Mem. Fac. Lit. & Sci, Shimane Univ., Nat. Sci. 4, pp. 10–26, March 15, 1971

On a Representation of The Group GL (k; R) and Its Spherical Functions

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(Received October 12, 1970)

In 1961, A. T. James [1] introduced the zonal polynomial of real positive definite matrix and he described some properties and a method of calculation of it. Lately, he [2] also showed that it is an eigenfunction of the Laplace-Beltrami operator. The zonal polynomial plays significant roles in distribution problems of eigenvalues related to the normal multivariate distribution [3]. In the present paper, we will describe a representation of the group $GL(k\,;R)$ and its spherical functions guided by N. J. Vilenkin [4]. Our assumptions 1 and 2 in the following may be satisfied with zonal polynomials. We also give a definition of (zonal) spherical function of the group $GL(k\,;R)$ guided by K. Maurin [5] and we show that our zonal spherical function is in agreement with the latter definition.

§ 1. Definition of spherical functions and some properties related to the representation $A_{(2f)}(g)$ of the group $GL(k\,;R)$

Let $\{\{g\}^{(2)}\}^{(f)}$, $g \in GL(k\,;R)$ be a representation of the group $GL(k\,;R)$ on the space V_f of homogeneous polynomials of degree f of the matrix $\mathfrak{S}_k^+ \ni S$, where $\mathfrak{S}_k^+$ is the space of real positive definite matrices of order k. Owing to Thrall [6]-Hua [7] 's result, the representation space space V_f is completely decomposable into irreducible invariant subspaces $V_{(f)}$ on which representation $A_{(2f_1,\dots,2f_k)}(g)$, $g \in GL(k\,;R)$, acts,

$$V_f = \bigoplus_{(f)} V_{(f)},$$

where $(f) = (f_1, \dots, f_k)$'s are partitions of f into k parts such that $f_1 \geqslant f_2 \geqslant \dots \geqslant f_k \geqslant 0$ and $f_1 + f_2 + \dots + f_k = f$.

On the irreducible invariant space $V_{(f)}$, we have the representation

$$A_{(2f)}(g)X(S) = A_{(2f_1,\dots,2f_k)}(g)X(S) = X(g^{-1}Sg^{-1'}), \tag{1}$$

where $g \in GL(k; R)$, g^{-1} is the inverse of g and $g^{-1'}$ is the transposed of the inverse of g and $X(S)$ is a vector of $V_{(f)}$.

The dimension of the space $V_{(f)}$ is

$N(2f) = N(2f_1, \dots, 2f_k) = D(2f_1+k-1, 2f_2+k-2, \dots, 2f_k)/D(k-1, \dots, 0)$, where $D(., \dots, .)$ is the difference product.

The infinitesimal operator of the representation $A_{(2f)}(g)$, $g \in GL(k; R)$, is given by

$$A_{ii}^{(f)} = -2 \sum_{\alpha=1}^{k} s_{i\alpha} \frac{\partial}{\partial s_{i\alpha}} \qquad (1 \leqslant i \leqslant k),$$

$$A_{ij}^{(f)} = -2 \sum_{\alpha=1}^{k} s_{j\alpha} \frac{\partial}{\partial s_{i\alpha}} \qquad (1 \leqslant i, j \leqslant k \cdot i \neq j).$$

Let $X_{(f)}^{(1)}(S), \dots, X_{(f)}^{(N)}(S)$ be a basis of the space $V_{(f)}$ and $X(S)$ be a vector of $V_{(f)}$; then we have the unique representation

$X(S) = \sum_{i=1}^{N} \alpha_i X_{(f)}^{(i)}(S)$, where $N = N(2f)$ and α_i $(1 \leqslant i \leqslant N)$ are complex numbers.

In particular,

$$A_{(2f)}(g) X_{(f)}^{(i)}(S) = \sum_{j=1}^{N} \chi_{ji}^{(f)}(g) X_{(f)}^{(j)}(S) \qquad (1 \leqslant i \leqslant N). \tag{2}$$

