha = 2$ for the case $1 < s < 3$, we still continue to use α since we would like to use the proof for the case $1 < s < 3$ to prove the case $3 \le s < 4$.

4.1 The case $1 < s < 2$

We put $\theta := s/2$, $\gamma := 2(n+2)/n$, $\alpha := 2$. Then $\rho = 2(n+2)/n$, $2 < q < \infty$, where $2 < q$ holds since it is rewritten as $n - 2s < 2(n+2)$ by $p = 1 + 4/(n-2s)$. So that, $2 < r < 2n/(n-2)$. We start from the basic estimate.

Claim 4.1. *The following estimates hold;*

$$\begin{aligned} \|\Phi(u)\|_{L^\infty L^2 \cap L^q L^r} &\lesssim \|u_0\|_{L^2} + \|u\|^{p-1}_{L^q B^s_{r,m(r,s)}} \|u\|_{L^q L^r}, \\ d(\Phi(u), \Phi(v)) &\lesssim \max_{w=u,v} \|w\|^{p-1}_{L^q B^s_{r,m(r,s)}} d(u,v) \end{aligned}$$

for any u and v.

Proof. By Lemma 1.1, we have

$$\|\Phi(u)\|_{L^\infty L^2 \cap L^q L^r} \lesssim \|u_0\|_{L^2} + \|f(u)\|_{L^{\gamma'} L^{\rho'}}.$$

By (4.2), we have

$$\|f(u)\|_{L^{\gamma'} L^{\rho'}} \lesssim \|u\|_{L^q L^{m(r,s)}}^{p-1} \|u\|_{L^q L^r} \lesssim \|u\|_{L^q B^s_{r,m(r,s)}}^{p-1} \|u\|_{L^q L^r} \tag{4.3}$$

by the Hölder inequality and Lemma 2.4. So that, we obtain the first inequality. We also obtain the second inequality similarly by

$$\|\Phi(u) - \Phi(v)\|_{L^\infty L^2 \cap L^q L^r} \lesssim \|f(u) - f(v)\|_{L^{\gamma'} L^{\rho'}} \lesssim \max_{w=u,v} \|w\|_{L^q B^s_{r,m(r,s)}}^{p-1} \|u - v\|_{L^q L^r}. \quad \square$$

We put $\bar{q} := \gamma'$. We define $\bar{r}$ by the equation $2/\bar{q} - \delta(\bar{r}) = 2(1-\theta)$. Since $\rho < \infty$, $1 < \bar{q} \le \alpha \le q$, and $1 < \bar{r} < \infty$, we use Lemmas 1.1, 1.2, and Theorem 1.3 to have

$$\|\Phi(u)\|_X \lesssim \|u_0\|_{H^s} + \|f(u)\|_{B^\theta_{\gamma',\alpha} L^{\rho'}} + \|f(u)\|_{\ell^\alpha L^{\bar{q}} L^{\bar{r}}}. \tag{4.4}$$

We estimate the second and third terms in the right hand side, respectively.

Claim 4.2. $\quad \|f(u)\|_{\ell^\alpha L^{\bar{q}} L^{\bar{r}}} \lesssim \|f(u)\|_{L^{\gamma'} B^0_{\bar{r},\alpha}} \lesssim \|u\|_{L^q B^s_{r,\alpha}}^p.$

Proof. We have $\|f(u)\|_{\ell^\alpha L^{\bar{q}} L^{\bar{r}}} \lesssim \|f(u)\|_{L^{\gamma'} B^0_{\bar{r},\alpha}}$ by $\bar{q} = \gamma' \le \alpha$. Let $\varepsilon > 0$ be a sufficiently small number. By the Sobolev embedding $B^\varepsilon_{m(\bar{r},-\varepsilon),\alpha} \hookrightarrow B^0_{\bar{r},\alpha}$, Lemma 2.1 with the equation $1/m(\bar{r},-\varepsilon) = (p-1)/m(r,s) + 1/m(r,s-\varepsilon)$, and the embedding $B^s_{r,\alpha} \hookrightarrow L^{m(r,s)} \cap B^\varepsilon_{m(r,s-\varepsilon),\alpha}$ by $\alpha \le m(r,s)$, we have

$$\|f(u)\|_{B^0_{\bar{r},\alpha}} \lesssim \|f(u)\|_{B^\varepsilon_{m(\bar{r},-\varepsilon),\alpha}} \lesssim \|u\|_{L^{m(r,s)}}^{p-1} \|u\|_{B^\varepsilon_{m(r,s-\varepsilon),\alpha}} \lesssim \|u\|_{B^s_{r,\alpha}}^p.$$

