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ON THE EQUIVALENCES OF THE AXIOMATIC SYSTEMS OF THE HARMONIC SPACES

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In the Brelot space the convergence axiom and the Harnack's inequality are equivalent, (M. Brelot [1], R. M. Hervé [1]). For the case that the topology of the space has a countable base, Mokobodzki has proved this equivalence, and P. A. Loeb and B. Walsh [1] have proved it without this restriction. In the Brelot space the convergence axiom is constructed by the boundedness of the upper envelope of the isotone family of harmonic functions on any one point of the space, and in the Bauer space it is done with the boundedness on the all points of any dense set of the space. Then, in the Bauer space the Harnack's inequality and the Harnack's principle of Bauer's type are gained as the result of the convergence axiom of Bauer's type, (H. Bauer [1]).

In this paper we shall append a condition for the absorption set of the harmonic space to the axiomatic system of harmonic space instead of the convergence axiom, and we shall show that the convergence axiom of Bauer's type and the Harnack's inequality and the Harnack's principle of Bauer's type are equivalent to each other.

The space X that we consider is a locally compact connected space with a countable base, and $\mathfrak{H}$ a sheaf on X, of real vector spaces of continuous functions, called *harmonic functions.* Let $\mathfrak{A}(X)$ (resp. $\mathfrak{A}_c(X)$) be a system of all non-empty open (resp. open relatively compact) sets in X.

A set $U \in \mathfrak{A}_c(X)$ is called *regular for* $\mathfrak{H}$, if the boundary ∂U of U is not empty and for any continuous real-valued function f on ∂U there exists a unique continuous extension to the closure $\bar{U}$ of U such that its restriction H_f^U to U is harmonic and non-negative if f is non-negative.

For any regular set U and for any point x of U, the map $f \to H_f^U(X)$ is a linear non-negative functional on the vector space $C(\partial U)$. Then the map $f \to H_f^U(x)$ defines a positive Radon measure on ∂U, and we denote it by μ_x^U

and call it *harmonic measure* relative to U and x.

We suppose that the sheaf $\mathfrak{H}$ satisfies the following **Axiom A$_1$**, **Axiom A$_2$** and **Axiom A$_3$**, then we say that the space X is a *preharmonic space.* Moreover if $\mathfrak{H}$ satisfies any one of **Axiom A$_C$**, **Axiom A$_I$** and **Axiom A$_P$**, defined afterward, we say that X is a *harmonic space.*

Axiom A$_1$: *The regular sets form a base of* X.

An extended real-valued function u defined on an open set U of X is called *hyperharmonic* in U, if u is a lower semicontinuous function in U, $-\infty < u(x) \leq +\infty$ for any $x \in U$, and for any regular set V such that $V \subset \overline{V} \subset U$ and for any continuous function f on ∂V such that $f \leq u$ on ∂V, $H_f^V \leq u$ on V.

A closed set A is called an *absorption set,* if for any point x of the set A and for any regular neighborhood V of x, the support $S_{\mu_x^V}$ of the harmonic measure μ_x^V is also contained in the set A.

Axiom A$_2$: *For any two points x and y of X $(x \neq y)$, there exist two hyperharmonic functions u and v such that $u(x)\, v(y) \neq u(y)\, v(x)$. And for any open relatively compact set U, there exists a harmonic function h such that h is strictly positive in U.*

Lemma 1. *There exists the space which satisfies* **Axiom A$_1$** *and* **Axiom A$_2$**, *but not the Bauer's convergence axiom* **A$_C$**.

Axiom A$_C$: *The upper enveloae $\sup\limits_n$ $\mathrm{h}_n(x)$ of any isotonne sequence (h_n) of harmonic functions h_n on an open set U is harmonic on U, if it is finite on any one dense subset of U.*

Proof. Let X be a $n+1$-dimensional real Euclidean vector space R^{n+1}, x_i be i-th coordinate of a point $x = (x_1, \ldots, x_i, \ldots, x_{n+1})$ of $R^{n+1}$and the Laplacian operator $\Delta_n = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_u^2}$. Let h be harmonic on $U \in \mathfrak{A}$ (R^{n+1}), if and only if h is continuous in U and for fixed x_{n+1}, the Laplace's equation $\Delta_n h = 0$ is fulfilled in U. All open balls $B_{a,r}$ of center $a = (a_1, \ldots, a_{n+1})$ and radius r form a regular base of $X = R^{n+1}$, and we denote $B_{a,r}$ by U. Then the harmonic measure μ_x^U is equal to the harmonic measure with respect to Laplace's equation $\Delta_n h = 0$ in case of subset Ix_{n+1} of R^{n+1} in which x_{n+1} is fixed, and the support $S_{\mu_x} U = \partial U \cap \{y = (y_1, \ldots, y_{n+1}) \mid y_{n+1} = x_{n+1}\}$. Therefore, in this space the smallest absorption set A_x containing x is $Ix_{n+1} = \{y \in R^{n+1} \mid y_{n+1} = x_{n+1}\}$. Since the set $\mathring{A}_x$ of all inner points of A_x is empty, Harnack's inequality and Harnack's principle of Bauer's type are fulfilled in this space.

