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On Some Quasigroups of Algebraic Models of Symmetric Spaces

Michihiko KIKKAWA

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In the previous paper [4], we have studied a characterization of linearly connected manifolds with parallel torsion and curvature by their tangent algebras. Lie algebra Lie group correspondence and Lie triple system symmetric space correspondence are found there in the special cases.

On the other hand, as a generalization of Lie group with $(-)$-connection of Cartan, we have a binary-systematic characterization [1] of linearly connected manifold in our minds. From such a view point, we shall try to present in this note a quasigroup, called a *symmetric loop*, as an algebraic model of symmetric space. In [5], O. Loos has introduced an axiomatic binary system in symmetric space and defined the symmetric space by means of the multiplication. We were motivated by this work to construct the symmetric loop.

At the last part of the present note, the family of all left translations of the symmetric loop will be observed on the lines of Lie triple family of transformations of T. Nôno [6].

1. Symmetric loops and quasigroups of reflection.

DEFINITION. [2] A *loop* (G.) is a quasigroup with the identity element e and with the multiplication denoted by $x.y$. A loop is said to be *power associative* (resp. *di-associative*) if every element (resp. every couple of elements) generates a subgroup. A power associative loop G is said to be *left di-associative* if, for every element $a \in G$, the left translation $f_a : x \longrightarrow a.x$ has the property $f_a \circ f_a = f_{(a^2)}$ and $(f_a)^{-1} = f_{(a^{-1})}$. A *right di-associative* loop is defined similarly. A loop is left and right di-associative if it is di-associative.

DEFINITION. A loop (G.) will be called a *symmetric loop* if it has the following properties:

(A. 1) G is left di-associative;

(A. 2) $(x.y)^{-1} = x^{-1}.y^{-1}$

(A. 3) $x.((y.y).z) = (x.y).(x.y).(x^{-1}.z)$

1) See [2] and [3].

2) See R. H. Bruck [1].

(A. 4) the mapping $x \longrightarrow x.x$ is a bijection of G onto itself.

REMARK. Under the assumption (A. 1), the axioms (A. 2) and (A. 3) can be replaced by a single axiom ;

(A. 2′) $x.((y.y).z^{-1}) = (x.y).(x.y)\ (x.z)^{-1}$.

ADDED IN PROOF. We found the fact that (A. 2) follows from (A. 1) and (A. 3).

PROPOSITION 1. *If a symmetric loop is a group, then it is an Abelian group.*

DEFINITION. A quasigroup (G_*) with the following properties will be called a *quasigroup of reflection*, whose multiplication will be denoted by x_*y :

(B. 1) $x_*x = x$

(B. 2) $x_*(x_*y) = y$

(B. 3) $x_*(y_*z) = (x_*y)_*(x_*z)$.

O. Loos ([5]) has defined a symmetric space as a differentiable manifold with a differentiable multiplication satisfying the axioms (B. 1), (B. 2), (B. 3) and

(B. 4) every element x has a neighborhood U such that $x_*y = y$ implies $y = x$ for all y in U.

Thus we have

PROPOSITION 2. *If a quasigroup of reflection, G, is a manifold and if the multiplication is differentiable, then G is a symmetric space.*

In the following few Theorems, we shall show the equivalency between the category of pointed quasigroups of reflection and the category of symmetric loops.

THEOREM 1. *Let (G_*) be a quasigroup of reflection whose multiplication is denoted by x_*y. Let e be an arbitrarily fixed element of G and denote $\bar{x}$ an element defined by the relation $\bar{x}_*e = x$ for every element x of G. A binary system $(G.)$ defined by $x.y = \bar{x}_*(e_*y)$ is a symmetric loop with the identity e.*

Proof of Theorem 1. The fact that e is the left and right identity is easily seen by (B. 1), (B. 2) and

LEMMA 1. $\bar{e} = e$.

An element e_*x which will be denoted hereafter by x^{-1} is an inverse element of x in the binary system $(G.)$ and the relations $x^{-1}.(x.y) = y$ and $(x.y)^{-1} = x^{-1}.\ y^{-1}$ are valid. These facts are proved by the following two lemmas :

LEMMA 2. $\overline{(e_*x)} = e_*\bar{x}$, *or equivalently*, $\overline{(x^{-1})} = (\bar{x})^{-1}$.

LEMMA 3. $\bar{x}_*x = e$.

Using the above results we can show that ;

LEMMA 4. $\bar{x}.\bar{x} = x$.

This lemma implies that the mapping $x \longrightarrow x.x$ is a permutation of G. It is also seen that the quasigroup $(G.)$ is left di-associative. Finally, the formula (A. 3) is proved by using the following relations ;

$x.(y_*z) = (x.y)_*(x.z)$,

$(x_*y)^{-1} = x^{-1}{}_*y^{-1}$

and

LEMMA 5. $x_*(y_*x) = (x_*y)_*x$.

Q. E. D.

The followings are also proved easily :

LEMMA 6. $\overline{\bar{x}_*\bar{x}^{-1}} = x$.

LEMMA 7. $(\bar{x}_*\bar{y})_*x = \bar{x}_*y$.