In the following we construct an scalar product for which the representation $A_{(2f)}(g)$, $g \in 0(k)$, is unitary. Let $X(S) = \sum_{i=1}^{N} \alpha_i X_{(f)}^{(i)}(S)$ and $Y(S) = \sum_{i=1}^{N} \beta_i X_{(f)}^{(i)}$ be vectors of $V_{(f)}$, and we define $(X(S), Y(S)) = \sum_{i=1}^{N} \alpha_i \bar{\beta}_i$, $\bar{\beta}_i$'s are complex conjugate of β_i 's, and

$$\langle X(S), Y(S) \rangle = \int_{0(k)} (A_{(2f)}(h)X(S), A_{(2f)}(h)Y(S))\, dV(h), \tag{3}$$

where $dV(\cdot)$ is the normalized Haar measure on the group $0(k)$. We easily find that the expression (3) defines a scalar product in the space $V_{(f)}$ and moreover it has a property

$$\langle A_{(2f)}(h)X(S), A_{(2f)}(h)Y(S)\rangle = \langle X(S), Y(S)\rangle, \qquad (4)$$
$$\text{for any } h \in 0(k).$$

Whence we have a proposition

Proposition 1

The representation $A_{(2f)}(h)$, $h \in 0(k)$ *is a unitary representation of the group* $0(k)$ *with respect to the scalar product* (3) *and it is a completely reducible representation.*

We will describe explicit evaluation of the scalar product (3). Since

$$A_{(2f)}(g)X(S) = \sum_{i=1}^{N} \alpha_i A_{(2f)}(g) X^{(i)}_{(f)}(S)$$
$$= \sum_{j=1}^{N} \Big(\sum_{i=1}^{N} \alpha_i \chi^{(f)}_{ji}(g) \Big) X^{(j)}_{(f)}(S)$$

we have an expression

$$\langle X(S), Y(S)\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} a^{(f)}_{ij}\, \alpha_i \bar{\beta}_j,$$

where

$$a^{(f)}_{ij} = \int_{0(k)} \sum_{l=1}^{N} \chi^{(f)}_{li}(h)\, \overline{\chi^{(f)}_{lj}(h)}\, dV(h) \qquad (1 \leqslant i, j \leqslant N).$$

The number $a^{(f)}_{ij}$'s are evaluated in the following way ;

$$a^{(f)}_{ij} = \int_{0(k)} \sum_{l=1}^{N} \chi^{(f)}_{li}(h)\overline{\chi^{(f)}_{lj}(h)}\, dV(h)$$
$$= \int_{S0(k)} \sum_{l=1}^{N} \chi^{(f)}_{li}(h)\, \overline{\chi^{(f)}_{lj}(h)}\, dV(h)$$
$$+ \int_{0\ (k)} \sum_{l=1}^{N} \chi^{(f)}_{li}(h)\, \overline{\chi^{(f)}_{lj}(h)}\, dV(h)$$

, where $SO(k)$ is the special orthogonal group and $O^-(k)$ is the improper part of the group $O(k)$.

In the integral over $O^-(k)$, let $h = J_k\tilde{h}$, where J_k is the special improper orthogonal involution such that

$$x_1' = x_1, x_2' = x_2, \ldots, x_k' = -x_k$$

and $\tilde{h} \in SO(k)$; since $dV(h)$ is the normalized Haar measure on a compact group $O(k)$, we have the following result,

let $\chi_{(f)}(h) = (\chi_{ji}^{(f)}(h)) = [\chi_1^{(f)}(h), \ldots, \chi_N^{(f)}(h)]$, $\chi_i^{(f)}(h)^*$ be the conjugate transposed of $\chi_i^{(f)}(h)$, and $\chi_{(f)}(h)^*$ be the adjoint matrix of the matrix $\chi_{(f)}(h)$;

$$a_{ij}^{(f)} = \frac{1}{\mathrm{Vol}(O(k))} \int_{SO(k)} \chi_j^{(f)}(h)^* \, [E_N + \chi^{(f)}(J_k)^* \, \chi^{(f)}(J_k)] \, \chi_i^{(f)}(h)\, dh, \qquad (5)$$

where Vol($O(k)$) is the total volume of the group $O(k)$, dh is the Haar measure on the group $O(k)$ and E_N is the unit matrix of order N.