Since $\bar{q} = \gamma' = q/p$, we obtain the required inequality. $\square$

Claim 4.3. $\quad \|f\|_{B^\theta_{\gamma',\alpha} L^{\rho'}} \lesssim \|u\|_{L^q B^s_{r,\alpha}}^{p-1} \|u\|_{B^\theta_{q,\alpha} L^r}.$

Proof. We use the equivalent norm (see [14] and [17, (2.3)])

$$\|f(u)\|_{B^\theta_{\gamma',\alpha} L^{\rho'}} = \|f(u)\|_{L^{\gamma'} L^{\rho'}} + \left\{ \int_0^\infty \left(\tau^{-\theta} \|f(u(\cdot)) - f(u(\cdot + \tau))\|_{L^{\gamma'} L^{\rho'}} \right)^\alpha \frac{d\tau}{\tau} \right\}^{1/\alpha}.$$

The first term in the right hand side is bounded by $\|u\|_{L^q B^s_{r,\alpha}}^{p-1} \|u\|_{L^q L^r}$ by (4.3). The second term is bounded by $\|u\|_{L^q B^s_{r,\alpha}}^{p-1} \|u\|_{B^\theta_{q,\alpha} L^r}$ by the inequality

$$|f(u(\cdot)) - f(u(\cdot + \tau))| \lesssim (|u(\cdot)| + |u(\cdot + \tau)|)^{p-1} |u(\cdot) - u(\cdot + \tau)|. \quad \square$$

By the above claims, we have obtained

$$\|\Phi(u)\|_X \le C\|u_0\|_{H^s} + C\|u\|_X^p \le C\|u_0\|_{H^s} + CR^p,$$
$$d(\Phi(u),\Phi(v)) \le C\max_{w=u,v}\|w\|_X^{p-1}d(u,v) \le CR^{p-1}d(u,v)$$

for any $u, v \in X(R)$ for some constant $C > 0$. Taking R such that $CR^{p-1} \le 1/2$ and $R \ge 2C\|u_0\|_{H^s}$ for sufficiently small u_0, Φ becomes a contraction mapping on X_R.

The last part of the theorem, the continuous dependence of the solutions to the initial data, follows easily. Indeed, for any solutions u and $v \in X$ for initial data u_0 and $v_0 \in H^s(\mathbb{R}^n)$, respectively, we have

$$d(u,v) \lesssim \|u_0 - v_0\|_{L^2} + \|f(u) - f(v)\|_{L^{\gamma'}L^{\rho'}} \lesssim \|u_0 - v_0\|_{L^2} + \max_{w=u,v}\|w\|_X^{p-1}d(u,v)$$

by the similar argument for Claim 4.1. So that, the flow mapping $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R}^n)$ to X.

4.2 The case $2 < s < 3$

We put $\theta := s/2-1$, $\gamma := 2$, and $\alpha := 2$. Then we have $1/\rho = 1/2-1/n$, $2 < q < \infty$, $2 < r < 2n/(n-2)$, where $2 < q$ holds by $q = 2p$ and $p > 1$. Claim 4.1 holds by the similar argument. By $\|\Phi(u)\|_{\dot{H}^s} = \|\Delta\Phi(u)\|_{\dot{H}^{s-2}}$ and the equation $\Delta\Phi(u) = i\,(\partial_t\Phi(u) - f(u))$, we have

$$\|\Phi(u)\|_{L^\infty H^s} \lesssim \|\Phi(u)\|_{L^\infty L^2} + \|f(u)\|_{L^\infty \dot{H}^{2\theta}} + \|\partial_t\Phi(u)\|_{L^\infty \dot{H}^{2\theta}}. \tag{4.5}$$

Similarly, we also have

$$\|\Phi(u)\|_{L^q B^s_{r,\alpha}} \lesssim \|\Phi(u)\|_{L^q L^r} + \|f(u)\|_{L^q \dot{B}^{2\theta}_{r,\alpha}} + \|\partial_t\Phi(u)\|_{L^q \dot{B}^{2\theta}_{r,\alpha}}. \tag{4.6}$$

The second terms in the right hand side in (4.5) and (4.6) are estimated as follows.

