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On the Quotient Topological Ordered Spaces.

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In the theory of general topology, the following theorem is well known (c. f. [2] or [4]). *For a topological space X, and an equivalence relation R on X, if the quotient space X/R is Hausdorff, then R is closed in the product space* X^2. *If the projection p of a space X onto the quotient space X/R is open, and R is closed in* X^2, *then X/R is a Hausdorff space.* The analogy of this theorem in a topological ordered space has been obtained in the case where X is a compact ordered space (c. f. [9] Proposition 9). In this paper, we shall study the sufficient conditions for X/R to be T_2-ordered, and give some examples. For the problem of this kind, S. D. McCartan studied in [6] a particular quotient ordered space (that is, a quotient ordered space by a particular equivalence relation) which inherites some interesting properties of the domain ordered space.

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§1. Preliminaries.

In this section, we shall present some definitions and propositions which are used in the later sections. Let X be a topological space and partially ordered space, then we call X a *topological ordered space.* The partial (or quasi) order is denoted by $\leqq$. Let R be an equivalence relation on X. The topology of the quotient space X/R is the usual quotient topology. Let p be a natural projection of X onto X/R. The order in the quotient X/R is variously considered (c. f. [1] §1 Exercise 2, [6] and [9]). In this paper, we adopt the definition of the order in [9] where it is denoted by $\leqslant_2$, i. e. $p(x) \leqq p(y)$ in X/R if and only if there exist $x' \in p^{-1}(p(x))$, $y' \in p^{-1}(p(y))$ such that $x' \leqq y'$. By this order, X/R is a quasi ordered space, but in general not necessarily a partially ordered space. In a partially ordered space X, for any $x, y \in X$, $x \| y$ means that $x \nleqq y$ and $y \nleqq x$.

Definition 1. (c. f. [7]) Let X be a partially ordered space, then $[x, \rightarrow]$ and $[\leftarrow, x]$ denote the sets $\{y \in X : x \leqq y\}$ and $\{y \in X : y \leqq x\}$ respectively. If $A \subset Y \subset X$, we put $i_Y(A) = \{\bigcup \{[a, \rightarrow] : a \in A\}\} \cap Y$ $d_Y(A) = \{\bigcup \{[\leftarrow, a] : a \in A\}\} \cap Y$. A is said to be *increasing* (*decreasing*) in Y if and only if $A = i_Y(A)$ $(A = d_Y(A))$.

Definition 2. (c. f. [7])Let X be a topological ordered space, then X is said to be T_1-*ordered* (T_2-*ordered*) if and only if for each pair $a, b \in X$ such that $a \not\leq b$, there exist an increasing neighbourhood U of a and a decreasing neighbourhood V of b such that $b \notin U$ and $a \notin V$ ($U \cap V = \phi$).

If X is T_1-ordered (T_2-ordered), then it is clear that X is a T_1-space (Hausdorff space). Also, X is T_2-ordered if and only if the partial order of X is closed, that is, its graph is closed in X^2 (c. f. [8] p. 26 Proposition 1).

In this paper, we use a notion of a *proper mapping*. For this, see [2] §10.

Next, we consider the following conditions in a topological ordered space X.

(C. I) $i_X(K)$ and $d_X(K)$ are closed for each compact set K of X.

(C. II) $i_X(F)$ and $d_X(F)$ are closed for each closed set F of X.

If X is Hausdorff, then (C. II) implies (C. I). The converse of this fact does not hold, even if X is locally compact normal and T_2-ordered. For this, see §3 Example 2. Also, if X is compact, then (C. I) implies (C. II). The converse of this fact does not hold. Indeed, although X in §3 Example 4 is compact, (C. II) does not imply (C. I).

The following propositions are useful in the next section.

Proposition 1. *Let X be a locally compact Hausdorff space. Then, X is T_2-ordered if and only if X satisfies* (C. I).

Proof. The necessity and the sufficiency are clear by [8] p. 44 Proposition 4, and [5] Theorem 3. 3 respectively. *Q. E. D.*

Remark 1. The necessity always holds by [8] p. 44 Proposition 4. However, if X is not locally compact Hausdorff, then the sufficiency does not necessarily hold. For this, see §3 Example 1.