To see effects of linear transformation of basis, let $Y_{(f)}^{(i)}(S)$'s $(1 \leqslant i \leqslant N)$ be another basis in the space $V_{(f)}$ and let

$$Y_{(f)}^{(i)}(S) = \sum_{j=1}^{N} \gamma_{ji}^{(f)} X_{(f)}^{(j)}(S) \qquad (1 \leqslant i \leqslant N) \qquad (6)$$

1) Let

$$A_{(2f)}(g) Y_{(f)}^{(i)}(S) = \sum_{j=1}^{N} \tilde{\chi}_{ji}^{(f)}(g) Y_{(f)}^{(j)}(S) \qquad (1 \leqslant i \leqslant N)$$

and define matrices $\Gamma^{(f)} = (\gamma_{ji}^{(f)})$ and $\tilde{\chi}^{(f)}(g) = (\tilde{\chi}_{ji}^{(f)}(g))$,

then we have the relation

$$\tilde{\chi}^{(f)}(g) = \overset{-1}{\Gamma}{}^{(f)} \, \chi^{(f)}(g) \Gamma^{(f)},$$

where $\overset{-1}{\Gamma}{}^{(f)}$ is the inverse matrix of $\Gamma^{(f)}$.

2) Let

$\langle Y^{(i)}_{(f)}(S),\ Y^{(j)}_{(f)}(S)\rangle = \tilde{a}^{(f)}_{ij}$ and $A^{(f)}$, $\tilde{A}^{(f)}$ be the matrices $(a^{(f)}_{ij})$, $(\tilde{a}^{(f)}_{ij})$ respectively, then we have the relation

$$\tilde{A}^{(f)} = \Gamma^{(f)'} A^{(f)} \bar{\Gamma}^{(f)},$$

where $\bar{\Gamma}^{(f)}$ is the conjugate matrix, $\Gamma^{(f)'}$ is the transposed matrix of $\Gamma^{(f)}$, respectively.

From these results, we have a proposition

Proposition 2

The scalar product (3) *is invariant under the linear transformation of basis.*

Now let $X^{(i)}_{(f)}(S)$'s $(1 \leqslant i \leqslant N)$ be an orthonormal basis of the space $V_{(f)}$ with respect to the scalar product (3) and be meeting the relation (2), then we have following results :

$$\langle A_{(2f)}(g)X^{(i)}_{(f)}(S),\ A_{(2f)}(g)X^{(j)}_{(f)}(S)\rangle$$
$$= \sum_{l=1}^{N} \chi^{(f)}_{li}(g)\bar{\chi}^{(f)}_{lj}(g) \text{ for any } g \in GL(k\,;R)$$

and since $A_{(2f)}(h)$, $h \in 0(k)$ is a unitary representation,

$$\sum_{l=1}^{N} \chi^{(f)}_{li}(h)\bar{\chi}^{(f)}_{lj}(h) = \delta_{ij} \qquad (1 \leqslant i, j \leqslant N) \quad \text{for any } h \in 0(k).$$

In this case, we have also results

$$\langle X(S),\ Y(S)\rangle = \sum_{i=1}^{N} \alpha_i \bar{\beta}_i,$$

$$\langle A_{(2f)}(g)X(S),\ A_{(2f)}(g)Y(S)\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \bar{\beta}_j \sum_{l=1}^{N} \chi^{(f)}_{li}(g)\,\bar{\chi}^{(f)}_{lj}(g)$$

for any $g \in GL(k\,;R)$.

In the following we assume

Assumption 1

In the space $V_{(f)}$ *there exists a vector* $X(S) \neq 0$ *such that*

$$A_{(2f)}(h)X(S) = X(S) \text{ for any } h \in O(k) \tag{8}$$

Under the assumption 1, let $X_{(f)}^{(1)}(S)$ in a basis be the above vector $X(S)$, then we see that $\chi_{11}^{(f)}(h) = 1$, $\chi_{i1}^{(f)}(h) = 0 (2 \leqslant i \leqslant N)$ for any $h \in O(k)$ and $a_{11}^{(f)} = 1$. Under the assumption 1, we see that our representation (1) is a representation of class one with respect to the subgroup $O(k)$ [4]. Let $X_{(f)}^{(i)}(S)$'s $(1 \leqslant i \leqslant N)$ be a basis of the space $V_{(f)}$ such that $X_{(f)}^{(1)}(S)$ satisfies the expression (8).

