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Construction of Inversive Semigroups

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This is a supplement to the previous papers [5] and [6] of the author in which the structure of strictly regular semigroups has been clarified. By using the results of [5], [6], at first we shall show how to construct every inversive semigroup. Secondly in the latter half of the paper we shall investigate the relations between quasi-direct products and spined products, and give another proof for the structure theorem for strictly inversive semigroups which has been established in the previous paper [4]. Any notation and terminology should be referred to [5], [6], unless otherwise stated.

§1. A construction theorem. Throughout the whole paper, the notation "a band $E \equiv \{E_\gamma : \gamma \in \Gamma\}$" means that "a band E which has Γ as its structure semilattice and whose structure decomposition is $E \sim \Sigma\{E_\gamma : \gamma \in \Gamma\}$". Hence, "$E \equiv \{E_\gamma : \gamma \in \Gamma\}$ is a band" means that "E is a band whose structure semilattice is Γ and whose structure decomposition is $E \sim \Sigma\{E_\gamma : \gamma \in \Gamma\}$".

Let Λ be an inverse semigroup, and Γ its basic semilattice. Let $E \equiv \{E_\gamma : \gamma \in \Gamma\}$ be a band. Define equivalence relations π_1, π_2 on E as follows:

(1.1) $e\,\pi_1 f$ if and only if $ef = f$ and $fe = e$,

(1.2) $e\,\pi_2 f$ if and only if $ef = e$ and $fe = f$.

For $e \in E$, $\tilde{e}$, $\tilde{\tilde{e}}$ will denote the π_1-class and the π_2-class containing e respectively. Further, let $\tilde{E} = \{\tilde{e} : e \in E\}$, $\tilde{\tilde{E}} = \{\tilde{\tilde{e}} : e \in E\}$, $\tilde{E}_\gamma = \{\tilde{e} : e \in E_\gamma\}$ and $\tilde{\tilde{E}}_\gamma = \{\tilde{\tilde{e}} : e \in E_\gamma\}$, $\gamma \in \Gamma$.

To each ordered pair (ξ, η) of elements ξ, η of Λ, let correspond a mapping $\rho_{(\xi,\eta)} : (\tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi^{-1}\xi}) \times (\tilde{E}_{\eta\eta^{-1}} \times \tilde{\tilde{E}}_{\eta^{-1}\eta}) \to \tilde{E}_{\xi\eta(\xi\eta)^{-1}} \times \tilde{\tilde{E}}_{(\xi\eta)^{-1}\xi\eta}$. According to [6], the system $\{\rho_{(\xi,\eta)}: \xi, \eta \in \Lambda\}$ of these mappings $\rho_{(\xi,\eta)}$ is said to be *a set of quasi-direct factors of E with respect to* Λ if it satisfies the following

$$(1.3)\begin{cases}(1)\ \text{If } \xi, \eta \in \Gamma, \text{ then } \rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)) = (\widetilde{ef}\ \widetilde{\widetilde{ef}}), \text{ where } e, f \text{ are elements of } E_\xi, E_\eta \text{ respectively such that } \tilde{e} = \tilde{e}_1, \tilde{\tilde{e}} = \tilde{\tilde{f}}_1, \tilde{f} = \tilde{e}_2 \text{ and } \tilde{\tilde{f}} = \tilde{\tilde{f}}_2.\\ (2)\ \rho_{(\xi\eta,\nu)} \circ_L \rho_{(\xi,\eta)} = \rho_{(\xi,\eta\nu)} \circ_R \rho_{(\eta\nu)} \text{ for all } \xi, \eta, \nu \in \Lambda.\\ (3)\ \text{For any } \xi \in \Lambda, e \in E_{\xi\xi^{-1}} \text{ and } f \in E_{\xi^{-1}\xi}, \text{ there exist } h \in E_{\xi^{-1}\xi} \text{ and } k \in E_{\xi\xi^{-1}} \text{ such that}\\ \quad \rho_{(\xi,\xi^{-1}\xi)} \circ_R \sigma_{(\xi^{-1},\xi)}((\tilde{e}, \tilde{\tilde{f}}), (\tilde{h}, \tilde{\tilde{k}}), (\tilde{e}, \tilde{\tilde{f}})) = (\tilde{e}, \tilde{\tilde{f}}).^{1)}\end{cases}$$

