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On Fully-Completeness in Topological Vector Spaces

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Introduction. Let E be a separated locally convex topological vector space and E' be its dual space. E is said to be fully complete provided any linear subspace L of E' is weakly closed in E' whenever $L \cap U^{\circ}$ is weakly closed for every neighbourhood U of zero in E. A fully complete space is also called B-complete [3]. E is said to be B_r-complete provided any weakly dense subspace L of E' is weakly closed in E' whenever $L \cap U^{\circ}$ is weakly closed for every neighbourhood U of zero in E [3]. A. Persson [2] introduced the notions of t-polar and weakly t-polar spaces. They are the spaces E which are obtained by replacing the neighbourhood U by a barrel T in the above definitions of a B-complete and a B_r-complete spaces respectively.

We shall study some generalizations and some relations of these notions. We introduce new spaces, an $\mathfrak{S}$-polar and a weakly $\mathfrak{S}$-polar spaces with $\mathfrak{S}$ a set of barrels in E. These are the spaces obtained by restricting every barrel T of E to that of $\mathfrak{S}$ in the definitions of t-polar and weakly t-polar spaces. Therefore, when $\mathfrak{S}$ is the family of all absolutely convex and closed neighbourhoods of zero (resp. all barrels) in E, an $\mathfrak{S}$-polar space is a fully complete (resp. t-polar) space and a weakly $\mathfrak{S}$-polar space is a B_r-complete (resp. weakly t-polar) space.

Notations. We denote by E and F separated locally convex spaces with the dual spaces E' and F' respectively and by u a linear mapping of E into F with the adjoint mapping u'. We also denote by $\mathfrak{U}_E$ the family of all the absolutely convex closed neighbourhoods of zero in E and by $\mathfrak{B}_E$ the family of all barrels in E. $\mathfrak{S}_E$ denotes a subfamily of $\mathfrak{B}_E$ which is a filterbasis, and $\mathfrak{T}_{\mathfrak{S}}$ denotes a locally convex topology such that $\mathfrak{S}$ is a basis of neighbourhoods of zero.

1. We start with the following

DEFINITION 1. We call a subset L of E' $\mathfrak{S}_E$-nearly closed if $L \cap T^{\circ}$ is

$\sigma(E', E)$-closed for every barrel T belonging to $\mathfrak{S}_E$.

DEFINITION 2. We call E an $\mathfrak{S}_E$-polar space (resp. weakly $\mathfrak{S}_E$-polar space) if any linear subspace L (resp. any $\sigma(E', E)$-dense linear subspace L) of E' is $\sigma(E', E)$-closed whenever L is $\mathfrak{S}_E$-nearly closed.

DEFINITION 3. We call $u : E \to F$ $\mathfrak{S}_E$-nearly open if $\overline{u(T)}$ is a neighbourhood of zero in F for every barrel T belonging to $\mathfrak{S}_E$. We also call $u : E \to F$ $\mathfrak{S}_F$-nearly continuous if $\overline{u^{-1}(T)}$ is a neighbourhood of zero in E for every barrel T belonging to $\mathfrak{S}_F$.

LEMMA 1. (a) *Let E be an $\mathfrak{S}_E$-polar space. If $E[\mathfrak{T}]' = E'$, then $E[\mathfrak{T}]$ is an $\mathfrak{S}_E$-polar space.*

(b) *Let $\mathfrak{S}_E \supset \mathfrak{S}'_E$. If E is an $\mathfrak{S}'_E$-polar space, then E is an $\mathfrak{S}_E$-polar space.*

(c) *Let $\mathfrak{S}_E \supset \mathfrak{S}'_E$ and let $E[\mathfrak{T}_{\mathfrak{S}_E}]' = E[\mathfrak{T}_{\mathfrak{S}'_E}]' = E'$. If $E[\mathfrak{T}_{\mathfrak{S}'_E}]$ is an $\mathfrak{S}'_E$-polar space, then $E[\mathfrak{T}_{\mathfrak{S}_E}]$ is an $\mathfrak{S}_E$-polar space.*

Proof. The statement (a) follows from the fact that the property of being a barrel in E depends only on the dual pair (E, E'). The statement (b) is obvious and statement (c) follows from (a) and (b).

2. In this section we shall mainly study the properties of $\mathfrak{S}_E$-nearly open mappings. We first show

LEMMA 2. *Let $u : E \to F$ be continuous and $\mathfrak{S}_E$-nearly open with the adjoint mapping u'. If L is a $\overline{u(\mathfrak{S}_E)}$-nearly closed linear subspace of F', then $u'(L)$ is $\mathfrak{S}_E$-nearly closed, where $\overline{u(\mathfrak{S}_E)}$ is the family of the closure $\overline{u(T)}$ of $u(T)$ for every barrel T belonging to $\mathfrak{S}_E$.*

Proof. Let $T \in \mathfrak{S}_E$. Since $u'^{-1}(T^\circ) = u(T)^\circ$, we have $u'(L) \cap T^\circ = u'(L \cap u'^{-1}(T^\circ)) = u'(L \cap u(T)^\circ) = u'(L \cap \overline{u(T)}^\circ)$.

