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On Locally Reductive Spaces and Tangent Algebras

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Introduction.

A linearly connected space is called locally reductive if both of the torsion tensor field and the curvature tensor field are parallel. A symmetric space with a canonical connection is such a space of vanishing torsion. On the other hand, every Lie group has a left invariant connection ((−)-connection) with parallel torsion and vanishing curvature. From this point of view, the geometry of locally reductive spaces has been studied by K. Nomizu in his paper [4] and he showed that a locally reductive space is determined, locally, by its torsion and curvature at a given point.

Observing the tangent algebras of these spaces, K. Yamaguti has introduced in [5] an algebraic system, called general Lie triple system, which is a generalization of both of Lie algebra and Lie triple system, and it has been studied, algebraically, by himself [6] and others.

In the present paper, we shall investigate a correspondence between certain locally reductive spaces and general Lie triple systems as their tangent algebras. In the case of connected, simply connected and complete locally reductive spaces, which can be regarded as homogeneous spaces (Theorem 2), a remarkable correspondence will be seen (Theorem 3). We shall also study certain subspaces of a locally reductive space and subsystems of its tangent algebra. Some results about symmetric spaces will be given as corollaries.

§ 1. General Lie triple systems.[1)]

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1) See [5] and [6].

A vector space $\mathfrak{M}$ (over a field K) with a bilinear multiplication denoted by $X \cdot Y$ and a trilinear multiplication denoted by $[X, Y, Z]$, for X, Y and Z in $\mathfrak{M}$, is called a *general Lie triple system* (*general Lts*) if the following axioms are satisfied :

$$X \cdot X = 0 \tag{1. 1}$$

$$D(X, X) = 0 \tag{1. 2}$$

$$\mathfrak{S}\{(X \cdot Y) \cdot Z + [X, Y, Z]\} = 0 \tag{1. 3}$$

$$\mathfrak{S}\, D(X \cdot Y, Z) = 0 \tag{1. 4}$$

$$D(X, Y)(Z \cdot W) = (D(X, Y)Z) \cdot W + Z \cdot (D(X, Y)W \tag{1. 5}$$

$$[D(X, Y), D(U, V)] = D(D)X, Y)U, V) + D(U, D(X, Y)V) \tag{1. 6}$$

for any X, Y, Z, U, V and W in $\mathfrak{M}$, where $D(X, Y)$ denotes the endomorphism $Z \longrightarrow [X, Y, Z]$ of $\mathfrak{M}$, $\mathfrak{S}$ denotes the cyclic sum with respect to the three elements X, Y and Z and the bracket in (1. 6) denotes the usual bracket of endomorphisms of a vector space. The axiom (1. 5) implies that, for X and Y in $\mathfrak{M}$, the endomorphism $D(X, Y)$ is a derivation of the binary multiplication of $\mathfrak{M}$, while the axiom (1. 6) implies that it is also a derivation of the ternary one. This endomorphism $D(X, Y)$ is called an *inner derivation* of $\mathfrak{M}$. In general, an endomorphism A of $\mathfrak{M}$ is called a *derivation* of the general Lts $\mathfrak{M}$ if the equations $A(X \cdot Y) = (AX) \cdot Y + X \cdot (AY)$ and $[A, D(X, Y)] = D(AX, Y) + D(X, AY)$ are valid for any X and Y in $\mathfrak{M}$.

A subsystem $\mathfrak{N}$ of a general Lts $\mathfrak{M}$ is a linear subspace of $\mathfrak{M}$ closed under the two kinds of multiplication, that is, $\mathfrak{N} \cdot \mathfrak{N}$ and $[\mathfrak{N}, \mathfrak{N}, \mathfrak{N}]$ are contained in $\mathfrak{N}$. A *homomorphism* of a general Lts into a general Lts is a linear mapping which preserves the binary and the ternary multiplications.

REMARKS. If the binary operation of a general Lie triple system vanishes identically, it is called a *Lie triple system.* On the other hand, if the ternary operation of a general Lts vanishes, it comes to a Lie algebra.

