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ON A LINEAR EXTRAPOLATION PROBLEM OF SOME HOMOGENEOUS RANDOM FIELDS

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§1. Introduction and Summary

Let V be the d-dimensional Euclidean space or d-dimensional sphere. Our random field $(X(\omega\,;z),\ z\in V)$ has the mean function 0 and the covariance function $R(z, z')$, and it belongs to the class $L^2(\Omega)$. We call it the *homogeneous random field* (*in the weak sense*) whenever $R(gz, gz')=R(z, z')$ for any $g\in G(V)$, where $G(V)$ is the full linear group of motions preserving distances. In this case, the covariance function is of the form $R(\rho(z, z'))$, where $\rho=\rho(z, z')$ is a distance between z and z'.

Moreover, we assume that the random field $X(z)$ is continuius in quadratic mean, then the covariance function $R(\rho(z, z'))$ is a positive definite function. As well known, the positive definite function has the spectral representation with spherical functions. In our case, it takes the following forms ;
for $V=E^d$ (Euclidean case),

$$R(\rho(z, z'))=\Gamma\left(\frac{d}{2}\right)\int_0^{\infty}\frac{J_{\frac{d-2}{2}}(\gamma\rho)}{\left(\frac{\gamma\rho}{2}\right)^{\frac{d-1}{2}}}dF(\gamma)$$

$$\text{, where } R(0)=F(+\infty), \tag{1}$$

for $V=S^d$ (Spherical space),

$$R(\rho(z, z'))=\Gamma(d-1)\sum_{\gamma=0}^{\infty}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+d-1)}c_\gamma^{\frac{d-1}{2}}(\cos\rho)\,W(\gamma)$$

$$\text{, where } w(\gamma)\geqslant 0 \text{ and } R(0)=\sum_{\gamma=0}^{\infty}w(\gamma).$$

We consider the following extrapolation problem ; our data is observations on n concentric spheres with center at coordinate origin, and using these observations, we will extrapolate any one point outside of these regions. Let an extrapolated value be $X(z)$; our extrapolation error is measured by

$$\delta^2(z) = E(X(z)-X(z))^2. \tag{2}$$

Our extrapolator $X(z)$ is of the form, when we denote a point z by coordinate (φ, θ), $\varphi \in S^{d-1}$, and $0 \leqslant \theta < +\infty (V = S^d)$,
$0 \leqslant \theta \leqslant \pi$ $(V = S^d)$,

$$\hat{X}(z)=\sum_{i=1}^{n} \int_{S^{d-1}} c^{(i)}(z;\ \varphi) X(\varphi;\ \theta_i)\, dv(\varphi), \tag{3}$$

, where $dv(\varphi)$ is the normalized Haar measure on S^{d-1}.

We will determine coefficients $c^{(i)}(z\ ;\ \varphi)(1 \leqslant i \leqslant n)$ to minimize the error $\delta^2(z)$ and evaluate the minimum value of $\delta^2(z)$. We assume that these coefficients belong to the class $L^2(S^{d-1})$.

In the following, we treat these problems and in particular, for the case of $n = 1$ and z is the origin, we will give an explicit evaluation. U. D. Popov (1968) treated the same problem for $V = E^2$ and U. D. Popov and M. I. Yadrenko (1969) treated the case of $V = E^d$ and $n = 1$. In the next paper, we will treat the case of hyperbolic space.

In the present paper, we use the following notations :

H^{md} ; the space of harmonic polynomials of degree m
$(m = 0, 1, 2, \ldots\ldots)$.

$h(m, d)$ = dimension of H.

$S_m^k(\varphi)$; an orthonormal basis of the space H^{md}
$(k = 1, 2, \ldots\ldots, h(m, d))$.

$L^2(S^{d-1}) = \sum_{m=0}^{\infty} H^{md}$ (direct sum).

Let the Fourier expansion of coefficients $c^{(i)}(z\ ;\ \varphi)$ be

$$c^{(i)}(z;\ \varphi) = \sum_{m+0}^{\infty} \sum_{k=1}^{h(m,d)} \hat{c}^{(i)}(z;\ m, k)\, S_m^k(\varphi), \tag{4}$$

§1. Euclidean space

For fixed $z_j's \in E^d$, a set of observation points, from (3) we have

$$R(\rho(z, z')) = \sum_{i=1}^{n} \int_{S^{d-1}} c^{(i)}(z;\ \varphi) R(\rho(z_i, z_j)) dv(\varphi). \tag{5}$$

Putting $z_i = (\theta_i, \varphi)$ and $z_j = (\theta_j, \varphi_j)$, we get

$$R(\rho(z_i, z_j)) = \Gamma^2\left(\frac{d}{2}\right)\sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)} a_m^{(d)}(\theta_i, \theta_j)\,\overline{S_m^k(\varphi)}\,S_m^k(\varphi_j) \tag{6}$$