We define spherical functions, zonal spherical function, and associate spherical function of this representation ;

(a) *The spherical function* is defined by

$$X(g) = \langle A_{(2f)}(g)X(S),\ X_{(f)}^{(1)}(S)\rangle, \quad g \in GL(k; R) \tag{9}$$
$$\text{for any vector } X(S) \in V_{(f)}.$$

We note that it has a property $X(hg) = X(g)$ for any $h \in O(k)$, that is, it may be considered a function on the left coset space $O(k)/GL(k; R)$. It must be a function of $g'g$ from results of the invariant theory, where g' is the transposed of g. In the explicit expression, we heve

$$X(g) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i a_{ij}^{(f)} \chi_{ji}^{(f)}(g) \qquad \text{for the vector } X(S) = \sum_{i=1}^{N} \alpha_i X_{(f)}^{(i)}(S).$$

(b) *The zonal spherical function* is the function

$$X_{(f)}^{(1)}(g) = \langle A_{(2f)}(g)X_{(f)}^{(1)}(S),\ X_{(f)}^{(1)}(S)\rangle$$
$$= \sum_{j=1}^{N} \bar{a}_{1j}^{(f)} \chi_{j1}^{(f)}(g), \quad g \in GL(k; R) \tag{10}$$

We note that it has a property

$$X_{(f)}^{(1)}(h_1 g h_2) = X_{(f)}^{(1)}(g) \text{ for any } h_1,\ h_2 \in O(k),$$

whence it must be a symmetric function of eigenvalues of $g'g$ and it may be considered a function constant-valued on the sphere $h(gg')h', h \in O(k)$, with center E_k (the unit matrix of order k) passing through the real positive definite matrix $gg' \in \mathfrak{S}_k^+$.

(c) *Associate spherical functions* are functions

$$\langle A_{(2f)}(g)X_{(f)}^{(1)}(S),\ X_{(f)}^{(i)}(S)\rangle \tag{11}$$

$$= \sum_{j=1}^{N} \bar{a}_{ij}^{(f)} \chi_{j1}^{(f)}(g) \qquad (2 \leqslant i \leqslant N)$$

These may be considered functions on the right coset space $GL(k; R)/O(k)$ and must be functions of gg'.

Since we have a result $\mathfrak{S}_k^+ \cong GL(k; R)/O(k)$, zonal spherical function and associate spherical functions are functions on the space $\mathfrak{S}_k^+$.

In particular, with respect to the above defined unitary basis, we have the following expression of these:

(a') The spherical function

$$X(g) = \sum_{i=1}^{N} \alpha_i \chi_{1i}^{(f)}(g). \tag{9'}$$

(b') The zonal spherical function

$$X_{(f)}^{(1)}(g) = \chi_{11}^{(f)}(g). \tag{10'}$$

(c') Associate spherical functions

$$\langle A_{(2f)}(g)X_{(f)}^{(1)}(S),\ X_{(f)}^{(i)}(S)\rangle = \chi_{i1}^{(f)}(g) \qquad (2 \leqslant i \leqslant N). \tag{11'}$$

To see effects of linear transformation of a basis on these definitions let $Y_{(f)}^{(1)}(S), \ldots, Y_{(f)}^{(N)}(S)$ be another basis such that $Y_{(f)}^{(1)}(S)$ satisfies the expression (8) and the transformation is given by the expression (6) ; we have following results,

(a) The spherical function

$$X(g) = \langle A_{(2f)}(g)X(S),\ \ Y^{(1)}_{(f)}(S)\rangle$$

$$= [\Gamma^{(f)*}A^{(f)'}\chi^{(f)}(g)\alpha]_{(1,1)}\,, \tag{12}$$

where $X(S) = \sum_{i=1}^{N} \alpha_i X^{(i)}_{(f)}(S) = \sum_{i=1}^{N} \tilde{\alpha}_i Y^{(i)}_{(f)}(S)$ and $[\ \cdot\ ]_{(1,1)}$

is the (1, 1) element of matrix $[\ \cdot\ ]$, $\alpha' = (\alpha_1, \ldots, \alpha_N)$

is the transposed of a vector α.

(b) The zonal spherical function

$$Y^{(1)}_{(f)}(g) = \langle A_{(2f)}(g)Y^{(1)}_{(f)}(S),\ \ Y^{(1)}_{(f)}(S)\rangle$$

$$= [\Gamma^{(f)*}A^{(f)'}\chi^{(f)}(g)\,\Gamma^{(f)}]_{(1,1)}. \tag{13}$$

(c) Associate spherical functions

$$\langle A_{(2f)}(g)\,Y^{(1)}_{(f)}(S),\ \ Y^{(i)}_{(f)}(S)\rangle \tag{14}$$

$$= [\Gamma^{(f)*}A^{(f)'}\chi^{(f)}(g)\,\Gamma^{(f)}]_{(i,1)},$$

where $[\ \cdot\ ]_{(i,1)}$ is the (i, 1) element of matrix $[\ \cdot\ ]$,
$(2 \leqslant i \leqslant N)$.