Here $\overline{u(T)}^\circ$ is $\sigma(F', F)$-compact, for u is $\mathfrak{S}_E$-nearly open. By our hypothesis, $L \cap \overline{u(T)}^\circ$ is $\sigma(F', F)$-closed and therefore $L \cap \overline{u(T)}^\circ$ is $\sigma(F', F)$-compact. Since u' is weakly continuous, $u'(L) \cap T^\circ$ is $\sigma(E', E)$-compact, that is, $\sigma(E',E)$-closed. Thus the proof is complete.

As an extesion of [1, Th 14] we now show the following

THEOREM 1. *Let $u : E \to F$ be surjective, continuous and $\mathfrak{S}_E$-nearly open. If E is an $\mathfrak{S}_E$-polar space, then F is a $\overline{u(\mathfrak{S}_E)}$-polar space.*

Proof. Let L be a $\overline{u(\mathfrak{S}_E)}$-nearly closed linear subspace of F'. Then, by Lemma 2, $u'(L)$ is $\mathfrak{S}_E$-nearly closed. Since E is an $\mathfrak{S}_E$-polar space, $u'(L)$ is $\sigma(E', E)$-closed. Therefore $L = u'^{-1}(u'(L))$ is $\sigma(F', F)$-closed, for u' is weakly continuous. Thus the proof is complete.

COROLLARY. *Let $u : E \to F$ be surjective and continuous, and let F be a* *barrelled space. If E is a t-polar space, then F is a fully complete space and therefore a t-polar space.*

PROPOSITION 1. *Let $u : E \to F$ be surjective and continuous, let the adjoint mapping $u' : F' \to E'$ be weakly closed. If E is an $\mathfrak{S}_E$-polar space, then F is a $\overline{u(\mathfrak{S}_E)}$-polar space.*

Proof. Since u is surjective, $u' : F' \to E'$ is injective, weakly continuous and weakly closed. Let $T \in \mathfrak{S}_E$ and let L be a $\overline{u(\mathfrak{S}_E)}$-nearly closed linear subspace. Then, we have $u'(L) \cap T^\circ = u'(L \cap u'^{-1}(T^\circ)) = u'(L \cap (u(T))^\circ) = u'(L \cap (\overline{u(T)})^\circ)$.

By assumption, $L \cap (\overline{u(T)})^\circ$ is $\sigma(F', F)$-closed. Therefore $u'(L) \cap T^\circ$ is $\sigma(E', E)$-closed, for u' is weakly closed. Since E is an $\mathfrak{S}_E$-polar space, $u'(L)$ is $\sigma(E', E)$-closed and so $L = u'^{-1}(u'(L))$ is $\sigma(F', F)$-closed, because u' is weakly continuous. Thus the proof is complete.

COROLLARY. *Let H be a closed subspace of E and let $k : E \to E/H$ be the canonical mapping. If E is an $\mathfrak{S}_E$-polar space, then the quotient space E/H is a $\overline{k(\mathfrak{S}_E)}$-polar space.*

Proof. $k : E \to E/H$ is surjective and continuous and $k' : (E/H)' \to E'$ is weakly closed. Therefore the statement follows from Proposition 1.

3. In this section we shall mainly study the properties of $\mathfrak{S}_F$-nearly continuous mappings.

LEMMA 3. *Let $u : E \to F$ be $\mathfrak{S}_F$-nearly continuous and let $u' : F' \to E^*$ be the adjoint mapping of u. If L is a subspace of E' which is $\overline{u^{-1}(\mathfrak{S}_F)}$-nearly closed, then $u'^{-1}(L)$ is $\mathfrak{S}_F$-nearly closed.*