PROPOSITION 1.[1] *Let* $\mathfrak{M}$ *be a general Lie triple system and let* $\mathfrak{H}$ *be a Lie*

1) K. Nomizu [4] and K. Yamaguti [5].

subalgebra of $\mathfrak{gl}(\mathfrak{M})$ *generated by all derivations of* $\mathfrak{M}$, *where* $\mathfrak{gl}(\mathfrak{M})$ *denotes the Lie algebra of all endomorphisms of* $\mathfrak{M}$. *Then the direct sum* $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ *forms a Lie algebra with respect to the bracket operations defined as follows* ;

$$(1.7) \qquad [X, Y] = X\cdot Y + D(X, Y) \quad \textit{for } X, Y \in \mathfrak{M},$$

$$(1.8) \qquad [A, X] = -[X, A] = AX \quad \textit{for } A \in \mathfrak{H} \textit{ and } X \in \mathfrak{M}$$

and

$$(1.9) \qquad [A, B] = AB - BA \quad \textit{for } A, B \in \mathfrak{H}.$$

Proof. Since $D(X, Y)$ is a derivation of $\mathfrak{M}$, the bracket $[X, Y]$ is well defined for $X, Y \in \mathfrak{M}$. The Jacobi's identity is derived from the axioms (1. 3) $\sim$(1. 6) of general Lts. Thus we see that $\mathfrak{G}$ is a Lie algebra, $\mathfrak{H}$ is a subalgebra of $\mathfrak{G}$ and a relation $[\mathfrak{H}, \mathfrak{M}] \subset \mathfrak{M}$ holds.

REMARK. To imbed a general Lts $\mathfrak{M}$ into a Lie algebra, we may take a subalgebra generated by all *inner* derivations of $\mathfrak{M}$, instead of the above subalgebra $\mathfrak{H}$. Such an imbedding is called a *standard imbedding* of $\mathfrak{M}$.

§ 2. Locally reductive space and its tangent algebra.

Let (M, ∇) be a differentiable manifold with a linear connection. The torsion tensor field and the curvature tensor field are denoted by T and R respectively. They are defined by the following formulas :

$$(2.1) \qquad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$(2.2) \qquad R(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

for any vector fields X, Y and Z on M.

The following identitis hold on (M, ∇) :

$$(2.3) \qquad \mathfrak{S}\{R(X, Y)Z\} = \mathfrak{S}\{T(T(X, Y), Z) + \nabla_X T)(Y, Z)\}$$

$$(2.4) \qquad \mathfrak{S}\{(\nabla_X R)(Y, Z) - R(T(X, Y), Z)\} = 0$$

(2. 5) *for any* $(1, k)$-tensor field K on M,

$$([\nabla_X, \nabla_Y]K)(Z_1, Z_2, \ldots, Z_k) - (\nabla_{[X, Y]}K)(Z_1, \ldots, Z_k)$$
$$= R(X, Y)(K(Z_1, \ldots, Z_k)) - \sum_{i=1}^{k} K(Z_1, \ldots, R(X, Y)Z_i, \ldots, Z_k)$$

where $X, Y, Z, Z_1, \dots$ and Z_k are vector fields on M and $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. The identities (2. 3) aned (2. 4) are known as the *Bianchi's* identities.

If $\nabla T = 0$ and $\nabla R = 0$ hold on M, (M, ∇) is called a *locally reductive space*. The following result will be used later.

LEMMA. *Let (M, ∇) and (M', ∇') be locally reductive spaces. Assume M to be simply connected and ∇' to be complete. If a linear mapping F of a tangent space M_{x_0} at a given point $x_0 \in M$ into a tangent space $M'_{x'_0}$ at $x'_0 \in M'$ preserves the curvature and the torsion, then there exists a unique affine mapping, f, of (M, ∇) into (M', ∇') such that $f(x_0) = x'_0$ and the tangent linear mapping of f at x_0 coincides with F.*

For the proof see 0. Loos [3], Chapter II, § 4.

PROPOSITION 2.[1] *Let (M, ∇) be a locally reductive space and let $\mathfrak{M} = M_{x_0}$ be a tangent space at a given point $x_0 \in M$. If a bilinear mapping and a trilinear mapping on $\mathfrak{M}$ are defined by*

(2. 6) $$X \cdot Y = T_{x_0}(X, Y)$$

(2. 7) $$[X, Y, Z] = R_{x_0}(X, Y)Z \quad \text{for } X,\ Y \text{ and } Z \text{ in } \mathfrak{M},$$

then $\mathfrak{M}$ forms a general Lie triple system.

We shall call $\mathfrak{M}$ the *tangent general Lie triple system* of (M, ∇) at x_0.