In the following, in addition to the assumption 1, we assume

Assumption 2

In the space $V_{(f)}$ *there exists only one normalized vector satisfyiug assumption* 1.

Let $X^{(i)}_{(f)}(S)$, $Y^{(i)}_{(f)}(S)$ $(1 \leqslant i \leqslant N)$ be two unitary bases of the space $V_{(f)}$ such that $X^{(1)}_{(f)}(S) = Y^{(1)}_{(f)}(S)$ satisfies the expression (8). In this case, the transformation matrix $\Gamma_{(f)}$ must be of the form

$$\Gamma^{(f)} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\Gamma}^{(f)} \end{pmatrix}, \quad \text{where } \tilde{\Gamma}^{(f)} \in U(N-1), \text{ since } \Gamma^{(f)} \in U(N).$$

Thus, we have following relations ;

(a) The spherical function

$$X(g) = [\chi^{(f)}(g)\alpha]_{(1,1)} = X(g). \tag{12'}$$

(b) The zonal spherical function

$$Y^{(1)}_{(f)}(g) = X^{(1)}_{(f)}(g). \tag{13'}$$

(c) Associate spherical functions

$$\langle A_{(2f)}(g)X^{(1)}_{(f)}(S),\ Y^{(i)}_{(f)}(S)\rangle = \sum_{j=2}^{N} \bar{\gamma}_{ij}\chi^{(f)}_{j1}(g) \tag{14'}$$

$$(2 \leqslant i \leqslant N)$$

From these results we have a proposition

Proposition 3

With respect to two unitary bases constructed as the above, spherical functions and the zonal spherical function of the representation (1) *of the group* $GL(k\,;R)$ *are invariant under the linear transformation of basis.*

Example

For partition $(2f) = (2f,\ 0^{k-1})$, $k = 2n$ or $k = 2n+1$, we have

$$\{2f\} = \{2f\}' + \{2f-2\}' + \ldots + \{2(f-i)\}' + \ldots + \{0\}',$$

where $\{2f\}$ is the character of representation $A_{(2f,\ 0^{k-1})}(g)$ of the group $GL(k\,;R)$ and $\{2f\}'$ and so on, are characters of represent ation $\langle P_0(2f,\ 0^{n-1})\rangle$ of the group $0(k)$ and so on.

Thus, in this case, our assumptions 1 and 2 are satisfied.

We give an explicit result for the case $k=2$, $f=1$. Let $X^{(1)}_{(f)}(S)=tr(S)$ $X^{(2)}_{(f)}=s_{12}$ ((1, 2) element ot the matrix S) and $X^{(3)}_{(f)}(S)=s_{22}$ ((2, 2) element of the matrix S) be a basis of $V_{(f)}$. At first we construct a unitary basis of $V_{(f)}$ from these vectors. We have matrices

$$\chi^{(f)}(J_2)=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \text{and}$$

$$\chi^{(f)}(h)=\chi^{(f)}(\theta)=\begin{pmatrix}1 & -\sin 2\theta/2 & \sin^2\theta\\ 0 & \cos 2\theta & -\sin 2\theta\\ 0 & \sin 2\theta & \cos 2\theta\end{pmatrix}$$

$$\text{for } h=h(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\in SO(2).$$

Whence using the formula (5), we have the matrix

$$A^{(f)}=\begin{pmatrix}1 & 0 & 1/2\\ 0 & 5/8 & 0\\ 1/2 & 0 & 3/4\end{pmatrix}$$

Thus, $\hat{X}^{(1)}_{(f)}(S)=tr(S)$, $\hat{X}^{(2)}_{(f)}(S)=\sqrt{8s_{12}^2/5}$, $\hat{X}^{(3)}_{(f)}=\sqrt{2}\ (s_{22}-tr(S)/2)$ is a unitary basis of $V^{(f)}$. Using this unitary basis, we have following results;

(1) The spherical function

Let $X(S)=\sum_{i=1}^{3}c_i\hat{X}^{(i)}_{(f)}(S)$, where $c_i\ (i=1,2,3)$ are complex numbers, then we have the corresponding spherical function

$$X(g) = c_1\ tr(g^{-1}g^{-1\prime})/2 + c_2\sqrt{\frac{2}{5}}\ (g^{-1}g^{-1\prime})_{12} + c_3\left(-\frac{1}{2\sqrt{2}}((g^{-1}g^{-1\prime})_{11}\right.$$

$$\left.-(g^{-1}g^{-1\prime})_{22})\right),\ g \in GL(2; R),$$ where $(\cdot)_{12}$ is the (1, 2) element of the matrix $(\cdot)$, and so on.

