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Remarks on Potential Theoretic Kernels

Takasi KAYANO

Faculty of Literature and Science, Shimane University
Matsue, Japan.

and

Shirô OGAWA

Department of Engineering, Kobe University
Kobe, Japan
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Introduction.

In the preceding paper [4], we treated the characterization of the family of potential theoretic measures $G^+(\phi)$. Generally considering the characterization of the family $G^+(\phi)$, we defined a concept "T-kernel" and proved that Newtonian kernel Φ_N and the kernel Φ_W associated with the heat equation are T-kernels. The present paper deals with two remarks on potential theoretic kernels which have continuous potentials; the first is concerned with T-kernel and the second is done with the domination principles for Φ_W-potential. In the first section we shall remark that α-kernel and Green kernel are also T-kernels. In the second section we shall introduce a definition of C-domination principle and we shall prove that the kernel Φ_W does not satisfy the ordinary domination principle but it satisfies the C-domination principle.

1. Preliminary.

Let Ω be a locally compact Hausdorff space and $\phi(x, y)$ a measurable function in $\Omega\times\Omega$. A kernel $\check{\phi}(x, y)$ defined by $\check{\phi}(x, y)=\phi(y, x)$ is called the adjoint kernel to $\phi(x, y)$. We denote $\phi^+(x,y)=sup(\phi(x, y), 0)$ and $\phi^-(x, y)=-inf(\phi(x, y), 0)$. Then $\phi(x, y)$ is equal to $\phi^+(x,y)-\phi^-(x,y)$. The ϕ-potential of a positive Radon measure μ in Ω is defined by $\phi\mu(x)=\int^* \phi(x, y)\, d\mu(y)$, provided that $\phi^+\mu(x)$ and $\phi^-\mu(x)$ are not infinity at the same time. A kernel

$\phi(x,y)$ is called S-kernel if there exists at least such a positive measure λ that the support S_λ is compact and the potentials $\phi^+\lambda(x)$ and $\phi^-\lambda(x)$ are continuous in Ω. In case that $\phi(x,y)$ is S-kernel, we define the following classes of measures,

$$F^+(\phi)=\{\lambda : \lambda\geqq 0,\ S_\lambda \textit{ compact, } \phi^+\lambda \textit{ and } \phi^-\lambda \textit{ continuous in } \Omega\},$$

$$G^+(\phi)=\{\mu : \mu\geqq 0,\ \int^* \check{\phi}^+\mu d\lambda \textit{ and} \int^* \check{\phi}^-\mu d\lambda<+\infty \textit{ for any } \lambda\in F^+(\phi)\}.$$

A kernel $\phi(x,y)$ is called T-kernel if $\phi(x,y)$ is a non-negative S-kernel and for any compact set K there exist such a point x_K in Ω, a relatively compact open set U_K containing K, and a positive constant M_K depending on x_K and U_K that $\check{\phi}(x,y)\leqq M_K\check{\phi}(x_K,y)$ for any x of K and any y of $\Omega\backslash U_K$, where $\Omega\backslash U_K$ denotes the complementary set of U_K. For a T-kernel ϕ and a compact set K, we shall denote by E_K the set of all points x_K with the above properties. And in [4], we obtained the following result.

Theorem 1. *Suppose that $\phi(x,y)$ is a T-kernel in Ω.*

If a non-negative measure μ is such a measure that, for any compact set K is Ω, there exists a point x_K in E_K that $\check{\phi}\mu(x_K)<+\infty$, then μ is an element of $G_+(\phi)$. If for any compact set K, E_k contains some open set, and there exists a positive measure λ of $F^+(\phi)$, of which the support S_λ is contained by E_k, then the converse holds.

2. α-kernel and Green kernel are T-kernels

α-kernel $\Phi^\alpha(x,y)$ in R^n is defined by $\Phi^\alpha(x,y)=\dfrac{1}{|x-y|^{n-\alpha}}$ $(0<\alpha<n)$.

In [4], applying the axiomatic theory of harmonic function, we proved that Φ_N and Φ_W are T-kernels, but for Φ^α we can not apply axiomatic method. Therefore we must consider directly behaviour of the kernel Φ^α in the neighborhood of the Alexandroff point ω of the space R^n. Now, let $\tilde{\Omega}$ be the compactification of Ω, adding the Alexandroff point ω of Ω.

