Mem. Fac. Lit. & Sci., Shimane Univ., Nat. Sci., **6**, pp. 1–7, March 10, 1973

Topology on the Dual Semigroups of Locally Compact Semigroups.

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(Received September 30, 1972)

In this paper we give some sufficient conditions for the dual semigroup of a commutative locally compact semigroup to be locally compact, and some relevant matters.

§ 1. Preliminaries.

In this section, we shall present some definitions and a lemma.

Definition 1. A *topological semigroup* is an ordered triple consisting of a nonempty set S, a function $(x, y) \to xy$ from $S \times S$ into S and a Hausdorff topology on S such that

(a) $x(yz) = (xy)z$ for all $x, y, z \in S$
(b) $(x,y) \to xy$ is continuous.

If a topological semigroup S is (locally) compact as a topological space, S is called a *(locally) compact semigroup*.

Definition 2. Let S be a commutative topological semigroup. By a *semicharacter* of S we mean a continuous homomorphism of S into the complex unit disc D with the ordinary multiplication, i.e., a complex-valued continuous function χ on S such that

(a) $|\chi(x)| \leqq 1$ for all $x \in S$
(b) $\chi(xy) = \chi(x)\chi(y)$ for $x, y \in S$

The set $\hat{S}$ of all semicharacters of S constitutes a commutative semigroup under the ordinary pointwise product $\chi\psi(x) = \chi(x)\psi(x)$.

The unit semicharacter χ^1 (i. e. $\chi^1(x) \equiv 1$) and the zero semicharacter χ^0 (i. e. $\chi^0(x) \equiv 0$) are the identity and the zero of $\hat{S}$, respectively. They are called *trivial semicharacters*. Throughout this paper we shall use the notations χ^1 and χ^0 as the unit and zero semicharacter, respectively.

Definition 3. Let S be a commutative topological semigroup, and $\chi \in \hat{S}$. Let C be a compact subset of S, $\varepsilon > 0$, and define

$$U(C, \varepsilon, \chi) = \{\psi \in \hat{S} : |\psi(x) - \chi(x)| < \varepsilon \text{ for all } x \in C\}$$

We now define a topology on $\hat{S}$ by requiring that all the sets $U(C, \varepsilon, \chi)$ be an open basis. We call this topology the *compact open topology*.

Then it is clear that $\hat{S}$ with this topology is a commutative topological semigroup. $\hat{S}$ is called the *dual semigroup* of S.

The following lemma is useful in the later sections.

Lemma. Let X be a topological space and Y a uniform space. Let $C(X, Y)$ be the family of all continuous mappings of X into Y and F a subfamily of $C(X, Y)$. Consider the following two conditions,

(a) *F is equicontinuous and $\overline{F(x)}$ is compact for each $x \in X$ where $\overline{F(x)}$ is closure of $F(x)$, and*

(b) *the closure of F in $C(X, Y)$ under the compact open topology is compact.*

Then (a) *implies* (b).

For this, see e. g. [1] 0. 4. 11.

§2. The dual semigroups of compact semigroups.

In this section, we give a sufficient condition for the dual semigroup of a commutative compact semigroup to be locally compact. The following Theorem has been proved by A. Pol in [3].

Theorem [A. Pol]. *Let S be a commutative compact semigroup with identity e and zero element 0, such that*

(a) *e has a basis of open connected neighborhoods,*

(b) *for every open set $U \subset S$ and every $x \in S$, $x \neq 0$, the set Ux is also open.*

Then the dual semigroup $\hat{S}$ is locally compact.

However, we need not the condition (a), and the theorem can be established under the more general situation of locally compact semigroups. Indeed we shall show the next theorem.

Theorem 1. *Let S be a commutative compact semigroup with identity e and zero element 0, such that*

(*) *for every open set $U \subset S$ and every $x \in S$, $x \neq 0$, the set Ux is also open.*

Then the dual semigroup $\hat{S}$ is locally compact.

Proof. We shall show the fact that $\hat{S}$ is closed in $C(S, D)$, and $\hat{S}$ has an equicontinuous neighborhood of each $\chi \in \hat{S}$. If this fact is shown, our Theorem 1 is a consequence of the above lemma and the fact that D is compact.