(2) The zonal spherical function

$$X^{(1)}_{(f)}(g) = tr(g^{-1\prime}g^{-1})/2,\ g \in GL(2; R).$$

(3) Associate spherical functions

$$\langle A_{(2f)}(g)\hat{X}^{(1)}_{(f)}(S),\ \hat{X}^{(2)}_{(f)}(S)\rangle = \sqrt{5/2}\ (g^{-1\prime}g^{-1})_{12},$$

$$\langle A_{(2f)}(g)\hat{X}^{(1)}_{(f)}(S),\ \hat{X}^{(3)}_{(f)}(S)\rangle = -\frac{1}{\sqrt{2}}\ ((g^{-1\prime}g^{-1})_{11} - (g^{-1\prime}g^{-1})_{22}).$$

Put $gg'^{-1} = T \in \mathfrak{S}^{+}_{k}$, then we have also

$$X^{(1)}_{(f)}(g) = tr(T)/2,$$

$$\langle A_{(2f)}(g)X^{(1)}_{(f)}(S),\ X^{(2)}_{(f)}(S)\rangle = \sqrt{\frac{5}{2}}\ t_{12},$$

$$\langle_{(2f)}(g)X^{(1)}_{(f)}(S),\ X^{(3)}_{(f)}(S)\rangle = -\frac{1}{\sqrt{2}}(t_{11} - t_{22}),$$

where t_{12} is the (1, 2) element of the matrix T and so on.

§ 2. Zonal spherical functions on the group $GL(k; R)$

Let G be the group $GL(k; R)$ and K be the subgroup $O(k)$. In this section, at first we prove that the pair (G, K) is a Gelfand pair, that is, $C_0(K\backslash G/K)$, the set of continuous functions defined on $K\backslash G/K$ with compact supports, is a commutative Banach algebra and after K. Maurin [5] we give an definition of zonal spherical function on the group $G=GL(k; R)$ and we prove that our zonal spherical function defined in § 1 satisfies this definition.

Let $C_0(G)$ be the set of continuous functions defined on G with compact supports ; for f_1, $f_2 \in C_0(G)$, we define a product $f_1 f_2$ by the convolution $f_1 * f_2$,

$$(f_1 f_2)(g_1) = (f_1 * f_2)(g_1) = \int_G f_1(g_1 g_2^{-1}) f_2(g_2)\, dg_2, \quad g_1 \in G,$$

where dg_2 is the bi-invariant measure on the group G.

Let $f \in C(K\backslash G/K)$, the set of continuous functions on $K\backslash G/K$;

$f(k_1 g) = f(gk_2) = f(k_1 g k_2) = f(g)$ for any $g \in GL(k; R)$, k_1, $k_2 \in 0(k)$,

whence owing to results in invariant theory, the function $f(g)$ must be a symmetric function of eigenvalues of gg' and also it must be a symmetric function of eigenvalues of $g'g$, that is, $C(K\backslash G/K)$ is the set of symmetric continuous functions of eigenvalues of real positive definite symmetric matrices.

Now let functions f_1, f_2 be elements of $C_0(K\backslash G/K)$, we have

$$\begin{aligned}(f_1 f_2)(k_1 g_1 k_2) &= \int_G f_1(k_1 g_1 k_2 g_2^{-1})\, f_2(g_2)\, dg_2 \\ &= \int_G f_1(g_1 k_2 g_2^{-1}) f_2(g_2)\, dg_2 \\ &= \int_G f_1(g_1 g^{-1}) f_2(gk_2)\, d(gk_2) = \int_G f_1(g_1 g^{-1}) f_2(g)\, dg \\ &= (f_1 f_2)(g_1), \quad k_1,\ k_2 \in K \text{ and } g_1 \in G.\end{aligned}$$

Thus, when f_1, $f_2 \in C_0(K\backslash G/K)$, $f_1 f_2 \in C_0(K\backslash G/K)$. Using the above remark, we see also that for f_1, $f_2 \in C_0(K\backslash G/K)$,

$$
\begin{aligned}
(f_1 f_2)(g_1) &= \int_G f_1(g_1 g_2^{-1}) f_2(g_2) dg_2 \\
&= \int_G f(g_1 g_2^{-1} g_2^{-1\prime} g_2') f_2(g_2' g_2)\, dg_2 \\
&= \int_G f_1(g' g) f_2(g_1' g^{-1} g^{-1\prime} g_1)\, dg = (f_2 f_1)(g_1),\ g_1 \in G.
\end{aligned}
$$