Proposition 2. *Let X be a regular space satisfying* (C. II). *Then, X is T_2-ordered.*

Proof. For $x, y \in X$, $x \not\leq y$, since $[\leftarrow, y]$ is closed, $X-[\leftarrow, y]$ is an increasing open neighbourhood of x. Since X is regular, there exists a closed neighbourhood U of x such that $U \subset X-[\leftarrow, y]$. Then, $i_X(U) \subset X-[\leftarrow, y]$, and $i_X(U)$ is a closed increasing neighbourhood of x. Therefore, $X-i_X(U)$ is an open decreasing neighbourhood of y. Thus, X is T_2-ordered. *Q. E. D.*

Remark 2. The converse of this proposition does not hold. For this, see §3 Example 2.

Remark 3. Let X be a T_2-ordered space satisfying (C. II). Then, the fact that X is regular does not necessarily hold. For this, see §3 Example 3.

Remark 4. This proposition does not hold by merely assuming that X is a T_1-ordered space satisfying (C. II). For this, see §3 Example 4. It remains an open question as to whether the assumption of Proposition 2 may be relaxed to the one that X is a Hausdorff space satisfying (C. II).

§ 2. Main theorems.

In this section, we prove the main theorems. H. A. Priestley proved the following theorem in [9].

Theorem [H. A. Priestley]. *Let X be a compact ordered space. If X/R is a topological ordered space, X/R is a compact ordered space if and only if X/R is a Hausdorff space.*

We study the sufficient conditions for X/R to be T_2-ordered.

Theorem 1. *Let X be a locally compact T_2-ordered space. Assume that p is a proper mapping. If X/R is a topological ordered space, then X/R is a locally compact T_2-ordered space.*

Proof. Since X is locally compact Hausdorff and p is proper, X/R is locally compact Hausdorff by [2] § 10 Proposition 9. Also, by § 1 Proposition 1, X satisfies the condition (C. I). Then, since $i_{X/R}(K) = p(i_X(p^{-1}(K)))$, $d_{X/R}(K) = p(d_X(p^{-1}(K)))$ for each compact set K of X/R, $p^{-1}(K)$ is compact by [2] § 10 Proposition 7, $i_X(p^{-1}(K))$ and $d_X(p^{-1}(K))$ are closed by that X satisfies (C. I), and $p(i_X(p^{-1}(K)))$ and $p(d_X(p^{-1}(K)))$ are closed by the assumption of p, therefore $i_{X/R}(K)$ and $d_{X/R}(K)$ are closed. Thus, X/R satisfies the condition (C. I). By § 1 Proposition 1, X/R is T_2-ordered. *Q. E. D.*

Remark 5. In this theorem, the condition that p is proper is essential. For this, see § 3 Example 5.

Remark 6. In this theorem, the condition that X is T_2-ordered is essential. Indeed, S. D. McCartan showed in [7], Example 6 the existence of a space which is a compact Hausdorff T_1-ordered space but not T_2-ordered space.

Remark 7. This theorem does not hold by merely assuming that p is a closed mapping. Indeed, let X be a locally compact Hausdorff space but not a normal space. (For instance, Tychonoff's example.) Then, by the same way as [3] § 4 Exercise 14 we can construct an equivalence relation R on X such that p is closed but not proper and X/R is not Hausdorff. If we introduce the discrete order as the partial order in X, then we see that Theorem 1 does not hold.

Remark 8. Without p not being proper or X not being T_2-ordered, X/R can be T_2-ordered. For these, see § 3 Example 6 and 7.

Theorem 2. *Let X be a regular space satisfying* (C. II). *Assume that p is a proper mapping. If X/R is a topological ordered space, then X/R is a regular space satisfying* (C. II). *Therefore, X/R is T_2-ordered by* § 1 *Proposition* 2.

Proof. Since X is regular and p is proper, X/R is regular. For this, see [2]

§ 10 Exercise 5 (a) and § 10 Corollaire 4 of Proposition 5. Next, since $i_{X/R}(F) = p(i_X(p^{-1}(F)))$ and $d_{X/R}(F) = p(d_X(p^{-1}(F)))$ for all closed set F of X/R, therefore $i_{X/R}(F)$ and $d_{X/R}(F)$ are closed. Thus, X/R is a regular space satisfying (C. II). *Q. E. D.*

Remark 9. In this theorem, the assumption of p is essential. For this, see § 3 Example 5.

Remark 10. Note that the following fact holds in general. *Let X be a Hausdorff space satisfying* (C. II). *Assume that p is proper. Then, X/R is a Hausdorff space satisfying* (C. II). Therefore, if an open question in Remark 4 be answered in the affirmative, then X/R can be T_2-ordered under the assumption of this remark.

§ 3. Examples.

In this section, we give some examples. We use N to denote the set of all natural numbers.