First we shall prove that $\hat{S}$ is closed in $C(S. D)$. For $x \in S$, let f_x be a mapping from $C(S, D)$ into D such that $f_x(\psi) = \psi(x)$, then f_x is a continuous mapping. If we define $f_x f_y$ for $x, y \in S$ as usual (i. e. $f_x f_y(\psi) = f_x(\psi) f_y(\psi)$), $f_x f_y$ is also a continuous mapping of $C(S, D)$ into D. So the set $\hat{S}(x, y) = \{\psi \in C(S, D) : f_{xy}(\psi) = f_x f_y(\psi)\}$ is closed in $C(S, D)$. Since $\hat{S} = \bigcap_{x,y} \hat{S}(x, y)$, $\hat{S}$ is closed in $C(S, D)$.

Next we shall prove that $\hat{S}$ has an equicontinuous neighborhood of each $\chi \in \hat{S}$. We shall note that χ^0 and χ^1 are isolated points in $\hat{S}$. In fact,

$$U(S, \tfrac{1}{2}, \chi^0) = \{\psi \in \hat{S} : |\psi(x)| < \tfrac{1}{2}, x \in S\} = \{\chi^0\}$$

since $\psi \neq \chi^0$ implies $\psi(e) = 1$, and

$$U(S, \tfrac{1}{2}, \chi^1) = \{\psi \in \hat{S} : |\psi(x) - 1| < \tfrac{1}{2}, x \in S\} = \{\chi^1\}$$

since $\psi \neq \chi^1$ implies $\psi(0) = 0$. So, let $\chi \in \hat{S}$, $\chi \neq \chi^0$, $\chi \neq \chi^1$ We shall show that the set $U_0 = U(S, \frac{1}{4}, \chi)$ is an equicontinuous, i. e.,

(**) $\forall \varepsilon > 0$, $\forall x_0 \in S$, $\exists V(x_0)$, $\forall \psi \in U_0$, $\forall x \in V(x_0)$, $|\psi(x) - \psi(x_0)| < \varepsilon$ where $V(x_0)$ denotes an open neighborhood of x_0. In order to prove (**), we distinguish three cases ; (I) $x_0 = 0$, (II) $x_0 = e$, (III) $x_0 \neq 0$, $x_0 \neq e$.

(I) $x_0 = 0$. Let $W = \{x \in S : |\chi(x)| < \frac{1}{4}\}$. Then W is an open neighborhood of 0 in S. For every $\psi \in U_0$ and for every $x \in W$, $|\psi(x)| \leqq |\psi(x) - \chi(x)| + |\chi(x)| < \frac{1}{2}$. For any $\varepsilon > 0$, there exists a natural number n such that $(\frac{1}{2})^n < \varepsilon$. Let $V = W^n = WW\ldots W$. By the given condition (*), V is an open neighborhood of 0. If $x \in V$ then x is of the form $x_1 x_2 \ldots x_n$ with $x_i \in W$, $(i = 1, 2, \ldots, n)$, and for every $\psi \in U_0$ we have

$$|\psi(x)| = |\psi(x_1) \ldots \psi(x_n)| < (\tfrac{1}{2})^n < \varepsilon.$$

This means that (**) is satisfied.

(II) $x_0 = e$. For any $\varepsilon > 0$, we select the natural number n such that if $z \in D$, $|z^k - 1| < \frac{1}{2}$ $(k = 1, 2, \ldots, n)$, then $|z - 1| < \varepsilon$. Then $V = \{x \in S : |\chi(x^k) - 1| < \frac{1}{4}, (k = 1, 2, \ldots, n)\}$ becomes a neighborhood of e. Since $\chi(e) = \psi(e) = 1$, for every $\psi \in U_0$ and for every $x \in V$,

$$|\psi(x^k) - 1| \leqq |\psi(x^k) - \chi(x^k)| + |\chi(x^k) - 1| < \tfrac{1}{2}, \ (k = 1, 2, \ldots, n).$$

Therefore, we have $|\psi(x) - \psi(e)| < \varepsilon$ by the way of determination of n. Thus, (**) is satisfied.

(III) $x_0 \neq 0$, $x_0 \neq e$. For any $\varepsilon > 0$, by part (II) there exists a neighborhood W of e such that $|\psi(x)-1| < \varepsilon$ for all $\psi \in U_0$ and all $x \in W$. Let $V = Wx_0$. By using (*), V is an open neighborhood of x_0. If $x \in V$, then $x = yx_0$ with $y \in W$ and

$$|\psi(x)-\psi(x_0)| = |\psi(x_0)| \cdot |\psi(y)-1| \leqq |\psi(y)-1| < \varepsilon$$

Thus, (**) is satisfied.

This concludes the proof of the Theorem 1. The proof of the parts (I) and (III) is essentially same to that of Theorem 1 in [3].

The condition (*) is essential in our Theorem 1 and it cannot be removed as the follwing example shows.