Claim 4.4. $\|f(u)\|_{L^\infty \dot{H}^{2\theta}} \lesssim \|u\|^p_{L^\infty \dot{H}^s}$.

Proof. Since we have $1/2 = (p-1)/m(2,s) + 1/m(2,2)$, $2 < m(2,2) < \infty$, and $2\theta < p$, we apply Lemma 2.1 to $f(u)$ and we obtain

$$\|f(u)\|_{\dot{H}^{2\theta}} \lesssim \|u\|^{p-1}_{L^{m(2,s)}}\|u\|_{\dot{B}^{2\theta}_{m(2,2),2}} \lesssim \|u\|^p_{\dot{H}^s}, \tag{4.7}$$

where we have used the embedding $\dot{H}^s \hookrightarrow L^{m(2,s)} \cap \dot{B}^{2\theta}_{m(2,2),2}$ for the last inequality. We obtain the required result taking L^∞ norm in time. □

Claim 4.5. $\|f(u)\|_{L^q \dot{B}^{2\theta}_{r,\alpha}} \lesssim \|u\|^{p-1}_{L^\infty \dot{H}^s}\|u\|_{L^q \dot{B}^s_{r,\alpha}}$.

Proof. We have $1/r = (p-1)/m(2,s) + 1/m(r,2)$, $r < m(r,2) < \infty$, and $2\theta < p$, where $m(r,2) < \infty$ holds by $m(r,2) < m(r,s)$ and (4.1). By Lemma 2.1, we have

$$\|f(u)\|_{\dot{B}^{2\theta}_{r,\alpha}} \lesssim \|u\|^{p-1}_{L^{m(2,s)}} \|u\|_{\dot{B}^{2\theta}_{m(r,2),\alpha}} \lesssim \|u\|^{p-1}_{\dot{H}^s} \|u\|_{\dot{B}^s_{r,\alpha}},$$

where we have used the embeddings $\dot{H}^s \hookrightarrow L^{m(2,s)}$ and $\dot{B}^s_{r,\alpha} \hookrightarrow \dot{B}^{2\theta}_{m(r,2),\alpha}$ for the last inequality. The required result follows from the Hölder inequality in time. $\square$

Next, we estimate the third terms in (4.5) and (4.6). Since $\partial_t \Phi(u)$ satisfies

$$(\partial_t + i\Delta)\, \partial_t \Phi(u) = \partial_t f(u), \tag{4.8}$$

we have

$$\partial_t \Phi(u)(t) = U(t) u_1 + \int_0^t U(t-t')(\partial_t f(u))(t') dt',$$

where $u_1 := -i\Delta u_0 + f(u_0)$. We put $\bar{r}_1 := \rho'$ and $\bar{q}_2 := \gamma'$, and we define $\bar{q}_1$ and $\bar{r}_2$ by the equations

$$\frac{2}{\bar{q}_j} - \delta(\bar{r}_j) = 2(1-\theta) \quad \text{for} \ \ j = 1, 2. \tag{4.9}$$

Then we have $1/\bar{q}_1 = 1/\gamma' - \theta = (3-s)/2$, $2 \le \bar{q}_1 < \infty$, $1/\bar{r}_2 = 1/\rho' - (s-2)/n = 1/2 + (3-s)/n$, $1 < \bar{r}_2 \le 2$. Since $\rho < \infty$, $1 < \bar{q}_2 \le \alpha \le q$, we use Lemmas 1.1, 1.2 and Theorem 1.3 to have

$$\|\partial_t \Phi(u)\|_{L^\infty H^{2\theta}} \lesssim \|u_1\|_{H^{2\theta}} + \|\partial_t f(u)\|_{B^\theta_{\gamma',2} L^{\rho'}} + \|\partial_t f(u)\|_{\ell^2 L^{\bar{q}_1} L^{\bar{r}_1}}. \tag{4.10}$$

$$\|\partial_t \Phi(u)\|_{L^q B^{2\theta}_{r,\alpha}} \lesssim \|u_1\|_{H^{2\theta}} + \|\partial_t f(u)\|_{B^\theta_{\gamma',\alpha} L^{\rho'}} + \|\partial_t f(u)\|_{\ell^\alpha L^{\bar{q}_2} L^{\bar{r}_2}}. \tag{4.11}$$

By (4.7), we have

$$\|u_1\|_{H^{2\theta}} \lesssim \|u_0\|_{H^s} + \|f(u_0)\|_{H^{2\theta}} \lesssim \|u_0\|_{H^s} + \|u_0\|^p_{H^s}. \tag{4.12}$$

The second terms in the right hand sides in (4.10) and (4.11) are bounded by $\|f(u)\|_{B^{\theta+1}_{\gamma',2} L^{\rho'}}$ since $\alpha \ge 2$.