Lemma. *Let $\phi(x,y)$ be a positive S-kernel in Ω. If for any compact set K, there exists such a point x_K in Ω that $\limsup_{y\to\omega}\check{\phi}(x,y)/\check{\phi}(x_K,y)$ is uniformly bounded with respect to all points x of K, then the kernel $\phi(x,y)$ is a T-kernel.*

Proof. According to the assumption, we have the finite supremum $M=\sup_{x\in K}\limsup_{y\to\omega}\check{\phi}(x,y)/\check{\phi}(x_K,y)$, and M is a finite positive constant. Therefore, for any $\varepsilon > 0$ there exists such a neighborhood V_ω of the Alexandroff point ω, that the inequality $\check{\phi}(x, y)/\check{\phi}(x_K, y) \leqq M+\varepsilon$ holds for any point x of the compact set K and for any point y of the neighborhood V_ω. Since the complementary set of V_ω is compact, there exists a relatively open set U_K which contains the compact set $(\Omega \backslash V_\omega) \cup K$ and of which the closure is contained by Ω. Then the complementary set $\Omega \backslash U_K$ is contained by the neighborhood V_ω. If we substitute M_K for $M+\varepsilon$, we have that $\check{\phi}(x, y)/\check{\phi}(x_K, y) \leqq M_K$ for any point y of $\Omega \backslash U_K$ and for any point x of K.

Theorem 2. *α-kernel Φ^α $(0 < \alpha < n)$ satisfies the conditions of the above lemma. Therefore α-kernel $\Phi^\alpha (0 < \alpha < n)$ is a T-kernel and E_K is identified with R^n for any compact set K contained in R^n.*

Proof. It is clear that α-kernel Φ^α $(0 < \alpha < n)$ is a positive symmetric S-kernel. The function $\varphi^\alpha(r) = 1/r^{n-\alpha}$ $(r > 0, 0 < \alpha < n)$ is monotonously decreasing with respect to r. For any point x_K of R^n, we set

$$R_k = \sup_{x \in K} |x - x_K| \text{ for given compact set } K, \text{ and}$$

$$B_{r, x_K} = \{x : |x - x_K| < r \quad \textit{for } r > R_K\}.$$

We have the following inequalities

$$0 < |x_K - y| - |x - x_K| \leqq |x - y|.$$

for any point x of K and for any point y of $R^n \backslash B_{r, x_K}$.

According to the monotonous decrease of the function φ^α, we have the following inequalities

$$\varphi^\alpha(|x-y|) \leqq \varphi^\alpha(|x_K - y| - |x - x_K|)$$

$$\text{and } \frac{\Phi^\alpha(x, y)}{\Phi^\alpha(x_K, y)} = \frac{\varphi^\alpha(|x-y|)}{\varphi^\alpha(|x_K-y|)} \leqq \frac{\varphi^\alpha(|x_K-y| - |x-x_K|)}{\varphi^\alpha(|x_K-y|)}$$

$$= \left(\frac{|x_K-y| - |x-x_K|}{|x_K-y|}\right)^{\alpha-n} = \left(1 - \left|\frac{x-x_K}{y-x_K}\right|\right)^{\alpha-n}$$

for any point x of K and for any point y of $R^n \backslash B_{r, x_K}$.

From the symmetricity of the α-kernel Φ^α, the inequality

$$\limsup_{y\to\omega} \frac{\check{\Phi}^{\alpha}(x, y)}{\check{\Phi}^{\alpha}(x_K, y)} \leqq 1$$ holds for any point x of K.

Consequently, by the lemma the kernel Φ^{α} is a T-kernel. Since we can take an arbitrary point of R^n as x_K, then E_K is identified with R^n for any compact set K.

In succession, we shall prove that Green kernel is a T-kernel.
Let Ω be a harmonic space satisfying BRELOT-BAUER 's axiom, and let function 1 be harmonic in Ω. Now we define Green kernel in domain $D \subset \Omega$ by the function $G(x, y)$ with the following properties ;

(1) $G(x, y)$ is positive in $D \times D$,

(2) $G(x, y)$ is continuous in $D \times D$ for $x \neq y$,

(3) $\lim_{x\to\omega\ x\in D} G(x, y) = 0$ for any y of D, where ω is Alexandroff point of D,

(4) $G(x, y)$ is superharmonic in D with respect to x, and $G(x, y)$ is harmonic in any subdomain V of D with respect to x, when V does not contain y.

Theorem 3. *The Green kernel $G(x, y)$ is a T-kernel and E_K is identified with the domain D for any compact subset K of D.*

Proof. For any compact subset K of D and for any point x_K of D, there exists such a relatively compact open set U_K, which contains the compact set K and the point x_K, and of which the closure is contained in D.

We set $\alpha = \sup_{x\in K,\ y\in\partial U_K} \check{G}(x, y)$, and $\beta = \inf_{y\in\partial U_K} \check{G}(x_K, y)$. Since ∂U_K and K are compact, $\check{G}(x, y)$ and $\check{G}(x_K, y)$ are positive by property (1) and $\check{G}(x, y)$ is continuous for $x \neq y$ in D by property (2), then both values α and β are finite and positive. Therefore $\frac{\alpha}{\beta} \check{G}(x_K, y) - \check{G}(x, y) \geqq 0$ is valid for any point x of K and for any point y of the boundary ∂K.
By the property (3), the equality

$$\lim_{y\to\omega} \left(\frac{\alpha}{\beta} \check{G}(x_K, y) - \check{G}(x, y)\right) = \lim_{y\to\omega} \frac{\alpha}{\beta} G(y, x_K) - \lim_{y\to\omega} G(y, x) = 0$$

holds for any point x of K.