Proof. In view of the definitions (2. 1) and (2. 2) of T and R, the axioms (1. 1) and (1. 2) are clearly satisfied. The remaining axioms follow from the Bianchi's identities (2. 3), (2, 4) and the identity (2. 5), under the assumptions ; $\nabla T = 0$ and $\nabla R = 0$.

A submanifold, N, of a linearly connected manifold (M, ∇) is called *auto-parallel*[2] if, for each tangent vector $X \in M_x$ at any point $x \in N$ and for each curve τ in N starting from x, the parallel displacement of X along τ (with respect to ∇) yields a vector tangent to N. An auto-parallel manifold has a linear connection induced naturally from ∇ and the torsion tensor field, the curvature tensor field and the covariant derivatives of restricted tensors in N

1) loc. cit.
2) For the details, see [2], Chapter VII (Vol. II).

are the restriction of those in M respectively. In particular, an auto-parallel submanifold of a locally reductive space is itself locally reductive with respect to the induced connection. If M has the zero torsion, a submanifold of M is auto-parallel if and only if it is totally geodesic.

COROLLARY. *Let* (M, ∇) *be a connected locally reductive space and* N *be an auto-parallel subspace of* M. *Then, at each point* x_0 *of* N, *the tangent general Lie triple system,* $\mathfrak{N}$, *of* N *is a subsystem of the tangent general Lie triple system,* $\mathfrak{M}$, *of* M *at* x_0.

Proof. Since the torsion tensor field and the curvature tensor field of N is the restriction to N of those in M, the multiplications in $\mathfrak{N}$ must be the restriction of (2. 6) and (2. 7) to $\mathfrak{N}$.

PROPOSITION 3. *In a connected locally reductive space* (M, ∇), *tangent general Lie triple systems* $\mathfrak{M} = M_{x_0}$ *and* $\mathfrak{M}' = M_{x_0'}$ *at any two points* x_0 *and* x_0' *of* M *are isomorphic.*

Proof. Let τ be a piecewise differentiable curve in M joining x_0 to x_0' and denote by τ^* the parallel displacement of tangent vectors along the curve τ. Then τ^* is a linear isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$. Moreover, since $\nabla T = 0$ and $\nabla R = 0$, $\tau^*(T_{x_0}(X, Y)) = T_{x_0'}(\tau^*(X), \tau^*(Y))$ and $\tau^*(R_{x_0}(X, Y)Z) = R_{x_0'}(\tau^*(X), \tau^*(Y))\tau^*(Z)$ are valid for any X, Y and Z in $\mathfrak{M}$. Hence, by the definitions (2. 6) and (2. 7), it is seen that the linear mapping τ^* is an isomorphism of general Lts.

From the above proof we have :

COROLLARY. *The holonomy group of a connected locally reductive space is a subgroup of the group of automorphisms of the tangent general Lie triple system of* M.

§ 3. Reductive homogeneous spaces.

Let $M = G/H$ be a homogeneous space of a connected Lie group G by a closed subgroup H. Suppose that the Lie algebra $\mathfrak{G}$ of G is decomposed into a direct sum $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ of a Lie algebra $\mathfrak{H}$ of H and a subspace $\mathfrak{M}$ of $\mathfrak{G}$. If, in addition, $ad(H)\mathfrak{M} \subset \mathfrak{M}$ holds, $M = G/H$ is called a *reductive homogeneous* *space*. A reductive homogeneous space G/H with a fixed direct sum decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ has a G-invariant connection, called the *canonical connection* of G/H, which is characterized by the following property :[1)]

(P) For any 1-parameter subgroup $g(t)$ of G generated by an element of $\mathfrak{M}$ and for a curve $c(t) = g(t)x_0$ in M starting from the origin $x_0 = p(H)$ of G/H, the parallel displacement of tangent vectors at x_0 along the curve $c(t)$ is the same as the tangent linear mapping at x_0 of the diffeomorphism $g(t)$ actig on M.

A reductive homogeneous space with the canonical connection is locally reductive and, by identifying the subspace $\mathfrak{M}$ with the tangent space at the origin x_0, the torsion tensor and the curvature tensor of the canonical connection are evaluated at x_0 as follows:[2)]

$$(3.\ 1) \qquad T_{x_0}(X, Y) = -[X, X]_{\mathfrak{M}}$$

$$(3.\ 2) \qquad R_{x_0}(X, Y) = -[X, Y]_{\mathfrak{H}} \quad \text{for } X, Y \in \mathfrak{M}$$

where $[\ ]_{\mathfrak{M}}$ and $[\ ]_{\mathfrak{H}}$ means the $\mathfrak{M}$-component and the $\mathfrak{H}$-component of the bracket in $\mathfrak{G}$ respectively.