Now, especially we shall consider the case where Λ is a weakly C-inversive (for the definitions of inversive semigroups and weakly C-inversive semigroups, see [5]).[2] In this case, for $\xi \in \Lambda$ the equality $\xi\xi^{-1} = \xi^{-1}\xi$ holds. Hence for any $(\tilde{e}, \tilde{\tilde{f}}) \in \tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi^{-1}\xi} = \tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi\xi^{-1}}$, there exists $(\tilde{h}, \tilde{\tilde{h}}) \in \tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi^{-1}\xi}$ such that $(\tilde{h}, \tilde{\tilde{h}}) = (\tilde{e}, \tilde{\tilde{f}})$ (in fact, $h = ef$). Hence, $\tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi^{-1}\xi} = \{(\tilde{e}, \tilde{\tilde{e}}) : e \in E_{\xi\xi^{-1}}\}$. Let $\Omega = \{\rho_{(\xi,\eta)}: \xi, \eta \in \Lambda\}$ be a set of quasi-direct factors of E with respect to Λ, and for each ordered pair (ξ, η) of elements ξ, η of Λ define a mapping $\delta_{(\xi,\eta)} : E_{\xi\xi^{-1}} \times E_{\eta\eta^{-1}} \rightarrow E_{\xi\eta(\xi\eta)^{-1}}$ by

(1.4) $\delta_{(\xi,\eta)}(e, f) = h$ if and only if $\sigma_{(\xi,\eta)}((\tilde{e}, \tilde{\tilde{e}}), (\tilde{f}, \tilde{\tilde{f}})) = (\tilde{h}, \tilde{\tilde{h}})$ Then, we can get the system $\{\delta_{(\xi,\eta)} : \xi, \eta \in \Lambda\} = \Delta$ of these mappings $\delta_{(\xi,\eta)}$. By using this Δ, (1.3) can be rewritten as follows:

$$(1.5)\begin{cases}(1)\ \text{If } \xi, \eta \in \Gamma, \text{ then } \delta_{(\xi,\eta)}(e, f) = ef.\\ (2)\ \delta_{(\xi\eta,\nu)} \circ_L \delta_{(\xi,\eta)} = \delta_{(\xi,\eta\nu)} \circ_R \delta_{(\eta,\nu)} \text{ for all } \xi, \eta, \nu \in \Lambda.\\ (3)\ \text{For any } \xi \in \Lambda, e \in E_{\xi\xi^{-1}}, \text{ there exists } h \in E_{\xi\xi^{-1}} \text{ such that}\\ \quad \delta_{(\xi,\xi\xi^{-1})} \circ_R \delta_{(\xi^{-1},\xi)}(e, h, e) = e.\end{cases}$$

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1) $\rho_{(\xi\eta,\nu)} \circ_L \rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2), (\tilde{e}_3, \tilde{\tilde{f}}_3))$ means $\rho_{(\xi\eta,\nu)}(\rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)), (\tilde{e}_3, \tilde{\tilde{f}}_3))$, and $\rho_{(\xi,\eta\nu)} \circ_R \rho_{(\xi,\nu)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2), (\tilde{e}_3, \tilde{\tilde{f}}_3))$ means $\rho_{(\xi,\eta\nu)}((\tilde{e}_1, \tilde{\tilde{f}}_1), \rho_{(\xi,\nu)}((\tilde{e}_2, \tilde{\tilde{f}}_2), (\tilde{e}_3, \tilde{\tilde{f}}_3)))$.

2) In Ljapin [3], a weakly C-inversive semigroup has been called a completely regular semigroup. It is well-known that a semigroup is weakly C-inversive if and only if it is a semilattice of groups. The structure of weakly C-inversive semigroups has been clarified by Clifford [1] (see also Clifford and Preston [2], p. 128).