Thus we have a proposition

Proposition 4

The pair $(GL(k\,;R), O(k))$ *is a Gelfand pair.*

Now we define the zonal spherical spherical function and spherical functions on the group G as follows (cf. [5], pp. 227);

Definition 1

A complex valued function $\omega(g)$ *on the group* G *is called a zonal spherical function if*

(1) $\omega \in C(K\backslash G/K)$; *that is,* $\omega(g)$ *is a symmetric continous function of eigenvalues of matrix* $S \in \mathfrak{S}_k^+$,

(2) ω *defines a homomorphism between* $C_0(K\backslash G/K)$ *and the complex number field* C *by*

$$\hat{\omega} : f \to \hat{\omega}(f) = \langle f, \omega \rangle = \int_G f(g^{-1})\omega(g) dg,\ f \in C_0(K\backslash G/K)\,; \quad \text{that is,}$$

ω *satisfies*

1) $\hat{\omega}(\alpha f_1 + \beta f_2) = \alpha\hat{\omega}(f_1) + \beta\hat{\omega}(f_2)$, $\alpha, \beta \in C$ and $f_1, f_2 \in C_0(K\backslash G/K)$,

$\hat{\omega}(f_1 f_2) = \hat{\omega}(f_1)\,\hat{\omega}(f_2)$

2) when $\|f_1 - f_2\| = \int_G |f_1 - f_2|\, dg \to 0$,

$|\hat{\omega}(f_1) - \hat{\omega}(f_2)| \to 0.$

K. Maurin ([5], Proposition 3, p. 230) proved that a function $\omega \in C(K\backslash G/K)$ is a zonal spherical function iff (1) $\omega(e)=1$, e is the unit of G and

(2) for any $f \in C_0(K\backslash G/K)$, the convolution equaton

$$(f*\omega)(g)=\lambda(f)\omega(g),\ \lambda(f)\in C$$

is satisfied.

A. T. James [3] mentioned that his zonal polynomial $C_{(f)}(S)/C_{(f)}(E_k)$, $S \in \mathfrak{S}_k^+$, E_k is the unit matrix of order k, belonging to $V_{(f)}$, has a property

$$\int_K C_{(f)}(S\,k\,T\,k')/C_{(f)}(E_k)dV(k)=C_{(f)}(S)/C_{(f)}(E_k)\cdot C_{(f)}(T)/C_{(f)}(E_k),\ \text{where } S,$$

$T \in \mathfrak{S}_k^+$.

Thus we have a proposition

Proposition 5

The function $C_{(f)}(S)/C_{(f)}(E_k)$ is a zonal spherical function on the group $GL(k;R)$ and it satisfies the above convolution equation for any $f \in C_0(K\backslash G/K)$.

Definition 2

A function $\varphi \in C(K\backslash G)$ is a spherical function if for any $f \in C_0(K\backslash G/K)$ it satisfies the convolution equation

$$(f*\varphi)(g)=\lambda(f)\varphi(g),\ \lambda(f)\in C.$$

A function $\psi \in C(G/K)$ is a left spherical function if for any $f \in C_0(K\backslash G/K)$ it satisfies the convolution equation

$$(\psi*f)(g)=\lambda(f)\psi(g),\ \lambda(f)\in C.$$

We note that these zonal spherical functions and left spherical functions are functions on the space $\mathfrak{S}_k^+$.

In the following, we prove that our zonal spherical function $X^{(1)}_{(f)}(g) = \langle A_{(2f)}(g)X^{(1)}_{(f)}(S), X^{(1)}_{(f)}(S)\rangle$ defined in § 1 satisfies those requirements in definition 1. Under our assumptions 1 and 2, $X^{(1)}_{(f)}(S)$ is the vector satisfying these assumptions. The space $V_{(f)}$ is a unitary space and $A_{(2f)}(g)$, $g \in GL(k\,;R)$ is a continuos representation of the group $GL(k\,;R)$ and $\|A_{(2f)}(g)X^{(1)}_{(f)}(S)\|$ is bounded on compact subset of the group G.