From (3. 1) and (3. 2), if we apply Proposition 2 to a reductive homogeneous space, we have

PROPOSITION 4. *Let $M = G/H$ be a reductive homogeneous space with a fixed direct sum decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$. Then the subspace $\mathfrak{M}$ forms a general Lie triple system with multiplications as follows*

$$(3.\ 3) \qquad X \cdot Y = -[X, Y]_{\mathfrak{M}}$$

$$(3.\ 4) \qquad [X, Y, Z] = -[[X, Y]_{\mathfrak{H}}, Z] \quad \textit{for } X, Y, Z \in \mathfrak{M}.$$

The converse of the above proposition is also true. That is ;

THEOREM 1. *Let $\mathfrak{M}$ be a real finite dimensional general Lie triple system. There exists a simply connected reductive homogeneous space $M = G/H$ with a direct sum decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$, where $\mathfrak{G}$ and $\mathfrak{H}$ are Lie algebras of G and H respectively.*

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1) See also [4], §10.
2) loc. cit.

Proof. Let $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ be an imbedding of a given general Lts $\mathfrak{M}$ into a Lie algebra $\mathfrak{G}$ constructed in Proposition 1. Since $\mathfrak{H}$ is a subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{M})$ of endomorphisms of $\mathfrak{M}$, it must be of finite dimension. Let G be a simply connected Lie group with Lie algebra $\mathfrak{G}$ and H be a connected subgroup of G generated by $\mathfrak{H}$. The group H is closed because it is the connected component of the group of automorphisms of the general Lts $\mathfrak{M}$. Thus we have a homogeneous space $M = G/H$, which is also simply connected. As remarked in the proof of Proposition 1, the direct sum $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ satisfies the condition $[\mathfrak{H}, \mathfrak{M}] \subset \mathfrak{M}$. Therefore, a reductive homogeneous space, $M = G/H$, with the desired properties is obtained.

THEOREM 2. *A connected, simply connected, complete and locally reductive space* (M, ∇) *is a reductive homogeneous space with the canonical connection.*

Proof. Let G_0 be the connected component of the group of affine transformations of (M, ∇). G_0 operates transitively on M. In fact, given any two points x and y in M, let $\tau(t)$ be a piecewise differentiable curve joining $x = \tau(t_0)$ to $y = \tau(t_1)$ and let $F(t)$ be a linear isomorphism of the tangent space M_x onto $M_{\tau(t)}$ defined by a parallel displacement of vectors along τ. Since M is simply connected and complete, by Lemma in § 2, $F(t)$ can be extended to a unique affine transformation, $f(t)$, of M for each value of t. Hence, there exists an affine transformation, $f(t_1)$, in G_0 which sends the point x to y.

Let G be a universal covering group of G_0. The action of an element of G is well defined by the action of its image in G_0 under the covering map. Given a fixed point x_0 in M, let H be an isotropy subgroup of G at x_0. Then M can be regarded as a homogeneous space G/H. For any tangent vector X at x_0, let $\gamma^*(t)$ be the parallel displacement of M_{x_0} onto $M_{\gamma(t)}$ along a geodesic $\gamma(t)$ tangent to X at $x_0 = \gamma(0)$. By extending each $\gamma^*(t)$ to an affine transformation on M, we have a 1-parameter subgroup $g(t)$ of G. The mapping of M_{x_0} into the Lie algebra $\mathfrak{G}$ of G which sends X to $\left(\frac{d}{dt}g(t)\right)_{t=0}$ is an injective linear map whose image we denote by $\mathfrak{M}$. Since an element of H is an affine transformation of M leaving x_0 fixed, it sends any geodesic starting from x_0 to a geodesic from x_0. Moreover, it commutes with the parallel displacements along these geodesics. Therefore we have $ad(H)\,\mathfrak{M} \subset \mathfrak{M}$. Thus $M = G/H$ is a reductive homogeneous space with a direct sum decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$. The canonical connection of G/H is a G-invariant connection characterized by the property (P) mentioned in §3. Since ∇ has this property, the uniqueness of the canonical connection implies that (M, ∇) is a reductive homogeneous space G/H with ∇ as the canonical connection.