Therefore, in case where Λ is weakly C-inversive, we shall call a system $\{\delta_{(\xi,\eta)} : \xi, \eta \in \Lambda\}$ of mappings $\delta_{(\xi,\eta)} : E_{\xi\xi^{-1}} \times E_{\eta\eta^{-1}} \rightarrow E_{\xi\eta(\xi\eta)^{-1}}$ *a set of quasi-direct factors of E with respect to* Λ if it satisfies (1.5).

Now, let S be an inversive semigroup and $E \equiv \{E_\gamma : \gamma \in \Gamma\}$ the band consisting of all idempotents of S. It is obvious from [4] that S is a union of subgroups S_γ's of S, and for any $a \in S$ there exists a unique element a^* of S such that $aa^*a = a$, $a^*aa^* = a^*$ and $aa^* = a^*a$. In this case, if a is an element of S_γ then a^* is the group inverse of a in S_γ. Hereafter, we shall denote such a^* by a^{-1}.[3)] Since the greatest inverse semigroup decomposition σ of S exists (see [6]), we can consider the factor semigroup $S/\sigma = \Lambda$ of S mod σ. The semigroup Λ is an inverse semigroup and is also inversive, and hence Λ is weakly C-inversive (see [5], [6]). Further, in this case it follows from [6] that (i) for any $e \in E_\gamma$, the σ-class containing e is E_γ ; (ii) the basic semilattice of S/σ is $E/\sigma_E = \{E_\gamma : \gamma \in \Gamma\}$, where σ_E is the restriction of σ to E, and hence the basic semilattice E/σ_E of S/σ is of course isomorphic to Γ. Hereafter, we shall denote the σ-class containing x by $\bar{x}$. Further when we regard $\bar{e}$, where $e \in E$, as a subset of E we denote it by $E_{\bar{e}}$. Hence, $E_{\bar{e}} = E_\gamma$, $\gamma \in \Gamma$, if and only if $\bar{e} \equiv \gamma$, *i. e.*, $E_\gamma \in e$. Then, the basic semilattice of S/σ is $\{E_{\bar{e}} : \bar{e} \in E/\sigma_E\}$, and the structure decomposition of E is $E \sim \Sigma\, \{E_{\bar{e}} : \bar{e} \in E/\sigma_E\}$.[4)]

Now, let define $\Delta = \{\delta_{(\bar{x},\bar{y})} : \bar{x}, \bar{y} \in \Lambda\}$ by

(1.6) $\delta_{(\bar{x},\bar{y})}(e, f) = uv(uv)^{-1}$, where $(e, f) \in E_{\bar{x}\bar{x}^{-1}} \times E_{\bar{y}\bar{y}^{-1}}$ and u, v are elements of S such that $\bar{u} = \bar{x}$, $\bar{v} = \bar{y}$, $uu^{-1} = e$ and $vv^{-1} = f$ (for given $\bar{x}, \bar{y} \in \Lambda$ and given $(e, f) \in E_{\bar{x}\bar{x}^{-1}} \times E_{\bar{y}\bar{y}^{-1}}$, such u, v are uniquely determined ; see [4]).

3) Let S be an inverse semigroup which is also inversive. Then for each $a \in S$, there exists a unique inverse a' of a (*i. e.*, an element a' such that $aa'a = a$ and $a'aa' = a'$). Since S is inversive, there exists also a unique element a^{-1} such that $aa^{-1}a = a$, $a^{-1}aa^{-1} = a^{-1}$ and $aa^{-1} = a^{-1}a$. Hence, necessarily $a' = a^{-1}$ and $aa' = a'a$ are satisfied in this case.

4) Hence, the structure semilattice of E can be considered as E/σ_E.

