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A NOTE ON G-VECTOR BUNDLES

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In the previous papers, [2], [3], the author has given classification theorems for isomorphism classes of G-vector bundles over a G-manifold in the category of continuous G-vector bundles. The technics used there, are justified only for continuous G-vector bundles. But by a theorem which is given here, the classification theorems make sense in the category of differentiable G-vector bundles. Let G be a compact Lie group and M be a compact G-manifold. We denote by $Vect_G^{\partial}(M)$ and $Vect_G^{C}(M)$ the semi-groups of isomorphism classes in the categories of differentiable G-vector bundles and continuous ones, respectively. Any differentiable bundle is also a continuous one, then we have a natural homomorphism $T : Vect_G^{\partial}(M) \longrightarrow Vect_G^{C}(M)$. Then we prove

Theorem. *T is an isomorphism of semi-groups.*

Lemma (An equivariant analogue of 3.12, [4]. *Let M be a compact G-manifold and A be its invariant closed subspace. Let N be an invariant compact submanifold of an Euclidean G-space W. Let $f : M \longrightarrow N$ be an equivariant continuous map which is differentiable on A. Then there exists $\bar{f} : M \longrightarrow N$ such that*

(1) *$\bar{f}$ is differentiable and equivariant*

(2) *$\bar{f}$ is equivariantly homotopic to f*

(3) $\bar{f}\,|A = f|A$.

Proof of the lemma. There exists an invariant neighborhood U of N in W and an equivariant retraction $\rho : U \longrightarrow N$. For any $\varepsilon > 0$, there exists a differentiable map $f' : M \longrightarrow N$ such that $|f(x) - f'(x)| < \varepsilon$ for all $x \in M$ and $f' \mid A = f \mid A$. Then for each $g \in G$ and $x \in M$,

$| gf(g^{-1}x) - gf'(g^{-1}x) \mid < \varepsilon$. Thus

$| f(x) - \int_G gf'(g^{-1}x)dg \mid \leq \int_G \mid gf(g^{-1}x) - gf'(g^{-1}x)dg \mid < \varepsilon.$

Define $\bar{f}(x)$ to be $\int_G gf'(g^{-1}x)dg$, then $\rho(t\bar{f}(x) + (1-t)f(x))$ gives an equivariant homotopy from f to $\bar{f}$.

Proof of Theorem. By 2, [1], for any n-dimensional G-vector bundle $E \longrightarrow M$, there exists a Grassmann G-manifold $G_n(V)$, where V is a G-module, and a continuous map $f : M \longrightarrow G_n(V)$ such that $E \cong f^*K$, where K is the universal G-vector bundle. By [5], there exists a G-module W and an equivariant imbedding $h : G_n(V) \subset M$. Then by the lemma above, we have an equivariant differentiable map $\bar{f} : M \longrightarrow G_n(V)$ such that $f \simeq \bar{f}$, G-homotopic. Thus $f^*K \cong \bar{f}^*K$, which is a smooth G-vector bundle. Hence T is an epimorphism.

Next let E, F be differentiable G-vector bundles and φ be an isomorphism between them as a continuous G-vector bundle. Then φ determines an equivariant continuous section $s(\varphi) : M \longrightarrow Hom(E, F)$. Since M is compact, min $\{| s(x) | ; x \in M\}$, say ε, is positive. By Theorem 6. 7, [6], we have a differentiable section $s' : M \longrightarrow Hom(E, F)$ such that $| s(\varphi)(x) - s'(x) | < \varepsilon/2$ for all $x \in M$, and so $| gs(\varphi)(g^{-1}x) - gs'(g^{-1}x) | < \varepsilon/2$ for all $g \in G$ and $x \in M$, where we choose a G-invariant Hermitian metric in $Hom(E, F)$. Thus $| s(\varphi)(x) - \int_G gs'(g^{-1}x) | dg < \varepsilon/2$. Hence $\bar{s}(x) = \int_G gs'((g^{-1}x)dg$ gives an equivariant differentiable section $\bar{s} : M \longrightarrow Hom(E, F)$, then we have an equivariant differentiable isomorphism $\varphi(\bar{s}) : E \longrightarrow F$, which proves that T is a monomorphism.

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Refernces

[1] M. F. Atiyah and G. Segal, *Equivariant K-Theory*. mimeograph note, Warwick, 1965.
[2] H. Matsunaga, *K_G-groups and invariant vector fields on special G-manifolds*. Osaka Jour. of Math. 9(1), (1972), 143-157.
[3] H. Matsunaga, *On G-vector bundles and invariant vector fields*, Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. 7, (1974)
[4] J. Milnor, *Differential topology*, Lecture note, Princeton, 1958.
[5] R. S. Palais, *Imbedding of Compact Differentiable Transformation Groups in Orthogonal Representations*, Jour. of Math. and Mech. 6(5), (1957), 673-678.
[6] N. E. Steenrod, *The topology of fibre bundles*, Princeton, (195[illegible].