Example 1. Let $X = [1, \infty)$, X^* be the one-point compactification of X, $Y = [1, \infty)$, Y^* the one-point compactification of Y, and D the complex unit disc. Then X and Y are the additive semigroups. We extend the addition to X^* and Y^* respectively by $r + \infty = \infty + r = \infty$ for all $r \in X$ or $r \in Y$ and $\infty + \infty = \infty$.

Let S be the set $X^* \cup Y^* \cup D$ in which ∞ of X^*, ∞ of Y^* and 0 of D are identified. In S, we take the sets $U_\varepsilon(0) \cup (X-C) \cup (X-C')$ as the neighborhoods of 0, where each $U_\varepsilon(0)$ is an usual ε-neighborhood in D, each C is a compact subset of X and each C' is a compact subset of Y, and as the neighborhoods of the other point x of S we take the ordinary neighborhoods of x in X, Y or D. By this topology, S becomes a compact space. Let $S^* = S \cup \{e\}$ be the adjunction of e to S.

Next we define multiplication on S^* as follows:

(a) for $(x), (x') \in X^*$: $(x)\circ(x') = (x')\circ(x) = (x+x') \in X^*$

(b) for $[y], [y'] \in Y^*$: $[y]\circ[y'] = [y']\circ[y] = [y+y'] \in Y^*$

(c) for $re^{i\theta}, r'e^{i\theta'} \in D$: $re^{i\theta}\circ r'e^{i\theta'} = r'e^{i\theta'}\circ re^{i\theta} = rr'e^{i(\theta+\theta')} \in D$

(d) for $(x) \in X, [y] \in Y$: $(x)\circ[y] = [y]\circ(x) = e^{-(\lambda x+\mu y)} \in D$ where $\lambda, \mu > 0$ and linearly independent irrationals,

(e) for $(x) \in X, re^{i\theta} \in D$: $(x)\circ re^{i\theta} = re^{i\theta}\circ(x) = e^{-\lambda x}re^{i\theta} \in D$

(f) for $[y] \in Y, re^{i\theta} \in D$: $[y]\circ re^{i\theta} = re^{i\theta}\circ[y] = e^{-\mu y}re^{i\theta} \in D$

(g) for $s \in S, e$: $e\circ s = s\circ e = s$ and $e\circ e = e$.

By this multiplication, S^* becomes a commutative compact semigroup with zero element 0 and identity e. However, the condition (*) in Theorem 1 is not satisfied.

Let $\psi \in \hat{S}^*$ be $\psi \neq \chi^1, \psi \neq \chi^0$.

By (a) $\psi((x)) = (ae^{2\pi\alpha i})^x$ where $a \in (0,1)$, $\alpha \in T^1$ (1-dim. torus).
By (b) $\psi([y]) = (be^{2\pi\beta i})^y$ where $b \in (0,1)$, $\beta \in T^1$.
By (c) $\psi(re^{i\theta}) = r^{\varepsilon+2\pi\tau i}\, e^{ik\theta}$ where $\varepsilon > 0$, $\tau \in T^1$, $k \in Z$ (integers).
By (d) $\psi((x))\psi([y]) = \psi(e^{-(\lambda x+\mu y)})$,

hence, (h) $\begin{cases} a = e^{-\lambda\varepsilon},\ b = e^{-\mu\varepsilon}, \\ \alpha = -\lambda\tau,\ \beta = -\mu\tau,\ (\text{mod. } 1) \end{cases}$

By (e) similarly,

$$\text{(i)} \begin{cases} a = e^{-\lambda\varepsilon} \\ \alpha = -\lambda\tau \ (\text{mod. } 1) \\ k : \text{arbitrary integer.} \end{cases}$$

By (f) similarly

$$\text{(j)} \begin{cases} b = e^{-\mu\varepsilon} \\ \beta = -\mu\tau \ (\text{mod. } 1) \\ k : \text{arbitrary integer.} \end{cases}$$

By (h), (i), (j), we have

$(a, b, \varepsilon) = (e^{-\lambda\varepsilon}, e^{-\mu\varepsilon}, \varepsilon)$: this is a curve,

$(\alpha, \beta, \tau) = (-\lambda\tau, -\mu\tau, \tau)$: this is a dense curve in 3-dimensional torus, k is an arbitrary integer.

Thus, $\hat{S}^*$ is not locally compact.

§ 3. The dual semigroups of locally compact semigroups.

In this section, we show that Theorem 1 is valid in the case where S is locally compact, and give some relevant matters.