Then, the following lemma is obvious from [6] :

Lemma 1. *Δ is a set of quasi-direct factors of* $E \equiv \{E_{\bar{x}} : \bar{x} \in E/\sigma_E\}$ *with respect to* $\Lambda = S/\sigma$, *and the quasi-direct product* $E \times \Lambda$ *of* E, Λ *determined by* Δ, *that is,* $E \times \Lambda$ *such that*

$$(1.7) \begin{cases} E \times \Lambda = \{(e, \bar{x}) : \bar{x} \in \Lambda, e \in E_{\bar{x}\bar{x}^{-1}}\} ; \\ \text{the multiplication in } E \times \Lambda : \\ (e, \bar{x})(f, \bar{y}) = (\delta_{(\bar{x},\bar{y})}(e, f), \bar{x}\,\bar{y}) \end{cases}$$

is isomorphic to S. *Especially,* $\varphi : S \rightarrow E \times \Lambda$ *defined by* $\varphi(x) = (xx^{-1}, \bar{x})$, $x \in S$, *is an isomorphism of* S *onto* $E \times \Lambda$.
Proof. Since Λ is weakly C-inversive, (4.1) of [6] can be rewritten into the form of (1.6) by using (1.4). Further since (4.2) of [6] can be also rewritten into the form of (1.7), this lemma follows from Theorem 3 of [6].

Next, we shall study about the Δ above.

Lemma 2.

(1.8) $\Lambda \ni \xi, \eta$ *and* $\xi\xi^{-1} = \eta\eta^{-1}$ *implies* $\delta_{(\xi,\eta)}(e, e) = e$ *for* $e \in E_{\xi\xi^{-1}} (= E_{\eta\eta^{-1}})$.

Proof. Let $\bar{x}$, $\bar{y}$ be elements of Λ such that $\bar{x}\bar{x}^{-1} = \bar{y}\bar{y}^{-1}$. $\delta_{(\bar{x},\bar{y})}(e, e) = uv(uv)^{-1}$ by (1.6), where u, v are elements of S such that $\bar{u} = \bar{x}$, $uu^{-1} = e$, $\bar{v} = \bar{y}$ and $vv^{-1} = e$. Since $uv(uv)^{-1} = e$, we have $\delta_{(\bar{x},\bar{y})}(e, e) = e$.

By using Lemmas 1 and 2, we have

Theorem 1. *Let S be an inversive semigroup, and E the band consisting of all idempotents of S. Let σ be the greatest inverse semigroup decomposition of S, and σ_E the restriction of σ to E.*
Then

(1) $\Lambda = S/\sigma$ *is a weakly C-inversive semigroup, and its basic semilattice is* $\Gamma = E/\sigma_E$. *Further, the structure decomposition of* E *is* $E \sim \Sigma\{E_{\bar{e}} : \bar{e} \in E/\sigma_E = \Gamma\}$;

(2) *if* $\delta_{(\bar{x},\bar{y})}: E_{\bar{x}\bar{x}^{-1}} \times E_{\bar{y}\bar{y}^{-1}} \to E_{\bar{x}\bar{y}(\bar{x}\bar{y})^{-1}}$ *is the mapping defined by* (1.6), *then the system* $\Delta = \{\delta_{(\bar{x},\bar{y})} : \bar{x}, \bar{y} \in \Lambda\}$ *is a set of quasi-direct factors of* E *with respect to* Λ *and satisfies* (1.8) ;

(3) *the quasi-direct product* $E \times \Lambda$ *determined by* Δ *is isomorphic to* S.

Now, let Λ be a weakly C-inversive semigroup and Γ the basic semilattice of Λ. Let $E \equiv \{E_\gamma : \gamma \in \Gamma\}$ be a band. For each ordered pair (ξ, η) of elements ξ, η of Λ, let $\delta_{(\xi,\eta)}$ be a mapping of $E_{\xi\xi^{-1}} \times E_{\eta\eta^{-1}}$ into $E_{\xi\eta(\xi\eta)^{-1}}$. Let $\Delta = \{\delta_{(\xi,\eta)} : \xi, \eta \in \Lambda\}$.

Lemma 3. *If* Δ *satisfies* (1), (2) *of* (1.5) *and* (1.8), *then* Δ *satisfies also* (3) *of* (1.5).

Proof. For $\xi \in \Lambda$ and $e \in E_{\xi\xi^{-1}}$, $\delta_{(\xi,\xi^{-1})}(e, e) = e$ by (1.8). Hence, $\delta_{(\xi,\xi\xi^{-1})} \circ_R \delta_{(\xi^{-1},\xi)}(e, e, e) = e$. Therefore, we can take e as h of (1.5).