, where $a_m^{(d)}(\theta_i, \theta_j) = \int_0^\infty \frac{J_{\frac{d-2}{2}+m}(v\theta_i)}{\left(\frac{v\theta_i}{2}\right)^{\frac{d-2}{2}}} \frac{J_{\frac{d-2}{2}+m}(v\theta_j)}{\left(\frac{v\theta_j}{2}\right)^{\frac{d-2}{2}}} dF(v)$

Substituting (4) and (6) into (5), we get

$$\begin{aligned} R(\rho(z_i, z_j)) &= \Gamma^2\left(\frac{d}{2}\right)\sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)}\sum_{i=1}^{n} a_m^{(d)}(\theta_i, \theta_j)\,\hat{c}^{(i)}(z; m, k) S_m^k(\varphi_i) \\ &= \sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)} \gamma_j^{(d)}(z; m, k)\, S_m^k(\varphi_j) \end{aligned} \tag{7}$$

, where $\gamma_j^{(d)}(z; m, k) = \Gamma^2\left(\frac{d}{2}\right)\sum_{i=1}^{n} a_m^{(d)}(\theta_i, \theta_j)\,\hat{c}^{(i)}(z; m, k)$

From the expression (7), we can determine coefficients $\gamma_j^{(d)}(z; m, k)$ uniquely and the matrix $A_m(d) = (a_m^{(d)}(\theta_i, \theta_j))$ is a positive definite symmetric matrix, so that whenever $\det(A_m(d)) \neq 0$, we can determine coefficients $c^{(i)}(z; \varphi)$'s uniquely. To evaluate the error $\delta^2(z)$, we need an evaluation of $\| \hat{X}(z) \|^2 = E(\hat{X}(z))^2$. While from (3) and (5), using (7), we get

$$\delta^2(z) = F(+\infty) - \| \hat{X}(z) \|^2,$$

$$\|\hat{X}(z)\|^2 = \Gamma^2\left(\frac{d}{2}\right)\sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)}\sum_{i=1}^{n}\sum_{j=1}^{n} a_m^{(d)}(\theta_i, \theta_j)\,\overline{\hat{c}^{(i)}(z; m, k)}\,\hat{c}^{(j)}(z; m, k)$$

$$= \sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)}\sum_{i=1}^{n} \overline{\hat{c}^{(i)}(z; m, k)}\,\gamma_i^{(d)}(z; m, k) \tag{8}$$

In particular, when $n = 1$, we have

$$\delta^2(z) = F(+\infty) - \sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)} \Gamma^2\left(\frac{d}{2}\right) \frac{[a_m^{(d)}(\theta_0, \theta_1)]^2}{a_m^{(d)}(\theta_1, \theta_1)} \left|S_m^k(\varphi_0)\right|^2$$

$$= F(+\infty) - \Gamma^2\left(\frac{d}{2}\right)\sum_{m=0}^{\infty} (m, d) \frac{[a_m^{(d)}(\theta_0, \theta_1)]^2}{a_m^{(d)}(\theta_1, \theta_1)} \geqslant 0 \tag{9}$$

Thus we have a theorem

Theorem 1 : In the Euclidean case, using an extrapolator of the form (3), when we have observations on n-concentric spheres with center at origin, the extrapolation error is given by the formula (8), and in particular when $n=1$, it is given by the formula (9).

Corollary : For $n=1$, the extrapolation error depends only on distance from origin.

For the case of $d=2$, the expression (9) becomes as

$$\delta^2(z)=F(+\infty)-\sum_{m=-\infty}^{\infty}\frac{\left[a_m^{(d)}(\theta_0,\theta_1)\right]^2}{a_m^{(d)}(\theta_1,\theta_1)}$$

Finally, we consider extrapolation of origin. In this case,

$$R(\rho(z,z_j))=R(\theta_j)=\sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)}\gamma_j^{(d)}(z;m,k)S_m^k(\varphi_j),$$

so that $\gamma_j^{(d)}(z;m,k)=\delta(m)\delta(k)R(\theta_j)$ and

$$\delta^2(z)=F(+\infty)-\Gamma^2\left(\frac{d}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}a_0^{(d)}(\theta_i,\theta_i)\hat{c}^{(i)}(z;o,o)\overline{\hat{c}^{(j)}(z;o,o)} \tag{10}$$

in particular, when $n=1$, we have

$$\delta^2(z)=F(+\infty)-\frac{\left[\Gamma\left(\frac{d}{2}\right)\int_0^{\infty}\frac{J_{\frac{d-2}{2}}(\nu\theta_1)}{(\nu\theta_1/2)^{\frac{d-2}{2}}}dF(\nu)\right]^2}{\int_0^{\infty}\left[\Gamma\left(\frac{d}{2}\right)\frac{J_{\frac{d-2}{2}}(\nu\theta_1)}{(\nu\theta_1/2)^{\frac{d-2}{2}}}\right]^2dF(\nu)}$$