Moreover, there exists the isotone sequence (h_n) of harmonic functions h_n

on U such that $\sup_n h_n(x)$ is finite on a some dense set D of U, but it is not always harmonic in U. Let D be set of all rational points of U, then $D \supset U$. For $h = \sup_n h_n(x)$ Laplace's equation $\Delta_n h = 0$ is fulfilled in $I_{x_{n+1}}$ of rational fixed x_{n+1} and semicontinuous in U but it may be discontinuous at any point in U as the upper envelope of isotone sequence of continuous functions. Then **Axiom** $\mathbf{A_C}$ is not satisfied in this space.

We define **Axiom** $\mathbf{A_I}$ and **Axiom** $\mathbf{A_P}$ (resp. axiom for Harnack's inequality and Harnack's principle) as followings.

Axiom $\mathbf{A_I}$: *For any point* x_0 of X *and for any compact set* K *of* A_{x_0}, *there exists a positive number* M_{K,x_0} *such that for any harmonic function* h, $\sup_{x \in K} h(x) \leq M_{K,x_0} h(x_0)$.

Axiom $\mathbf{A_P}$: *Let* (h_n) *be an isotone sequence of harmonic functions* h_n. *For any one point* x_0 *of* X, *if* $h(x_0) = \sup_n h_n(x_0)$ *is finite*, $h = \sup_n h_n$ *is harmonic on* $\mathring{A}_{x_0}$.

From **Lemma 1**, in order that **Axiom** $\mathbf{A_C}$ be equivalent to **Axiom** $\mathbf{A_I}$ and **Axiom** $\mathbf{A_P}$, we need some condition for the space and the sheaf. Then we take a following axiom with respect to the absorption sets.

Axiom $\mathbf{A_3}$: *For any point* x *of* X, *for any dense subset* D *of* X *and for any neighborhood* V *of* x, *there exists a point* x_0 *of* $D \cap V$ *such that* $x \in \mathring{A}_{x_0}$.

Lemma 2. *We suppose that* **Axiom** $\mathbf{A_1}$, **Axiom** $\mathbf{A_2}$ and **Axiom** $\mathbf{A_3}$ are *fulfilled, i. e. that* X *is preharmonic with respect to the sheaf* $\mathfrak{H}$. *Then,* **Axiom** $\mathbf{A_P}$. *implies* **Axiom** $\mathbf{A_C}$.

Proof. Let (h_n) be an isotone sequence of harmonic functions h_n on $U \in \mathfrak{A}(X)$ and D dense subset of U such that $\sup_n h_n(x)$ is finite for any x of D. Since for any point x of U the neighborhood V of x exists and $V \subset U$, **Axiom** $\mathbf{A_3}$ implies that there exists a point x_0 of $V \cap D$ such that $x \in \mathring{A}_{x_0}$. Owing **Axiom** $\mathbf{A_P}$, $\sup_n h_n$ is harmonic on $\mathring{A}_{x_0}$ and on any open neighborhood $W \subset \mathring{A}_{x_0}$ of x. By the definition of the sheaf, $\sup_n h_n$ is a harmonic function on U.

H. Bauer [1] has proved that **Axiom** $\mathbf{A_C}$ implies **Axiom** $\mathbf{A_I}$ and that **Axiom** $\mathbf{A_I}$ implies **Axiom** $\mathbf{A_P}$ with **Axiom** $\mathbf{A_1}$ and **Axiom** $\mathbf{A_2}$. Then we take the following theorem.

Theorem 1. *We suppose the condition of* **Lemma 2.** *Then,* **Axiom** $\mathbf{A_C}$,

Axiom $\mathrm{A_I}$ *and* **Axiom** $\mathrm{A_P}$ *are equivalent to each other.*

Conversely, if we assume that **Axiom** $\mathrm{A_1}$, **Axiom** $\mathrm{A_2}$ and **Axiom** $\mathrm{A_I}$ are satisfied, the type of harmonic space is defined by the assumption with respect to the absorption set *i. e.* the assumption of the set $E_x = \{y \in X \mid \mathring{A}_y \ni x\}$. We take weaker axiom $\mathrm{A_3'}$ for the absorption set than **Axiom** $\mathrm{A_3}$ and weaker convergence axiom $\mathrm{A_C'}$ than **Axiom** $\mathrm{A_C}$ as followings.

Axiom $\mathrm{A_3'}$: *For any point x of X and for any neighborhood V of x, there exists a point x_0 of V such that $x \in \mathring{A}_{x_0}$.*

Axiom $\mathrm{A_C'}$: *The upper envelope $h = \sup\limits_n h_n$ of isotone sequence (h_n) of harmonic functions h_n on $U \in \mathfrak{A}(X)$ is harmonic on U, if $\sup\limits_n h_n(x)$ is finite for all points x of U,* (H. Bauer [2], N. Boboc and D. Mustata [1], C. Constantinescu [1]).

Theorem 2. *We suppose that* **Axiom** $\mathrm{A_1}$, **Axiom** $\mathrm{A_2}$ *and* **Axiom** $\mathrm{A_3'}$ *are fulfilled. Then,* **Axiom** $\mathrm{A_I}$ *implies* **Axiom** $\mathrm{A_C'}$.

Proof. Owing **Axiom** $\mathrm{A_1}$ and **Axiom** $\mathrm{A_2}$, **Axiom** $\mathrm{A_I}$ implies **Axiom** $\mathrm{A_P}$. It is easily proved that **Axiom** $\mathrm{A_P}$ implies **Axiom** $\mathrm{A_C'}$ with the method similar to the proof of **Lemma 2.**

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