By using the lemma above, we obtain the following construction theorem :

Theorem 2. *Let* Λ *be a weakly C-inversive semigroup, and* Γ *the basic semilattice of* Λ. *Let* $E \equiv \{E_\gamma : \gamma \in \Gamma\}$ *be a band. Let* $\Delta = \{\delta_{(\xi,\eta)} : \xi, \eta \in \Lambda\}$ *be a system of mappings* $\delta_{(\xi,\eta)} : E_{\xi\xi^{-1}} \times E_{\eta\eta^{-1}} \to E_{\xi\eta(\xi\eta)^{-1}}$ *satisfying* (1.8) *and* (1), (2) *of* (1.5).

Then

(1) Δ *is a set of quasi-direct factors of* E *with respect to* Λ ;

(2) *the quasi-direct product* $E \times \Lambda$ *of* E, Λ, *that is,* $E \times \Lambda$ *such that*

$$\begin{cases} E \times \Lambda = \{(e, \xi) : \xi \in \Lambda,\ e \in E_{\xi\xi^{-1}}\} ; \\ \textit{the multiplication in } E \times \Lambda : \\ (e, \xi)(f, \eta) = (\delta_{(\xi,\eta)}(e, f), \xi\eta) \end{cases}$$

is an inversive semigroup. Further every inversive semigroup is constructed in this fashion, up to isomorphism.

Proof. The part (1) of the first half of the assertion is obvious from Lemma 3. Hence, we shall next prove the part (2). At first, it is clear that the quasi-direct product $E \times \Lambda$ is strictly regular (see [6]). Take any element (e, ξ) from $E \times \Lambda$. Then, we have $(e, \xi)(e, \xi^{-1}) = (\delta_{(\xi,\xi^{-1})}(e, e), \xi\xi^{-1}) = (e, \xi\xi^{-1})$, $(e, \xi^{-1})(e, \xi) = (\delta_{(\xi^{-1},\xi)}(e, e), \xi^{-1}\xi) = (e, \xi^{-1}\xi)$. Since $\xi\xi^{-1} = \xi^{-1}\xi$,

we have $(e, \xi)(e, \xi^{-1}) = (e, \xi^{-1})(e, \xi)$. Further, $(e, \xi)(e, \xi^{-1})(e, \xi) = (e, \xi)(e, \xi^{-1}\xi) = (\delta_{(\xi,\xi^{-1}\xi)}(e, e), \xi\xi^{-1}\xi) = (e, \xi)$.
Similarly, $(e, \xi^{-1})(e, \xi)(e, \xi^{-1}) = (e, \xi^{-1})$. Hence, $E\times\Lambda$ is inversive.
The latter half of the assertion follow sfrom Theorem 1 and Lemma 3.

§ 2. **Spined products.** Let Λ be a weakly C-inversive semigroup, and Γ the basic semilattice of Λ. Let $E \equiv \{E_\gamma : \gamma \in \Gamma\}$ be a band. Since Λ is weakly C-inversive and Γ is the basic semilattice of Λ, the structure decomposition of Λ is $\Lambda \sim \Sigma\{\Lambda_\gamma : \gamma \in \Gamma\}$, where each Λ_γ is a subgroup (of Λ) containing γ (see [4], [5]).[5] Now for each ordered pair (ξ, η) of $\xi, \eta \in \Lambda$, define a mapping $\delta_{(\xi,\eta)} : E_{\xi\xi^{-1}} \times E_{\eta\eta^{-1}} \to E_{\xi\eta(\xi\eta)^{-1}}$ as follows :

(2.1) $\delta_{(\xi,\eta)}(e, f) = ef$ for $(e, f) \in E_{\xi\xi^{-1}} \times E_{\eta\eta^{-1}}$.

Since $ef \in E_{\xi\xi^{-1}} E_{\eta\eta^{-1}} \subset E_{\xi\xi^{-1}\eta\eta^{-1}} = E_{\xi\eta(\xi\eta)^{-1}}$, this $\delta_{(\xi,\eta)}$ is well-defined.

If we put $\Delta = \{\delta_{(\xi,\eta)} : \xi, \eta \in \Lambda\}$, then

Lemma 4.