In the latter case, the lower bound 0 is attained iff the spectral distribution function $F(\nu)$ jumps at ν-values such that

$$\Gamma\left(\frac{d}{2}\right)\frac{J_{\frac{d-2}{2}}(\nu\theta_1)}{\left(\frac{\nu\theta_1}{2}\right)^{\frac{d-2}{2}}}=C,\ 0<C<1.$$

§ 2. Spherical space

In this case, corresponding to (6), we have

$$R(\rho(z_i,z_j))=\frac{2^{d-2}}{\pi}\Gamma^2\left(\frac{d}{2}\right)\sum_{m=0}^{\infty}\sum_{k=1}^{h(m,d)}a_m^{(d)}(\theta_i,\theta_j)S_m^k(\varphi_i)\overline{S_m^k(\varphi_j)}, \tag{11}$$

where $a_m^{(d)}(\theta_i,\theta_j)=\sum_{\nu=m}^{\infty}\Bigg[W(\nu)\frac{\Gamma(\nu+1)}{\Gamma(\nu+d-1)}\frac{2^{2m+d+2}(\nu-m)!\,\Gamma^2(\frac{d-1}{2}+m)}{\Gamma(\nu+m+d+1)}\cdot$

$$C_{\nu-m}^{\frac{d-1}{2}+m}(\cos\theta i)\sin^m(\theta i)\,C_{\nu-m}^{\frac{d-1}{2}+m}(\cos\theta j)\sin^m(\theta j)\Bigg]$$

and corresponding to (7) we have

$$\delta^2(z) = R(0) - \|\hat{X}(z)\|^2; \tag{12}$$

$$\|\hat{X}(z)\|^2 = \frac{2^{d-2}}{\pi} \Gamma^2\left(\frac{d}{2}\right) \sum_{m=0}^{\infty} \sum_{k=1}^{h(m,d)} \sum_{i=1}^{n} \sum_{j=1}^{n} a_m^{(d)}(\theta_i, \theta_j)\, \hat{c}^{(i)}(z; m, k)\, \overline{\hat{c}^{(j)}(z; m, k)} \tag{13}$$

and for $n = 1$ we have

$$\delta^2(z) = R(0) - \frac{2^{d-2}}{2} \Gamma^2\left(\frac{d}{2}\right)_{m=0} \left(\frac{d}{2}\right) \sum_{m=0}^{\infty} \sum_{k=1}^{h(m,d)} \frac{[a_m^{(d)}(\theta_0, \theta_1)]^2}{a_m^{(d)}(\theta_1, \theta_1)} \left| S_m^k(\varphi_0) \right|^2$$

$$= R(0) - \frac{2^{d-2}}{\pi} \Gamma^2\left(\frac{d}{2}\right) \sum_{m=0}^{\infty} h(m, d) \frac{[a_m^{(d)}(\theta_0, \theta_1)]^2}{a_m^{(d)}(\theta_1, \theta_1)} \geqslant 0. \tag{14}$$

Thus we get a theorem

Theorem 2 : In the spherical case, using an extrapolator of the form (3), when we have observations on n-concentric spheres with center at origin (north pole), the extrapolation error is given by the formula (12), and in particular when $n = 1$, it is given by the formula (14).

Corollary : For the case of $n = 1$, the extrapolation error depends only on distance from origin.

We note that this result parallels the result in the Euclidean case.

Finally we consider extrapolation of the origin. We get

$$\|\hat{X}(z)\|^2 = \frac{2^{d-2}}{\pi} \Gamma^2\left(\frac{d}{2}\right) \sum_{i=1}^{n} \sum_{j=1}^{n} a_0^{(d)}(\theta_i, \theta_j)\, \hat{c}^{(i)}(z; o, o)\, \overline{\hat{c}^{(j)}(z; o, o)}$$

and for $n = 1$,

$$\delta^2(z) = R(0) - \frac{\left[\Gamma(d-1) \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu+1)}{\Gamma(\nu+d-1)} C_\nu^{\frac{d-1}{2}}(\cos\theta_1) W(\nu)\right]}{\sum_{\nu=0}^{\infty} \left[\Gamma(d-1) \frac{\Gamma(\nu+1)}{\Gamma(\nu+d-1)} C_\nu^{\frac{d-1}{2}}(\cos\theta_1)\right]^2 W(\nu)}$$

The lower bound of error 0 is attained, in this case, iff spectrum distributes on ν-values such that $\frac{\Gamma(d-1)\Gamma(\nu+1)}{\Gamma(\nu+d-1)} C_\nu^{\frac{d-1}{2}}(\cos\theta_1) = C,\ 0 < C \leqslant 1.$

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