(1) For any $k_1, k_2 \in 0(k)$, $g \in GL(k\,;R)$, we have $X^{(1)}_{(f)}(k_1gk_2) = X^{(1)}_{(f)}(g)$, whence $X^{(1)}_{(f)}(g) \in C(K\backslash G/K)$.

(2) $\hat{X}^{(1)}_{(f)} \overset{\text{def.}}{=} \int_G f(g^{-1})X^{(1)}_{(f)}(g)dg$ for $f \in C_0(K\backslash G/K)$. Then we have results

a) for any $f_1, f_2 \in C_0(K\backslash G/K)$ and $\alpha, \beta \in C$,

$$\hat{X}^{(1)}_{(f)}(\alpha f_1 + \beta f_2) = \alpha\hat{X}^{(1)}_{(f)}(f_1) + \beta\hat{X}^{(1)}_{(f)}(f_2),$$

b) for any $f_1, f_2 \in C_0(K\backslash G/K)$,

$$|\hat{X}^{(1)}_{(f)}(f_1) - \hat{X}^{(1)}_{(f)}(f_2)| = |\int_G f_1(g^{-1})X^{(1)}_{(f)}(g)dg - \int_G f_2(g^{-1})X^{(1)}_{(f)}(g)dg|$$

$$\leqslant \int_G |f_1(g^{-1}) - f_2(g^{-1})|\ |X^{(1)}_{(f)}(g)|dg$$

$$\leqslant M \int_G |f_1(g^{-1}) - f_2(g^{-1})|dg$$

$$= M \int_G |f_1(g) - f_2(g)|dg,$$

where $\sup\{\|A_{(2f)}(g)X^{(1)}_{(f)}(S)\|\,;\, g \in Car(f_1) \cup Car)(f_2)\} \leqslant M$.

Thus, $\hat{X}^{(1)}_{(f)}(f)$, $f \in C_0(K\backslash G/K)$, is a $\|\cdot\|_1$ -continuous function.

c) for any $f_1, f_2 \in C_0(K\backslash G/K)$, we have

$$\hat{X}^{(1)}_{(f)}(f_1 * f_2) = \int_G (f_1 * f_2)(g^{-1}) X^{(1)}_{(f)}(g) dg$$

$$= \int_G \int_G f_1(g^{-1} g_1^{-1}) f_2(g_1) X^{(1)}_{(f)}(g) dg_1 dg$$

$$= \int_G \int_G f_1(g^{-1}) f_2(g_1) X^{(1)}_{(f)}(g_1^{-1} g) dg_1 dg$$

$$= \int_G \int_G f_1(g^{-1}) f_2(g_1) \chi^{(f)}_{11}(g_1^{-1} g) dg_1 dg$$

$$= \int_G \int_G f_1(g^{-1}) f_2(g_1) \sum_{i=1}^{N} \chi^{(f)}_{1i}(g_1^{-1}) \chi^{(f)}_{i1}(g) dg_1 dg$$

$$= \sum_{i=1}^{N} \int_G f_1(g^{-1}) \chi^{(f)}_{i1}(g) dg \int_G f_2(g_1) \chi^{(f)}_{1i}(g_1^{-1}) dg_1$$

$$= \sum_{i=1}^{N} \int_G f_1(g^{-1}) \chi^{(f)}_{i1}(g) dg \int_G f_2(g_1^{-1}) \chi^{(f)}_{1i}(g_1) dg_1 ;$$

since $f \in C_0(K\backslash G/K)$, we have

$$\int_G f_1(g_1) \chi^{(f)}_{i1}(g_1) dg_1 = \int_G f_1(g_1^{-1}) \chi^{(f)}_{i1}(k_1 g_1 k_2) dg_1$$

$$= \int_G f_1(g_1^{-1}) dg_1 \int_{0(k)} \int_{0(k)} \chi^{(f)}_{i1}(k_1 g_1 k_2) dV(k_1) dV(k_2)$$

$$= \delta_{i1} \int_G f_1(g_1^{-1}) \chi^{(f)}_{11}(g_1) dg_1, \text{ whence}$$

we get the relation $\hat{X}^{(1)}_{(f)}(f_1 * f_2) = \hat{X}^{(1)}_{(f)}(f_1) \hat{X}^{(1)}_{(f)}(f_2)$.

Summarizing these results, we have

Proposition 6

Under the assumptions 1 *and* 2, *the function* $X^{(1)}_{(f)}(g)$ *is a zonal spherical function in the sense of Definition* 1.

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