(1) *Δ satisfies* (1.8) *and* (1), (2) *of* (1.5), *and hence Δ is a set of quasi-direct factors of E with respect to Λ ;*

(2) *the quasi-direct product $E\times\Lambda$ of E, Λ determined by Δ is a strictly inversive semigroup (see* [4]).[6]

(3) $E\times\Lambda = \{(e, \xi) : e \in E_{\xi\xi^{-1}}, \xi \in \Lambda\} = \{(e, \xi) : e \in E_\varepsilon, \xi \in \Lambda_\varepsilon, \varepsilon \in \Gamma\}$
$= \cup\{E_\varepsilon \times \Lambda_\varepsilon : \varepsilon \in \Gamma\}$, *and the multiplication in $E\times\Lambda$ is as follows :*
$(e, \xi)(f, \eta) = (ef, \xi\eta)$ *for* $(e, \xi), (f, \eta) \in E\times\Lambda$.

Proof. The part (1) is obvious from the definition of the mappings $\delta_{(\xi,\eta)}$. Next we shall show the part (2) : It is obvious from Theorem 2 that $E\times\Lambda$ is an inversive semigroup. Take an element $x = (h, \xi)$ from $E\times\Lambda$, and let $xx^{-1} = e$. Let f be an idempotent of $E\times\Lambda$ such that $f \leqq e$. Since f is an element of $E\times\Lambda$, $f = (g, \eta)$ for some $g \in E$, $\eta \in \Lambda$. It is clear that η is an

5) In this case, $\Lambda \sim \Sigma\{\Lambda_\gamma : \gamma \in \Gamma\}$ implies that $\Lambda = \Sigma\{\Lambda_\gamma : \gamma \in \Gamma\}$ (Σ means disjoint sum) and $\Lambda_\lambda\Lambda_\tau \subset \Lambda_{\lambda\tau}$ for all $\lambda, \tau \in \Gamma$.

6) A semigroup S is said to be strictly inversive if S is inversive and satisfies the following : If e, f are idempotents of S such that $f \leqq e$ and if $xx^{-1} = e$, then $xf = fx$. This concept was firstly introduced by [4].

idempotent since f is an idempotent. Further it follows that $xx^{-1}=(h, \xi\xi^{-1}) = e$, since $x^{-1}=(h, \xi^{-1})$. Now, $f\leqq e$ implies $(g, \eta)(h, \xi\xi^{-1}) = (g, \eta)$. Hence, $(gh, \eta\xi\xi^{-1})=(g, \eta)$. Similarly, $(hg, \xi\xi^{-1}\eta)=(g, \eta)$. Therefore, $gh = hg = g$ and $\eta\cdot\xi\xi^{-1} = \xi\xi^{-1}\cdot\eta = \eta$, that is, $\eta \leqq \xi\xi^{-1}$. Since any weakly C-inversive semigroup is strictly inversive (see [4]), $\xi\eta = \eta\xi$ holds. Thus, we have $xf=(h, \xi)(g, \eta)=(g, \xi\eta)=(g, \eta\xi) = (g, \eta)(h, \xi) = fx$. Therefore, $E\times\Lambda$ is strictly inversive. The part (3) is obvious.

The special quasi-direct product $E\times\Lambda$ given in Lemma 4 is called the spined product of E, Λ with respect to Γ, and denoted by $E\bowtie\Lambda(\Gamma)$.[7] By using the lemma above, we obtain the following result which has been established also in the previous paper [4] of the author :

Theorem 3. *Let S be a strictly inversive semigroup, and E the band consisting of all idempotents of S. Let σ be the greatest inverse semigroup decomposition of S and σ_E the restriction of σ to E. Denote the σ-class containing x by $\bar{x}$. Then*

(1) *$\Lambda = S/\sigma$ is a weakly C-inversive semigroup, and the basic semilattice of Λ is $\Gamma = E/\sigma_E$. Further, the structure decomposition of E is $E\sim\Sigma\{E_{\bar{e}} : \bar{e}\in\Gamma\}$;*

(2) *S is isomorphic to the spined product of E, Λ with respect to Γ, i. e., $S\cong E\bowtie\Lambda(\Gamma)$.*

Conversely, let Λ be a weakly C-inversive semigroup and Γ the basic semilattice of Λ. Let $E\equiv\{E_\gamma : \gamma\in\Gamma\}$ be a band. Then, the spined product of E, Λ with respect to Γ is a strictly inversive semigroup.