Theorem 2. *Let S be a commutative locally compact semigroup with identity e and zero element* 0, *such that the condition* (*) *of Theorem* 1 *is satisfied. Then the dual semigroup* $\hat{S}$ *is locally compact.*

Proof. That $\hat{S}$ is closed in $C(S, D)$ is shown by the same way in the proof of Theorem 1. Hence it is sufficient that we prove the fact that $\hat{S}$ has an equicontinuous neighborhood of each $\chi \in \hat{S}$.

First we note that χ^0 and χ^1 are isolated points in $\hat{S}$. In fact, we can select the neighborhoods W_0 and W_e of 0 and e respectively such that $\overline{W}_0$ and $\overline{W}_e$ are compact. Then,

$$U(\overline{W}_e, \tfrac{1}{2}, \chi_0) = \{\chi^0\}$$

since $\psi \neq \chi^0$ implies $\psi(e) = 1$, and

$$U(\overline{W}_0, \tfrac{1}{2}, \chi^1) = \{\chi^1\}$$

since $\psi \neq \chi^1$ implies $\psi(0) = 0$. Therefore, let $\chi \in \hat{S}$, $\chi \neq \chi^0$, $\chi \neq \chi^1$. We shall show that the set $U_0 = U(\overline{W}_0 \cup \overline{W}_e, \frac{1}{4}, \chi)$, where W_0 and W_e are the above neighborhoods, is an equicontinuous, i. e., the condition (**) is satisfied. To prove (**), we distinguish three cases; (I) $x_0 = 0$, (II) $x_0 = e$, (III) $x_0 \neq 0$, $x_0 \neq e$.

(I) $x_0 = 0$. Let $V_1 = W_0 \cap \{x \in S : |\chi(x)| < \frac{1}{4}\}$. Then V_1 is an open neighborhood of 0 in S. For any $\varepsilon > 0$, there exists a natural number n such that $(\frac{1}{2})^n < \varepsilon$. By the continuity of the multiplication, there exists a neighborhood V_2 of 0 such that $(V_2)^n \subset W_0$. The set $V_3 = V_1 \cap V_2$ is a neighborhood of 0. Let $V = (V_3)^n$. By the given condition (*), V is a neighborhood of 0. For every $\psi \in U_0$ and for every $x \in V_3$, since $x \in W_0$, $|\psi(x)| \leqq |\psi(x) - \chi(x)| + |\chi(x)| < \frac{1}{2}$. If $x \in V$, then x is of the form $x_1 x_2 \dots x_n$ with $x_i \in V_3$, and for every $\psi \in U_0$, we have

$$|\psi(x)| = |\psi(x_1) \dots \psi(x_n)| < (\tfrac{1}{2})^n < \varepsilon.$$

Thus, (**) is satisfied.

(II) $x_0 = e$. For any $\varepsilon > 0$, we select the natural number n such that if $z \in D$, $|z^k - 1| < \frac{1}{2}$ $(k = 1, 2, \dots n)$, then $|z - 1| < \varepsilon$. By the continuity of the multiplication, there exists a neighborhood V_1 of e such that $(V_1)^n \subset W_e$. Let $V = V_1 \cap \{x \in S : |\chi(x^k) - 1| < \frac{1}{4}, (k = 1, 2, \dots, n)\}$. Then V is a neighborhood of e. For every $\psi \in U_0$ and for every $x \in V$, since $x^k \in W_e$ $(k = 1, 2, \dots, n)$,

$$|\psi(x^k) - 1| \leqq |\psi(x^k) - \chi(x^k)| + |\chi(x^k) - 1| < \tfrac{1}{2}, \ (k = 1, 2, \dots, n).$$

Therefore, we have $|\psi(x) - 1| < \varepsilon$. Thus, (**) is satisfied.

(III) $x_0 \neq 0$, $x_0 \neq e$. In this case, (**) is proved by the same way as (III) of the proof of Theorem 1.

This concludes the proof of Theorem 2.

We shall give the following example in which without the condition (*), Theorem 2 does not hold any longer.

Example 2. Let X, Y and D be the same sets of Example 1. Let $S = X \cup Y \cup D$. We define the topology on S by the ordinary topologies in X, Y and D. By this topology, S becomes locally compact. Let $S^* = S \cup \{e\}$ be the adjunction of e to S.

Next we define multiplication in S^* by the same way as Example 1. By this multiplication, S^* becomes a commutative locally compact semigroup with zero element 0 and identity e. But the condition (*) is not satisfied. By the same way as Example 1, $\widehat{S^*}$ is not locally compact.

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