Claim 4.6. $\|f(u)\|_{B^{\theta+1}_{\gamma',2} L^{\rho'}} \lesssim \|u\|^{p-1}_{L^q L^{m(r,s)}} \|u\|_{B^{\theta+1}_{q,2} L^r} \lesssim \|u\|^{p-1}_{L^q B^s_{r,m(r,s)}} \|u\|_{B^{\theta+1}_{q,2} L^r}.$

Proof. We use the equivalent norm

$$\|f(u)\|_{B^{\theta+1}_{\gamma',2} L^{\rho'}} = \|f(u)\|_{L^{\gamma'} L^{\rho'}} + \left\{ \int_0^\infty \left(\tau^{-\theta-1} \|f(u(\cdot)) - 2f(u(\cdot+\tau)) + f(u(\cdot+2\tau))\|_{L^{\gamma'} L^{\rho'}} \right)^2 \frac{d\tau}{\tau} \right\}^{1/2} \tag{4.13}$$

(see [14, p22, Remark 1, p27, Theorem 3]). The first term in the right hand side is bounded by $\|u\|_{L^qL^{m(r,s)}}^{p-1}\|u\|_{L^qL^r}$ by the same argument in (4.3). To estimate the second term, we put $v(\cdot) := u(\cdot+\tau)$ and $w(\cdot) := u(\cdot+2\tau)$. We use the inequality

$$|f(u)-2f(v)+f(w)| \lesssim (|u|+|v|+|w|)^{p-1}|u-2v+w| + \begin{cases} (|u-v|+|v-w|)^p & \text{if } \ p<2, \\ (|u|+|v|+|w|)^{p-2}(|u-v|+|v-w|)^2 & \text{if } \ p\geq 2 \end{cases}$$

to have

$$\|f(u)-2f(v)+f(w)\|_{L^{\gamma'}L^{\rho'}} \lesssim \|u\|_{L^qL^{m(r,s)}}^{p-1}\|u-2v+w\|_{L^qL^r} + \begin{cases} \|u-v\|_{L^qL^{p\rho'}}^p & \text{if } \ p<2, \\ \|u\|_{L^qL^{m(r,s)}}^{p-2}\|u-v\|_{L^qL^{r_*}}^2 & \text{if } \ p\geq 2, \end{cases}$$

where $2/r^* := 1/m(r,s)+1/r$. So that, the second term in the right hand side in (4.13) is bounded by

$$\|u\|_{L^qL^{m(r,s)}}^{p-1}\|u\|_{B^{\theta+1}_{q,2}L^r} + \begin{cases} \|u\|_{B^{(\theta+1)/p}_{q,2p}L^{p\rho'}}^p & \text{if } \ p<2, \\ \|u\|_{L^qL^{m(r,s)}}^{p-2}\|u\|_{B^{(\theta+1)/2}_{q,4}L^{r_*}}^2 & \text{if } \ p\geq 2. \end{cases}$$

By Lemma 2.2, we have

$$\begin{aligned} \|u\|_{B^{(\theta+1)/p}_{q,2p}L^{p\rho'}} &\lesssim \|u\|_{L^qL^{m(r,s)}}^{1-1/p}\|u\|_{B^{\theta+1}_{q,2}L^r}^{1/p}, \\ \|u\|_{B^{(\theta+1)/2}_{q,4}L^{r_*}} &\lesssim \|u\|_{L^qL^{m(r,s)}}^{1/2}\|u\|_{B^{\theta+1}_{q,2}L^r}^{1/2}, \end{aligned}$$

where we have used $L^q \hookrightarrow B^0_{q,\infty}$. Therefore, by $B^s_{r,m(r,s)} \hookrightarrow L^{m(r,s)}$ in Lemma 2.4, we obtain the required result. □

Claim 4.7. $\|\partial_t f(u)\|_{\ell^2L^{\bar{q}_1}L^{\bar{r}_1}} \lesssim \|f(u)\|_{B^{\theta+1}_{\gamma',2}L^{\rho'}}$.