Proof. The part (1) of the first half of the assertion is obvious from Theorem 1. Next, we shall prove the part (2). For each ordered pair $(\bar{x}, \bar{y})$ of $\bar{x}, \bar{y}\in\Lambda$, define $\delta_{(\bar{x},\bar{y})} : E_{\bar{x}\bar{x}^{-1}}\times E_{\bar{y}\bar{y}^{-1}}\rightarrow E_{\bar{x}\bar{y}(\bar{x}\bar{y})^{-1}}$ by (1.6). Then the system $\Delta = \{\delta_{(\bar{x},\bar{y})} : \bar{x}, \bar{y}\in\Lambda\}$ of these $\delta_{(\bar{x},\bar{y})}$ is a set of quasi-direct factors of E

7) The concept of the spined product of a band and a weakly C-inversive semigroup was firstly introduced by [4].

with respect to Λ, and it follows from Theorem 1 that the quasi-direct product $E \times \Lambda$ of E, Λ determined by Δ is isomorphic to S. To show $E \times \Lambda = E \bowtie \Lambda(\Gamma)$, by Lemma 4 it is need only to show that Δ satisfies (2.1). Hence, we shall show that $\delta_{(\bar{x},\bar{y})}(e,f)=ef$ for $e \in E_{\bar{x}\bar{x}^{-1}}$, $f \in E_{\bar{y}\bar{y}^{-1}}$. By the definition of $\delta_{(\bar{x},\bar{y})}$, $\delta_{(\bar{x},\bar{y})}(e, f)=uv(uv)^{-1}$, where u, v are elements of S such that $\bar{u}=\bar{x}$, $\bar{v}=\bar{y}$, $uu^{-1}=e$ and $vv^{-1}=f$. Since S is strictly inversive, there exist a band Π and a subgroup S_τ of S for each $\tau \in \Pi$ such that $S=\Sigma\{S_\tau : \tau \in \Pi\}$ (Σ means disjoint sum) and $S_\tau S_\nu \subset S_{\tau\nu}$ for $\tau, \nu \in \Pi$ (see [4]). Suppose that $u \in S_\tau$ and $v \in S_\nu$. Then $uv \in S_{\tau\nu}$, and hence $uv(uv)^{-1}$ is the identity of the group $S_{\tau\nu}$. On the other hand, $uu^{-1} \in S_\tau$ and $vv^{-1} \in S_\nu$. Hence $uu^{-1}vv^{-1} \in S_{\tau\nu}$. Since $uu^{-1}vv^{-1}$ is an idempotent contained in $S_{\tau\nu}$, $uu^{-1}vv^{-1}=uv(uv)^{-1}$, that is, $uv(uv)^{-1}=ef$. Consequently, we have $\delta_{(\bar{x},\bar{y})}(e, f)=ef$. Thus, Δ satisfies (2.1). Therefore, the part (2) of the first half is satisfied. The latter half of the assertion is obvious from Lemma 4.

Remarks. 1. Let Λ be a weakly C-inversive semigroup, and $\Lambda \sim \Sigma\{\Lambda_\gamma : \gamma \in \Gamma\}$ the structure decomposition of Λ (see [5]). Let $E \equiv \{E_\gamma : \gamma \in \Gamma\}$ be a band. In this case, we can also obtain the following result :

A quasi-direct product of E, Λ *which is strictly inversive is uniquely determined up to isomorphism, and it is the spined product of* E, Λ *with respect to* Γ.

2. The existence of a semigroup which is inversive but not strictly inversive is easily seen from simple examples. For example, consider the full transformation semigroup $\mathscr{T}_X$ on a set $X=\{a, b\}$. Then, $\mathscr{T}_X$ is inversive but not strictly inversive.

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