Proof. Let $H = u'^{-1}(L)$. Then we have to show that $H \cap T^\circ$ is $\sigma(F', F)$-closed for every $T \in \mathfrak{S}_F$. Let $W = u^{-1}(T)$. Then $\overline{W}$ is a neighbourhood of zero in E, since u is $\mathfrak{S}_F$-nearly continuous. Then $L \cap W^\circ$ is $\sigma(E^*, E)$-closed. In fact, $\overline{W}^\circ$ is $\sigma(E^*, E)$-compact and L is $\overline{u^{-1}(\mathfrak{S}_F)}$-nearly closed. Therefore $L \cap \overline{W}^\circ$ is $\sigma(E', E)$-compact and so $L \cap \overline{W}^\circ$ is $\sigma(E^*, E)$-closed. Consequently $u'^{-1}(L \cap \overline{W}^\circ)$ is $\sigma(F', F)$-closed, since $u' : F' \to E^*$ is weakly continuous. Now $u'^{-1}(L \cap \overline{W}^\circ) = u'^{-1}(L \cap W^\circ) = H \cap u'^{-1}(W^\circ) = H \cap u(W)^\circ = H \cap (T \cap u(E))^\circ$. However $T^\circ \subset (T \cap u(E))^\circ$. Therefore $H \cap T^\circ = H \cap (T \cap u(E))^\circ \cap T^\circ$. Hence $H \cap T^\circ$ is $\sigma(F', F)$-closed. Thus the proof is complete.

LEMMA 4. *Let $u : E \to F$ be a continuous mapping with the adjoint u'. If a* *subspace* L *of* E' *is* $\overline{u^{-1}(\mathfrak{S}_F)}$-nearly closed, then $u'^{-1}(L)$ *is* $\mathfrak{S}_F$-nearly closed.

Proof. Let $T \in \mathfrak{S}_F$, $W = \overline{u^{-1}(T)}$ and $H = u'^{-1}(L)$. Then, by our hypothesis, $L \cap W^{\circ}$ is $\sigma(E', E)$-closed and u' is weakly continuous. Here

$$u'^{-1}(L \cap W^{\circ}) = u'^{-1}(L) \cap u'^{-1}(W^{\circ}) = H \cap u'^{-1}(W^{\circ}) = H \cap (u(W))^{\circ}.$$

Therefore $H \cap (u(W))^{\circ}$ is $\sigma(F', F)$-closed. However $u(W) \subset T$. Hence $H \cap T^{\circ} = H \cap (u(W))^{\circ} \cap T^{\circ}$ and $H \cap T^{\circ}$ is $\sigma(F', F)$-closed. Thus the proof is complete.

As a dual of Theorem 1, we here show the following theorem.

THEOREM 2. *Let* $u: E \to F$ *be* $\mathfrak{S}_F$*-nearly continuous, let* $u': F' \to E^*$ *be weakly open and let* $u'(F') \supset E'$. *Then, if* F *is an* $\mathfrak{S}_F$*-polar space,* E *is a* $\overline{u^{-1}(\mathfrak{S}_F)}$*-polar space.*

Proof. $u'^{-1}(0)$ being weakly closed, the space $F'/u'^{-1}(0)$ is separated under the quotient topology. Then we can write $u' = s \circ k$ where k is the canonical mapping of F' onto $F'/u'^{-1}(0)$ and s is an injective mapping of $F'/u'^{-1}(0)$ into E^*. Let L be a $\overline{u^{-1}(\mathfrak{S}_F)}$-nearly closed linear subspace of E'. Then, putting $H = u'^{-1}(L)$, H is $\mathfrak{S}_F$-nearly closed by Lemma 3. Since F is an $\mathfrak{S}_F$-polar space, H is $\sigma(F', F)$-closed. Here $k(H)$ is weakly closed, since k is the canonical mapping. Therefore $k(H) = s^{-1}(L)$ is weakly closed. Since s is weakly open and $L \subset E'$, L is $\sigma(E', E)$-closed. Thus the proof is complete.

As a dual of Proposition 1, we here show the following proposition.

PROPOSITION 2. *Let* $u: E \to F$ *be continuous (resp. surjective and continuous) and let* $u': F' \to E'$ *be surjective and weakly open. Then if* F *is an* $\mathfrak{S}_F$*-polar (resp. weakly* $\mathfrak{S}_F$*-polar) space,* E *is a* $u^{-1}(\mathfrak{S}_F)$*-polar (resp. weakly* $u^{-1}(\mathfrak{S}_F)$*-polar) space.*

Proof. This can be proved by the same way as the previous theorem by using Lemma 4 instead of Lemma 3. So we omit the proof.