Proof. By the Sobolev embeddings $B^\theta_{\gamma',2} \hookrightarrow L^{\bar{q}_1}$ and $L^{\rho'} \hookrightarrow B^0_{\rho',2}$, we have

$$\|\partial_t f(u)\|_{\ell^2L^{\bar{q}_1}L^{\bar{r}_1}} \lesssim \|\partial_t f(u)\|_{\ell^2B^\theta_{\gamma',2}L^{\rho'}} \lesssim \|\partial_t f(u)\|_{B^\theta_{\gamma',2}B^0_{\rho',2}} \lesssim \|f(u)\|_{B^{\theta+1}_{\gamma',2}L^{\rho'}} \tag{4.14}$$

as required. □

Claim 4.8. *The following estimate holds;*

$$\begin{aligned} \|\partial_t f(u)\|_{\ell^\alpha L^{\bar{q}_2}L^{\bar{r}_2}} &\lesssim \|u\|_{L^qL^{m(r,s)}}^{p-1}\|\partial_t u\|_{L^qB^{2\theta}_{r,m(r,2\theta)}} \\ &\lesssim \|u\|_{L^qB^s_{r,m(r,s)}}^{p-1}\|\partial_t u\|_{L^qB^{2\theta}_{r,m(r,2\theta)}}. \end{aligned}$$

Proof. Since $1 < \bar{r}_2 \le 2$ and $\alpha \ge 2$, we have $L^{\bar{r}_2} \hookrightarrow B^0_{\bar{r}_2,\alpha}$ (see [2, Theorem 6.4.4]). By $\bar{q}_2 = \gamma' \le \alpha$, $1/\bar{r}_2 = (p-1)/m(r,s) + 1/m(r,2\theta)$, we have

$$\begin{aligned}\|\partial_t f(u)\|_{\ell^\alpha L^{\bar{q}_2} L^{\bar{r}_2}} &\lesssim \|\partial_t f(u)\|_{L^{\bar{q}_2} B^0_{\bar{r}_2,\alpha}} \lesssim \|\partial_t f(u)\|_{L^{\bar{q}_2} L^{\bar{r}_2}} \\ &\lesssim \|u\|^{p-1}_{L^q L^{m(r,s)}} \|\partial_t u\|_{L^q L^{m(r,2\theta)}} \lesssim \|u\|^{p-1}_{L^q L^{m(r,s)}} \|\partial_t u\|_{L^q B^{2\theta}_{r,m(r,2\theta)}},\end{aligned}$$

where we note $2 \le r < m(r,2\theta) < m(r,s) < \infty$. By the embedding $B^s_{r,m(r,s)} \hookrightarrow L^{m(r,s)}$ in Lemma 2.4, we obtain the required result. □

By the above claims, (4.12), and $\alpha \le m(r,2\theta) < m(r,s)$, we obtain

$$\begin{aligned}\|\Phi(u)\|_X &\lesssim \|u_0\|_{H^s} + \|u_0\|^p_{H^s} + \|u\|^p_X. \\ d(\Phi(u),\Phi(v)) &\lesssim \max_{w=u,v} \|u\|^{p-1}_X d(u,v)\end{aligned}$$

for any u and $v \in X$. So that, Φ is a contraction mapping on $X(R)$ for some $R > 0$ if $\|u_0\|_{H^s}$ is sufficiently small.

4.3 The case $3 \le s < 4$

We put $\theta := s/2 - 1$. We note $\theta \ge 1/2$ by $s \ge 3$. Let γ be any number which satisfies

$$\theta < \frac{1}{\gamma'} < \min\left\{1,\ \frac{p}{2}\right\}, \qquad \text{and} \qquad \frac{(n-2s+4)p}{2(n+2)} \le \frac{1}{\gamma'} \le \frac{p(p-1)}{2}, \tag{4.15}$$

where the last condition $1/\gamma' \le p(p-1)/2$ is required only for the case $p < 2$. We note that there exists γ which satisfies (4.15). Indeed, $\theta < \min\{1, p/2\}$ holds by $s < 4$ and $s - 2 < s/2 < p$. Let us consider the case $p \ge 2$. Since $1 \le p/2$, it suffices to check $(n-2s+4)p/2(n+2) < 1$, which is equivalent to

$$3 \le s < \min\left\{4,\ 1 + \frac{\sqrt{(n-6)(n+2)}}{2}\right\} = \begin{cases} 1+\sqrt{5} = 3.23\cdots & \text{if } n = 8, \\ 1+\frac{\sqrt{33}}{2} = 3.87\cdots & \text{if } n = 9, \\ 4 & \text{if } n \ge 10, \end{cases} \tag{4.16}$$