COROLLARY.[1] *A connected, simply connected and complete locally affine symmetric space is globaly symmetric.*

Proof. A locally affine symmetric space is, by definition, a linearly connected space satisfying the conditions; $T = 0$ and $\nabla R = 0$. Thus, the result of the corollary follows easily from Theorem 2.

THEOREM 3. *The category of connected, simply connected, complete and locally reductive space with base points is equivalent to the category of real finite dimensional general Lie triple systems.*

Proof. Let $\mathscr{R}$ denote the category of connected, simply connected, complete and locally reductive spaces, (M, ∇, x), as *objects* and affine mappings as *morphisms*, and let $\mathscr{L}$ denote the category of real finite dimensional general Lts' and general Lts-homomorphisms. We shall construct covariant functors; Φ, from $\mathscr{R}$ to $\mathscr{L}$, and Ψ, from $\mathscr{L}$ to $\mathscr{R}$, respectively, such that the composite functors $\Psi\circ\Phi$ and $\Phi\circ\Psi$ are equivalences of $\mathscr{R}$ and $\mathscr{L}$ respectively.

To a given object (M, ∇, x) of $\mathscr{R}$, we assign its tangent general Lts, $\mathfrak{M} = M_x$, at $x \in M$. Let f be an affine mapping of (M, ∇) into (M', ∇') which sends x to a point x' in M. Then, since f sends the torsion tensor field, T, and the curvature tensor field, R, on M to the torsion tensor field, T', and the curvature tensor field, R', on M$'$ respectively, its tangent linear mapping, f^*, at x satisfies the following relations:

$$(3.5) \qquad f^*(T_x(X, Y)) = T'_{x'}(f^*(X), f^*(Y))$$

$$(3.6) \qquad f^*\circ R_x(X, Y) = R'_{x'}(f^*(X), f^*(Y))\circ f^*$$

for X and Y in M_x. Hence, by (2.6) and (2.7), f^* is a general Lts-homomor-

1) O. Loos, [3], Chapter II, §4.

phism of the tangent general Lts $\mathfrak{M} = M_x$ at x into the tangent general Lts $\mathfrak{M}' = M'_{x'}$ at x'. Let Φ be this assignment of objects and morphisms of $\mathscr{R}$ to those of $\mathscr{L}$. Then it is easily seen that Φ is a covariant functor from $\mathscr{R}$ to $\mathscr{L}$.

Next, to any finite dimensional general Lie triple system $\mathfrak{M}$, we assign a connected and simply connected reductive homogeneous space ($M = G/H$, ∇, x_0) with a canonical connection, ∇, and its origin, x_0, which is given in Theorem 1. By Lemma in § 2, a homomorphism, F, of a general Lts $\mathfrak{M}$ to another general Lts $\mathfrak{M}'$ can be extended to a global affine mapping, f, of corresponding reductive homogeneous spaces $M = G/H$ and $M' = G'/H'$. Thus we can define a covariant functor, Ψ, from $\mathscr{L}$ to $\mathscr{R}$ which assigns each object, $\mathfrak{M}$, in $\mathscr{L}$ to a reductive homogeneous space, G/H, and each morphism, F, in $\mathscr{L}$ to an affine mapping, f, respectively as above.

An object, (M, ∇, x), in $\mathscr{R}$ and its image under the composite functor $\Psi \circ \Phi$ has the same tangent general Lts $\mathfrak{M} = \Phi(M, \nabla, x)$. Hence, as a global extension of the identity isomorphism of $\mathfrak{M}$, we have an affine isomorphism from (M, ∇, x) to $\Psi \circ \Phi (M, \nabla, x) = (M', \nabla', x')$. On the other hand, since the tangent general Lts of given reductive homogeneous space is the general Lts, $\mathfrak{M}$, obtained from the associated direct sum decomposition, $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$, we see that the functor $\Phi \circ \Psi$ of $\mathscr{L}$ into itself is an identity of $\mathscr{L}$. Thus the proof is completed.

By restricting ourselves to the case of locally symmetric spaces, we have ;

COROLLARY.[1] *The category of connected, simply connected and complete locally symmetric space with the base points is equivalent to the category of real finite dimensional Lie triple systems.*

§ 4. Auto-parallel subspaces and their tangent algebras.

Now, we shall turn our attention to certain subspaces of a locally reductive space.