4. In this section, we shall study the properties of closed linear mappings.

THEOREM 3. *Let* $\mathfrak{S}_F$ *contain* $\mathfrak{U}_F$ *and let* $u: E \to F$ *be* $\mathfrak{S}_F$*-nearly continuous with a closed graph. If* F *is a weakly* $\mathfrak{S}_F$*-polar space, then* u *is continuous.*

Proof. We have to show that u is weakly continuous. For this it is sufficient to show that $u'^{-1}(E') = F'$, for we then have $u'(F') \subset E'$. Now $E' \cap \overline{u^{-1}(T)}^{\circ}$ is $\sigma(E', E)$-closed for every $T \in \mathfrak{S}_F$. Hence $u'^{-1}(E')$ is $\mathfrak{S}_F$-nearly closed, by Lemma 2. Since F is a weakly $\mathfrak{S}_F$-polar space and $u'^{-1}(E')$ is weakly dense in F', $u'^{-1}(E') = F'$. Thus u is weakly continuous. Let $V \in \mathfrak{U}_F$. Then since u is $\mathfrak{S}_F$-nearly continuous, $\overline{u^{-1}(V)} \in \mathfrak{U}_E$. u being weakly continuous, $\overline{u^{-1}(V)} =$

$u^{-1}(V)$ and so $u^{-1}(V)$ is a neighbourhood of zero in E. Thus the proof is complete.

As a special case of Theorem 3, we have the following result which is [2, Theorem 1].

COROLLARY. *If u is a closed linear mapping of a barrelled space into a weakly t-polar space, then u is continuous.*

PROPOSITION 3. *Let $u: E \to F$ be bijective and $\mathfrak{S}_E$-nearly open with $\mathfrak{S}_E \supset \mathfrak{U}_E$ and have a closed graph. If E is a weakly $\mathfrak{S}_E$-polar space, then u is open.*

Proof. u^{-1} is $\mathfrak{S}_E$-nearly continuous with a closed graph. Since E is a weakly $\mathfrak{S}_E$-polar space, by Theorem 3 u^{-1} is continuous, that is, u is open.

COROLLARY 1. *Let $u: E \to F$ be bijective and continuous. If E is a weakly t-polar space and F is a barrelled space, then u is a topological isomorphism.*

COROLLARY 2. *A weakly t-polar space cannot have a strictly coarser separated barrelled topology.*

COROLLARY 3. *Let $u: E \to F$ be bijective, continuous and $\mathfrak{B}_E$-nearly open. Then E is a weakly t-polar space if and only if F is a B_r-complete barrelled space.*

Proof. u is open. Therefore u is a topological isomorphism. The conclusion now follows from the fact that u is $\mathfrak{B}_E$-nearly open.

5. We shall finally apply the results in Sections 2 and 4 for the study of closed linear relations.

A is called a closed linear relation if it is a closed linear subspace of $E \times F$. Here we put

$A^{-1} = \{(y, x) : (x, y) \in A\}$, $Ax = \{y : (x, y) \in A\}$,

$D(A) = \{x : (x, y) \in A \text{ for some } y\}$, $R(A) = D(A^{-1})$.

Moreover, A is said to be continuous if $A^{-1}(V) = \{x : Ax \cap V \neq \phi\}$ is open in $D(A)$ for every open set $V \subset F$, and also A is said to be open, if A^{-1} is continuous. Then we have the following theorem which is an extension of [2, Theorem 2].

THEOREM 4. *Let $\mathfrak{S}_F$ contain $\mathfrak{U}_F$ and let a linear relation $A: E \to F$ be closed and $\mathfrak{S}_F$-nearly continuous. If F is a $\mathfrak{S}_F$-polar space, then A is continuous.*

Proof. A being closed, $A(0)$ is a closed linear subspace of F. Let k be the canonical mapping of F onto $F/A(0)$. If we set $(kA)(x) = k(A(x))$ for any x in E, kA is a linear mapping of E into $F/A(0)$. Here A is continuous if and only if kA is continuous and also kA is closed linear mapping. Let $T \in \mathfrak{S}_F$.

Since $A^{-1}(T) \subset (kA)^{-1}(k(T)) \subset (kA)^{-1}(\overline{k(T)})$, if A is $\mathfrak{S}_F$-nearly continuous then kA is $\overline{k(\mathfrak{S}_F)}$-nearly continuous. F being an $\mathfrak{S}_F$-polar space, $F/A(0)$ is a $\overline{k(\mathfrak{S}_F)}$-polar space by Corollary to Proposition 1. Therefore, by Theorem 3, kA is continuous, that is, A is continuous. Thus the proof is complete.

The following theorem which is an extension of [2, Theorem2′] follows from the previous theorem.

THEOREM 5. *Let $\mathfrak{S}_E$ contain $\mathfrak{U}_E$. If A is a closed $\mathfrak{S}_E$-nearly open linear relation of an $\mathfrak{S}_E$-polar space E onto a locally convex space F, then A is open.*

PROPOSITION 4. *Let $\mathfrak{S}_E$ contain $\mathfrak{U}_E$ and let $u : E \to F$ be a surjective and $\mathfrak{S}_E$-nearly open mapping with a closed graph. If E is an $\mathfrak{S}_E$-polar space, then u is open.*

COROLLARY *If u is a closed linear mapping of a t-polar space onto a barrelled space, then u is open.*

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