and the condition (4.16) is satisfied since $s_2 < 1 + \sqrt{(n-6)(n+2)}/2$. Here, s_2 is defined by (1.12). Therefore, if $p \ge 2$, then there exists γ which satisfies (4.15). Let us next consider the case $p < 2$, so that $p(p-1)/2 < p/2 < 1$. In this case, we should check $(n-2s+4)p/2(n+2) \le p(p-1)/2$ and $\theta < p(p-1)/2$. By Remark 1.7, it suffices to consider the case $11 \le n \le 13$. The inequality $(n-2s+4)p/2(n+2) \le p(p-1)/2$ holds if $3 \le s < 4$ for $n = 11$, $7 - \sqrt{15}\,(= 3.12\cdots) \le s < 4$ for $n = 12$, and $7/2 \le s < 4$ for $n = 13$. The inequality $\theta < p(p-1)/2$ holds if $3 \le s < 4$ for $n = 11$, and

$3 \le s < 5-\sqrt{3}\,(=3.26\cdots)$ for $n=12$; this inequality fails if $3 \le s < 4$ for $n=13$. So that, there exists the above γ under the condition (1.11).

We put $\alpha := q = \bar{q}_2$. We have $2 < \rho < 2n/(n-2)$ by $1/2 < 1/\gamma' < 1$. We also have $2 < q < \infty$ by $0 < 1/\gamma' < p/2$. So that, $2 < r < \infty$. We put $\bar{r}_1 := \rho'$, and we define $\bar{q}_1$ and $\bar{r}_2$ by (4.9). We have $2 \le \bar{q}_1 < \infty$ since $\bar{q}_1$ is written as $1/\bar{q}_1 = 1/\gamma' - \theta$ and $1/\gamma' < 1$, $1/2 \le \theta < 1/\gamma'$. We also have $1 < \bar{r}_2 < \infty$ by $\bar{q}_2 = q$ and the equation (4.9). Claim 4.1, and Claims 4.4, 4.5, 4.6, 4.7 hold in this setting by the analogous arguments. In the case $2 < s < 3$, we have put $1/\bar{r}_2 = 1/2 + (3-s)/n$ by (4.9), and the property $1 < \bar{r}_2 \le 2$ has been used in the proof of Claim 4.8. Since the definition of $\bar{r}_2$ in this subsection does not always satisfy this condition, we need to modify Claim 4.8 as follows.

Claim 4.9. *The following estimate holds;*

$$\|\partial_t f(u)\|_{\ell^\alpha L^{\bar{q}_2} L^{\bar{r}_2}} \lesssim \|u\|^{p-1}_{L^\infty H^s} \|\partial_t u\|_{L^q B^{2\theta}_{r,\alpha}}.$$

Proof. By $\alpha \ge \bar{q}_2$ and the Sobolev embedding $B^{\varepsilon}_{m(\bar{r}_2,-\varepsilon),\alpha} \hookrightarrow B^0_{\bar{r}_2,\alpha}$, we have

$$\|\partial_t f(u)\|_{\ell^\alpha L^{\bar{q}_2} L^{\bar{r}_2}} \lesssim \|\partial_t f(u)\|_{L^{\bar{q}_2} B^{\varepsilon}_{m(\bar{r}_2,-\varepsilon),\alpha}}.$$

By $r < m(r,2\theta) < m(r,s) < \infty$, the equation $1/m(\bar{r}_2,-\varepsilon) = (p-1)/m(2,s) + 1/m(r,2\theta-\varepsilon)$ and the embedding $B^{2\theta}_{r,\infty} \hookrightarrow B^{2\theta-\varepsilon}_{r,m(r,2\theta-\varepsilon)}$, we have

$$\|\partial_t f(u)\|_{L^{m(\bar{r}_2,-\varepsilon)}} \lesssim \|u\|^{p-1}_{H^s} \|\partial_t u\|_{B^{2\theta}_{r,\infty}}. \tag{4.17}$$