THEOREM 4. *Let $M = G/H$ be a reductive homogeneous space with the origine*

1) O. Loos, loc. cit.

x_0 and let $\mathfrak{M} = M_{x_0}$ be its tangent general Lie triple system at x_0. For any given subsystem, $\mathfrak{N}$, of $\mathfrak{M}$, there exists an auto-parallel subspace, N, of M containing x_0 and such that the tangent general Lie triple system of N at x_0 coincides with $\mathfrak{N}$.

REMARK. The converse of this theorem has been proved earlier in § 2 as a corollary to Proposition 2.

Proof. Let $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ be a fixed direct sum decomposition of the Lie algebra, $\mathfrak{G}$, of G associated with the reductive homogeneous space $M = G/H$. Denote by $D(\mathfrak{N}, \mathfrak{N})$ the subalgebra of the inner derivation algebra, $D(\mathfrak{M}, \mathfrak{M})$, of $\mathfrak{M}$ generated by $[\mathfrak{N}, \mathfrak{N}]_{\mathfrak{H}}$ and set $\mathfrak{G}' = \mathfrak{N} + D(\mathfrak{N}, \mathfrak{N})$. Then $\mathfrak{G}'$ is a Lie subalgebra of $\mathfrak{G}$. In fact, for any X and Y in $\mathfrak{N}$, $[X, Y] = X \cdot Y + D(X, Y)$[1] belongs to $\mathfrak{G}'$. Let G' be a connected subgroup of G with Lie algebra $\mathfrak{G}'$ and set $H' = H \cap G'$. Then $N' = G'/H'$ is a reductive homogeneous space with the Lie algebra decomposition $\mathfrak{G}' = \mathfrak{N} + D(\mathfrak{N}, \mathfrak{N})$.

We shall define a mapping, f, of N' into M which is an affine imbedding with respect to the respective canonical connections. For each point $x' = g'H'$ we define $f(x') = g'H$. Let i denote the inclusion mapping of G' into G and let p (resp. p') denote the projection of G (resp. G') onto $M = G/H$ (resp. $N' = G'/H'$). Then the following diagram

$$\begin{array}{ccc} G' & \xrightarrow{\;i\;} & G \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ N' = G'/H' & \xrightarrow{\;f\;} & M = G/H \end{array}$$

is commutative and hence we see that both of the mapping f and its tangent linear mapping f^* at the origin are injective. Therefore, it is seen that the image $N = f(N')$ of f is a submanifold of M, the origin x_0' of N' being sent to the origine x_0 of M, and that the tangent space N_{x_0} at x_0 can be identified with the subspace $\mathfrak{N}$. With respect to the canonical connection $\varDelta$ of M, any geodesic starting from x_0 and tangent to a vector, X, in $N_X = \mathfrak{N}$ is of the form $\tau(t) = exp\ tX(x_0)$. Since $exp\ tX$, for $X \in \mathfrak{N}$, is contained in G', this

1) See (1. 7)

geodesic is a curve in N, which is an image of a geodesic $exp\ tX(x_0')$ in N' under the imbedding f. Thus we see that N is totally geodesic in M. By means of the map f, we can induce a linear connection ∇' on N from the canonical connection of $N'=G'/H'$, which has the property (P) mentioned in § 3 for the group G' and the direct sum decomposition $\mathfrak{G}'=\mathfrak{N}+D(\mathfrak{N},\mathfrak{N})$. From this fact, we can easily seen that any ∇'-geodesic in N is also a ∇-geodesic and that a ∇'-parallelism of tangent vectors of N along a geodesic in N coincides with a ∇-parallelism in M. Hence, for any vector fields X and Y on N, we have $\nabla_X Y=\nabla'_X Y$. Therefore, we can conclude that the submanifold N is an auto-parallel subspace of M with ∇' as an induced linear connection. Moreover, the multiplications of the tangent general Lts, N_{x_0}, of (N, ∇') is the restriction of ones in $\mathfrak{M}$. Thus N_{x_0} can be identified with $\mathfrak{N}$ as a subsystem of the general Lts $\mathfrak{M}$.

PROPOSITION 5. *Let (M, ∇) be a connected and simply connected locally reductive space and let $\mathfrak{M}$ be a tangent general Lie triple system of M at a given point x_0. If $\mathfrak{N}$ is an ideal of $\mathfrak{M}$, there exists an auto-parallel submanifold of M passing through any point x in M, whose tangent general Lie triple system at every point is isomorphic to $\mathfrak{N}$.*

Proof. We shall give here an outline of the proof by using the results obtained in [1]. For the details, see Proposition 1 and 2 in [1].