Example 1. Let X be a real line. We define the topology of countable complements on X by declaring open all sets whose complements are countable, together with ϕ and X. Next, we introduce the discrete order as the partial order in X. By the above topology and order, X is a topological ordered space. Then, we easily see that the only compact sets are finite subsets of X, and a finite subset of X is closed. Therefore, X satisfies the condition (C. I). However X is not T_2-ordered.

Example 2. Let X be a set $\{(a, x, y) : a = 0 \text{ or } 1, x \in [0, 1], y \text{ is a real number}\}$. The topology on X is the usual topology. Next, we define a partial order in X as follows : $(a, x, y) \leqq (b, u, v)$ if and only if $a = 0$, $b = 1$, $x = u \neq 0$, $y = \frac{1}{x}$ or $a = b$, $x = u$, $y = v$. By the above topology and order, X is a locally compact normal space and T_2-ordered. Then, $F = \left\{(0, \frac{1}{n}, n) : n \in N\right\}$ is closed in X, but $i_X(F) = F \cup \left\{(1, \frac{1}{n}, y) : n \in \mathrm{N}, y \text{ is a real number}\right\}$ is not closed. Therefore, X does not satisfy (C. II).

Example 3. Let X be a set $\{(a, x) : a = 0 \text{ or } 1, x \in [0, 1]\}$. We define the topology on X as follows : the neighbourhood system of $(a, 0)$ ($a = 0$ or 1) is $\left\{U_\varepsilon(a, 0) - \left\{(a, \frac{1}{n}) : n \in N\right\} : 0 < \varepsilon < 1, U_\varepsilon(a, 0) \text{ is an open ball of } (a, 0)\right\}$ and the neighbourhoods of other points as usual. Next, we define the partial order on X as follows : $(a, x) \leqq (b, y)$ if and only if $a = 0$, $b = 1$, $x = y = \frac{1}{n}$, $n \in \mathrm{N}$, or $a = b$, $x = y$. By these, X is T_2-ordered and satisfies (C. II).

However, X is not regular.

Example 4. Let X be a countable set. We define the topology of finite complements on X by declaring open all sets whose complements are finite, together with ϕ and X. Next, we introduce the discrete order as the partial order in X. Then, all subsets of X are compact, and all closed subsets of X except ϕ and X are finite subsets. By the above topology and order, X is a T_1-ordered space satisfying (C. II). However, X is not T_2-ordered.

Example 5. Let X be a set $\{(a, x, y) : a = 0$ or $1, x \in [0, 1]$, and y is real number$\}$. We define an equivalence relation R on X as follows : $(a, x, y) R (b, u, v)$ if and only if $a = b$, $x = u$. The topology on X is the usual topology. We define the partial order in X as follows : $(b, u, v) \leqq (a, x, y)$ if and only if $a = 1$, $b = 0$, $x = u \neq 0$, $y = v = \frac{1}{x}$; or $a = b$, $x = u$, $y = v$. Then, X is a locally compact T_2-ordered space satisfying (C. II), but p is not proper. If we denote $p((a, x, y)) = (a, x)^*$, $(0, 0)^* \| (1, 0)^*$ in X/R. Then, there do not exist an increasing neighbourhood U of $(0, 0)^*$ and a decreasing neighbourhood V of $(1, 0)^*$ such that $U \cap V = \phi$. Therefore, X/R is not T_2-ordered.

Example 6. Let X be a real plane. The topology on X is the natural topology. We define a partial order in X as follows : $(u\ v) \leqq (x, y)$ if and only if $y = v = 0$, $u \leqq x$, x and u are real numbers, or $y = v = 1$, $u \leqq x$, x and u are rational numbers, or $x = u$, $y = v$. Next, we define an equivalence relation R on X as follows : $(x, y) R (u, v)$ if and only if $x = u$. Then, X is locally compact Hausdorff T_1-ordered but not T_2-ordered, and p is not proper. However, X/R is T_2-ordered.

Example 7. Let X be $\{a, b, c\}$. We define the topology on X as follows : $\overline{\{a\}} = \{a\}$, $\overline{\{b\}} = \overline{\{c\}} = \{b, c\}$ where $\overline{\{a\}}$ is a closure of $\{a\}$, etc. We introduce the discrete order as the partial order in X. Next, we define an equivalence relation R on X as follows : $x R y$ if and only if $x = y = a$ or $\{x, y\} = \{b, c\}$. Then, X is not even T_1-ordered, however X/R is T_2-ordered.

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