PROPOSITION 3. *Let* (G_*) *be a quasigroup of reflection. If* $(G.)$ *and* $(G\circ)$ *are the symmetric loops associated with different base points (identity elements)* e *and* e' *respectively, as are given in Theorem* 1. *Then the mapping of* G *onto itself defined by the left translation by* e' *in* $(G.)$ *is an isomorphism of* $(G.)$ *onto* $(G\circ)$, i. e. *the following relation holds* :

$$(e'.x)\circ(e'.y) = e'.(x.y).$$

Proof. For every element x, denote $\bar{x}$ and $\widetilde{x}$ the elements defined by the relations $\bar{x}_*e = x$ and $\widetilde{x}_*e' = x$ respectively. By the definitions of the multiplications in $(G.)$ and $(G\circ)$, each hand side of the formula in the Proposition is represented by means of the $_*$-multiplication, respectively, as follows ;

$$(e'.x)\circ(e'.y) = \widetilde{\overline{e'}_*(e_*x)}_*[e'_*(\overline{e'}_*(e_*y))],$$
$$e'.(x.y) = [e'_*(\overline{e'}_*\bar{x})]_*[e'_*(\overline{e'}_*(e_*y))].$$

Thus, to prove the Proposition, it is sufficient to show that

$$(e'_*(\overline{e'}_*\bar{x}) = \widetilde{\overline{e'}_*(e_*x)}.$$

Using Lemma 5, we see that this is equivalent to the relation

$$(e'_*\bar{x})_*\overline{e'} = \overline{e'}_*x,$$

which is always valid (Lemma 7).

Q. E. D.

THEOREM 2. *Let* $(G.)$ *be a symmetric loop with the identity* e. *A binary system* (G_*) *defined by* $x_*y = (x.x).y^{-1}$ *is a quasigroup of reflection and the associated symmetric loop with the identity* e *coincides with the original symmetric loop* $(G.)$.

Proof. It is easily seen that (G_*) is a quasigroup. The axiom (A. 1) implies (B. 1). Also (B. 2) follows from (A. 1) and (A. 2). Finally, (B. 3) is proved by the following

LEMMA 8. $x.(y_*z) = (x.y)_*(x.z)$,
$(x_*y)^{-1} = x^{-1}{}_*y^{-1}$.

This Lemma can be proved by using (A. 3).

The second part of the theorem is easily seen by calculating the product of x and y.

Q. E. D.

THEOREM 3. *Let* (G_*) *be a quasigroup of reflection and let* $(G.)$ *be a symmetric loop associated with the identity element* $e \in G$. *The quasigroup of reflection obtained from the symmetric loop* $(G.)$ *by means of Theorem* 2 *coincides with the original quasigroup of reflection* (G_*).

Proof. For any two elements x, $y \in G$, their product in the new quasigroup of reflection is represented by means of the original $_*$-product as follows :

$$x.x.y^{-1} = (\overline{x}_*e)_*(e_*(e_*y)).$$

It is easily seen that the right hand side of this formula is equal to x_*y.

Q. E. D.

As we have already remarked, O. Loos [5] characterizes a symmetric space as a differentiable manifold with a germ of local quasigroups of reflection. From the above Theorems we recognize that a symmetric space can be characterized also as a differentiable manifold with a germ of local symmetric loops. 3) However, this note is concerned only with purely algebraic aspects of symmetric spaces.

2. Triple family of transformations of symmetric loops.

DEFINITION. Let G be a set. A family $\mathscr{T}(G) = \{f_x\}_{x \in I}$ of mappings of G onto itself is called a *triple famly of transformations* 4) of G if it satisfies the following axioms:

(C. 1) every $f_x \in \mathscr{T}$ is a bijection of G onto itself;

(C. 2) there exists a mapping $\alpha: I \times I \longrightarrow I$ such that $f_x \circ f_y \circ f_x = f_{\alpha(x,y)}$, for $x, y \in G$;

(C. 3) there exists a mapping $\beta: I \longrightarrow I$ such that $f_x^{-1} = f_{\beta(x)}$, for $x \in G$;

(C. 4) the identity mapping Id_G of G belongs to $\mathscr{T}$.

REMARK. The concept of triple family of transformations have been introduced by T. Nôno in [6], where the set G is a differentiable manifold, $\mathscr{T}$ is a family of diffeomorphisms and the set of indices I is an open neighborhood of 0 in r-dimensional real numerical space R^r. T. Nôno has shown that a system $\mathbb{T}$ of infinitesimal transformations on a manifold G generates a triple family of local transformations of G if and only if $\mathbb{T}$ is a Lie triple system, which is known 5) as a tangent algebra of a symmetric space,

Now we shall present a Theorem which may be expected in the above remark.

THEOREM 4. *Let $(G.)$ be a symmetric loop. A family $\mathscr{T} = \{f_x\}_{x \in G}$ of all left translations of G is a triple family of transformations of G.*

Proof. For any elements x, y and z in G, the element $f_x \circ f_y \circ f_x(z)$ is represented as follows, by using the axioms (A. 1), (A. 2) and (A. 3).

$$\begin{aligned} f_x \circ f_y \circ f_x(z) &= x.(y.(x.z)) \\ &= (x.\bar{y}).(x.\bar{y}).(x.(x^{-1}.z^{-1}))^{-1} \\ &= (x.\bar{y}).(x.\bar{y}).z, \end{aligned}$$

where $\bar{y}.\bar{y} = y$. Hence we have

$$\text{(C. 2)} \quad f_x \circ f_y \circ f_x = f_{\alpha(x,y)},$$

where $\alpha(x, y) = (x.\bar{y}).(x.\bar{y})$ for any $x, y \in G$.

The remaining axioms are clearly satisfied since $(G.)$ is a left di-associative quasigroup. Q. E. D.

Department of Mathematics
Shimane University

3) In [2], we have considered certain local loops in linearly connected manifolds. In the application of the above loop theory to [2], it should be remarked that the order of the product is interchanged. Therefore, for instance, the word 'left di-assoicative' of Theorem 2 in [2] should be read 'right di-associative' in the present sense.

4) T. N no [6].

5) See, for instance, [4] or [5].

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