We put $1/m^* := (2+\varepsilon)/n$. By the equation $1/m(\bar{r}_2,-\varepsilon) = (p-1)/m(2,s) + 1/m(r,2\theta-\varepsilon)$, we have

$$\begin{aligned}
\|\partial_t f(u)\|_{\dot{B}^{\varepsilon}_{m(\bar{r}_2,-\varepsilon),\alpha}} &\lesssim \|u\|^{p-1}_{L^{m(2,s)}} \|\partial_t u\|_{\dot{B}^{\varepsilon}_{m(r,2\theta-\varepsilon),\alpha}} + \|f'(u)\|_{\dot{B}^{\varepsilon}_{m^*,\alpha}} \|\partial_t u\|_{L^{m(r,2\theta)}}.\\
&\lesssim \left(\|u\|^{p-1}_{\dot{H}^s} + \|f'(u)\|_{\dot{B}^{\varepsilon}_{m^*,\alpha}}\right) \|\partial_t u\|_{\dot{B}^{2\theta}_{r,\alpha}},
\end{aligned}$$

where, for the last inequality, we have used the embeddings $\dot{H}^s \hookrightarrow L^{m(2,s)}$, $\dot{B}^{2\theta}_{r,\alpha} \hookrightarrow \dot{B}^{\varepsilon}_{m(r,2\theta-\varepsilon),\alpha}$, $\dot{B}^{2\theta}_{r,\alpha} \hookrightarrow \dot{B}^{2\theta}_{r,m(r,2\theta)} \hookrightarrow L^{m(r,2\theta)}$. Here, we note that $\alpha \le m(r,2\theta)$ follows from $(n-2s+4)p/2(n+2) \le 1/\gamma'$ for $\dot{B}^{2\theta}_{r,\alpha} \hookrightarrow \dot{B}^{2\theta}_{r,m(r,2\theta)}$. By the equivalent norm, the assumption (1.10), and the equation $1/m^* = (p-1)/m(2,s) + \varepsilon/n$, we have

$$\begin{aligned}
\|f'(u)\|_{\dot{B}^{\varepsilon}_{m^*,\alpha}} &\lesssim \left\{\int_0^\infty \left(\tau^{-\varepsilon} \sup_{|y|<\tau} \|f'(u(x+y)) - f'(u(x))\|_{L^{m^*}}\right)^\alpha \frac{d\tau}{\tau}\right\}^{1/\alpha}\\
&\lesssim \begin{cases} \|u\|^{p-2}_{L^{m(2,s)}} \|u\|_{\dot{B}^{\varepsilon}_{m(2,s-\varepsilon),\alpha}} & \text{if } p \ge 2,\\ \|u\|^{p-1}_{\dot{B}^{\varepsilon/(p-1)}_{m^*(p-1),\alpha(p-1)}} & \text{if } p < 2, \end{cases}\\
&\lesssim \|u\|^{p-1}_{\dot{H}^s},
\end{aligned}$$

where we have used the embeddings $\dot{H}^s \hookrightarrow L^{m(2,s)} \cap \dot{B}^{\varepsilon}_{m(2,s-\varepsilon),\alpha} \cap \dot{B}^{\varepsilon/(p-1)}_{m^*(p-1),\alpha(p-1)}$. Here, we note that, when $p < 2$, the inequalities $m^*(p-1) \geq 1$ and $\alpha(p-1) \geq 2$ follow from the equation $1/m^*(p-1) = 1/2 - s/n + \varepsilon/n(p-1)$ and the condition $1/\gamma' \leq p(p-1)/2$, respectively. So that, we obtain

$$\|\partial_t f(u)\|_{\dot{B}^{\varepsilon}_{m(\bar{r}_2,-\varepsilon),\alpha}} \lesssim \|u\|^{p-1}_{\dot{H}^s} \|\partial_t u\|_{\dot{B}^{2\theta}_{r,\alpha}}. \tag{4.18}$$

By (4.17), (4.18), and $\bar{q}_2 = q$, we have $\|\partial_t f(u)\|_{L^{\bar{q}_2} B^{\varepsilon}_{m(\bar{r}_2,-\varepsilon),\alpha}} \lesssim \|u\|^{p-1}_{L^\infty \dot{H}^s} \|\partial_t u\|_{L^q B^{2\theta}_{r,\alpha}}$. Therefore, we have obtained the required estimate. □

By Claim 4.9 instead of Claim 4.8, we are able to show that Φ is a contraction mapping analogously to the case $2 < s < 3$. We note that we have to put $\alpha = q = \bar{q}_2$ since we need $\alpha \leq q$ for (4.11) by Theorem 1.3, and $\alpha \geq \bar{q}_2 = q$ to prove Claim 4.9.

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