It is well known that if the function 1 is harmonic, we have the following minimum principle.

Minimum Principle. *If u is a superharmonic function in the domain D, $D \subset \bar{D} \subset \Omega$, and satisfies $\liminf_{x \in D\, x \to y} u(x) \geqq 0$ for any y of ∂D, then u is non-negative in D.*

By the property (4) $\check{G}(x, y)$ and $\check{G}(x_K, y)$ are harmonic with respect to y in the domain $D \backslash U_K$, because $D \backslash U_K$ does not contain the points x and x_K. Then $\frac{\alpha}{\beta}\check{G}(x_K, y) - \check{G}(x, y)$ is superharmonic with respect to y in $D \backslash U_K$. Therefore by the minimum principle we obtain the following inequality $\frac{\alpha}{\beta}\check{G}(x_K, y) - \check{G}(x, y) \geqq 0$ for any point x of K and for any point y of $D \backslash U_K$. Setting $M_K = \frac{\alpha}{\beta}$, we have the following inequality
$\check{G}(x, y) \leqq M_K \check{G}(x_K, y)$ for any point x of K and for any point y of $D \backslash U_K$. Since x_K is an arbitrary point of D, then E_K is identified with D for any compact set K.

3. Domination Principles. In this section we use the following definition concerning with domination principles

Definition. *We say that S-kernel ϕ satisfies the domination principle* (resp. C-*domination principle*), *if for any positive measure λ* (resp. *of* $F^+(\phi)$) *with compact support S_λ and for any positive measure μ, the inequality $\phi\lambda(x) \leqq \phi\mu(x)$ in the whole space follows from the same inequality $\phi\lambda(x) \leqq \phi\mu(x)$ on the support S_λ.*

It is well known that α-kernel Φ^α satisfies the domination principle, but we have the following theorem concerning with the kernel Φ_W.

Theorem 4. *The kernel Φ_W does not satisfy the domination principle, but it satisfies the* C-*domination principle.*

Proof. We use a compact subset $K = \{x = (x_1, \ldots, x_n) : a_i \leqq x_i \leqq b_i\ (i = 1, 2, \ldots, n-1),\ x_n = c$ for constants a_i, b_i, $(a_i < b_i)$ and $c\}$ and a domain $\mathrm{D} = \{x = (x, \ldots, x_n) : x_n < c\}$. Let λ' be a positive measure placed on K and μ a positive measure with compact support in D. Then $\Phi_W\mu(x_0)$ and $\Phi_W\lambda'(x_0)$ are finite for an arbitrary point x_0 of $R^n \backslash \bar{D}$. The potential $\Phi_W\lambda'(x)$ vanishes on K and $\Phi_W\mu(x)$ is positive on K. Consequently, we have the inequality $\Phi_W\lambda'(x) \leqq \Phi_W\mu(x)$ on S_λ. Both values of $\Phi_W\mu(x_0)$ and $\Phi_W\lambda'(x_0)$ are positive and finite. Then we can take such a positive number M that $M\ \Phi_W\lambda'(x_0) > \Phi_W\mu(x_0)$. If we set $\lambda = M\lambda'$, we have the following inequalities,

$$\Phi_W\lambda(x) \leqq \Phi_W\mu(x) \quad \text{on } S_\lambda \text{ and } \Phi_W\lambda(x_0) > \Phi_W\mu(x_0).$$

This shows that Φ_W does not satisfy the domination principle.

Now, we show that Φ_W satisfies the C-domination principle. Let λ be a measure of $F^+(\Phi_W)$, μ a positive measure and suppose that we have the inequality $\Phi_W\lambda(x) \leqq \Phi_W\mu(x)$ on S_λ. In order to show that Φ_W satisfies the C-domination principle, it is sufficient to prove that the inequality $\Phi_W\lambda(x) \leqq \Phi_W\mu(x)$ holds in $R^n \backslash S_\lambda$. The function $\Phi_W\mu(x) - \Phi_W\lambda(x)$ is lower semi-continuous and we have the following inequalities

$$\liminf_{y \to x} \{\Phi_W\mu(y) - \Phi_W\lambda(y)\} \geqq \Phi_W\mu(x) - \Phi_W\lambda(x) \geqq 0,$$

where x is a boundary point of S_λ and y is a point of $R^n \backslash S_\lambda$. On the other hand, by the property of the kernel Φ_W, $\Phi_W\lambda(x)$ is an element of the class C_0, where C_0 denotes the set of all continuous functions tending to zero at the Alexandroff point ω. Since the function $\Phi_W\mu(x) - \Phi_W\lambda(x)$ is a superharmonic function in $R^n \backslash S_\lambda$, by the minimum principle, we have immediately the desired inequalty.

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