The definition of the multiplications in the tangent general Lts $\mathfrak{M}=M_{x_0}$ at x_0 implies that $T_{x_0}(\mathfrak{N},\mathfrak{N})$ is contained in $\mathfrak{N}$ and that $R_{x_0}(\mathfrak{M},\mathfrak{M})$ leaves the subspace $\mathfrak{N}$ invariant (see (2. 6) and (2.7)). Since M is simply connected, we see that the holonomy group at x_0, whose Lie algebra is the inner derivation algebra of $\mathfrak{M}$, also leaves the subspace $\mathfrak{N}$ invariant. Therefore, we are able to construct a parallel differential system, Σ, on M by means of the parallel displacements of the subspace $\mathfrak{N}$ along curves from x_0 to arbitrary points in M. From the above conditions, we see that Σ is completely integrable. Hence, there exists a connected integral manifold, N, of Σ passing through an arbitrarily given point in M. It is also seen that N is auto-parallel. Since a parallel displacement along any curve in M induces an isomorphism of tangent general Lts', by the definition of Σ, submanifold N has, as a locally reductive subspace of M, a tangent general Lts which is isomorphic to $\mathfrak{N}$.

THEOREM 5. *Let (M, ∇) be a connected, simply connected and complete locally reductive space. If a tangent general Lie triple system, $\mathfrak{M}$, of M is semi-simple, M can be decomposed into an affine product of locally reductive subspaces whose tangent general Lts' are ideals appearing in a direct sum decomposition*

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots + \mathfrak{M}_k. \tag{4. 1}$$

Proof. Since each $\mathfrak{M}_i$ is an ideal of $\mathfrak{M}$, it is valid that $[\mathfrak{M}, \mathfrak{M}, \mathfrak{M}_i] \subset \mathfrak{M}_i$, $\mathfrak{M}_i \cdot \mathfrak{M}_i \subset \mathfrak{M}_i$ and $\mathfrak{M}_i \cdot \mathfrak{M}_j = 0$ for $i \neq j$. Hence from the axiom (1. 3) we have $[\mathfrak{M}_i, \mathfrak{M}_j, \mathfrak{M}] = 0$ for $i \neq j$. Then on account of the formulas (2. 6) and (2. 7), the torsion tensor and the curvature tensor must satisfy the following conditions at x_0;

$$R_{x_0}(\mathfrak{M}, \mathfrak{M})\mathfrak{M}_i \subset \mathfrak{M}_i \tag{4. 2}$$

$$T_{x_0}(\mathfrak{M}_i, \mathfrak{M}_i) \subset \mathfrak{M}_i \tag{4. 3}$$

$$R_{x_0}(\mathfrak{M}_i, \mathfrak{M}_j) = 0 \quad \text{for } i \neq j, \tag{4. 4}$$

These conditions imply the conclusion of the theorem according to the following lemma obtained in our previous paper.

LEMMA.[1] *Let (M, ∇) be a connected locally reductive space. Suppose that, at a point x_0 of M, the following conditions are satisfied:*

(1) *The tangent space M_{x_0} is decomposed into a direct sum of subspaces S'_{x_0} and S''_{x_0} each of which is invariant under the holonomy group at x_0.*

(2) *$R_{x_0}(X, Y) = 0$ for $X \in S'_{x_0}$ and $Y \in S''_{x_0}$.*

(3) *The torsion tensor T is completely inducible at x_0, i. e. $T_{x_0}(S'_{x_0}, S'_{x_0}) \subset S'_{x_0}$, $T_{x_0}(S''_{x_0}, S''_{x_0}) \subset S''_{x_0}$ and $T_{x_0}(S'_{x_0}, S''_{x_0}) = 0$. Then the poiut x_0 has a neighborhood which is locally affine isomorphic to an affine product of two locally reductive subspaces M' and M'' tangent to S'_{x_0} and S''_{x_0} at x_0 respectively. Moreover, if M is simply connected and complete, it is globally affine isomorphic to the above affine product.*

From Theorem 5, we have

1) See corollary to Theorem 1 and Theorem 2 in [1].

COROLLARY. *Let M be a simply connected symmetric space. If the tangent Lie triple system,* $\mathfrak{M}$, *of M is semisimple, then M is decomposed into a direct product of symmetric subspaces tangent to ideals in a direct sum decomposition of* $\mathfrak{M}$.

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