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Some Remarks on Regular Extensions of Cliffordian Semigroups.

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0. Introduction.

One of the authors has shown in [3] the following results :
Let $K = \Sigma\{K_\lambda : \lambda \in \Lambda\}$ be a right Cliffordian semigroup (see [3] ; throughout this paper, any terminology and notation should be referred to [3], unless otherwise stated), and $\Gamma(\Lambda)$ an inverse semigroup having Λ as its basic semilattice[1]. For each $\lambda \in \Lambda$, let I_λ be an $\mathscr{L}$-class of K_λ. Then, $\mathscr{I} = \Sigma\{I_\lambda : \lambda \in \Lambda\}$ (Σ means the disjoint sum) is a lower partial chain of left groups $\{I_\lambda : \lambda \in \Lambda\}$[2]. Since K is a right Cliffordian semigroup, there exists an $\mathscr{R}$-class J_λ of K_λ for each $\lambda \in \Lambda$ such that $\mathscr{E} = \Sigma\{E_\lambda : \lambda \in \Lambda\}$, where E_λ is the set of idempotents of J_λ, is an upper partial chain of right zero semigroups $\{E_\lambda : \lambda \in \Lambda\}$. Now, let u_λ be a representative of K_λ for each $\lambda \in \Lambda$ and put $U = \{u_\lambda : \lambda \in \Lambda\}$. Let $\Delta = \{\alpha_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\} \cup \{\beta_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ be a collection of mappings $\alpha_{(\gamma,\tau)}$ and $\beta_{(\gamma,\tau)}$ such that

(0. 1) (I) $D(\alpha_{(\gamma,\tau)}) = D(\beta_{(\gamma,\tau)}) = E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}}$; $R(\alpha_{(\gamma,\tau)}) \subset I_{\gamma\tau(\gamma\tau)^{-1}}$ and $R(\beta_{(\gamma,\tau)}) \subset E_{(\gamma\tau)^{-1}\gamma\tau}$, where $D(\xi)$, $R(\xi)$ denote the domain and the range of ξ respectively ;

(II) for $q \in E_{\gamma^{-1}\gamma}$, $t \in I_{\tau\tau^{-1}}$, $h \in E_{\tau^{-1}\tau}$ and $v \in I_{\delta\delta^{-1}}$,
$(q, t)\alpha_{(\gamma,\tau)}((q, t)\beta_{(\gamma,\tau)}h, v)\alpha_{(\gamma\tau,\delta)} = (q, t(h, v)\alpha_{(\tau,\delta)})\alpha_{(\gamma,\tau\delta)}$
and $(q, t(h, v)\alpha_{(\tau,\delta)})\beta_{(\gamma,\tau\delta)}(h, v)\beta_{(\tau,\delta)} = ((q, t)\beta_{(\gamma,\tau)}h, v)\beta_{(\gamma\tau,\delta)}$;

(III) for $\gamma \in \Gamma$, $p \in I_{\gamma\gamma^{-1}}$ and $q \in E_{\gamma^{-1}\gamma}$, there exist $k \in I_{\gamma^{-1}\gamma}$ and $n \in E_{\gamma\gamma^{-1}}$ such that $p(q, k)\alpha_{(\gamma,\gamma^{-1})}((q, k)\beta_{(\gamma,\gamma^{-1})}n, p)\alpha_{(\gamma\gamma^{-1},\gamma)} = p$.

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1) The set B of idempotents of an inverse semigroup G constitutes a semilattice. This semilattice B is called the basic semilattice of G. An inverse semigroup G having B as its basic semilattice is sometimes denoted by $G(B)$.

2) Let S be a partial groupoid which is a union of a collection of pairwise disjoint subsemigroups $\{T_\delta : \delta \in \Lambda\}$ where Λ is a semilattice. If $x \in T_\gamma$, $y \in T_\delta$, and $\delta \leq \gamma$ [$\gamma \leq \delta$] (in Λ) imply xy is defined (in S) and $xy \in T_\delta$ [$xy \in T_\gamma$], and, if $\xi \leq \delta$ [$\delta \leq \xi$] and $z \in T_\xi$ imply $(xy)z = x(yz)$, then S is called a lower [upper] partial chain of the semigroups $\{T_\delta : \delta \in \Lambda\}$.

If further the collection $\Omega = \{\alpha_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\} \cup \{\beta_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\}$ satisfies the following

(IV) $u_\lambda jku_\delta = (j, k)\alpha_{(\lambda,\delta)} u_{\lambda\delta} (j, k)\beta_{(\lambda,\delta)}$ for $\lambda, \delta \in \Lambda$, $j \in E_\lambda$ and $k \in I_\delta$, then $S = \{(i, \gamma, j) : \gamma \in \Gamma,\ i \in I_{\gamma\gamma^{-1}},\ j \in E_{\gamma^{-1}\gamma}\}$ becomes a regular extension of $K(\Lambda)$ by $\Gamma(\Lambda)$ under the multiplication defined by

(0. 2) $(i, \gamma, j)(k, \tau, h) = (i(j, k)\alpha_{(\gamma,\tau)}, \gamma\tau, (j, k)\beta_{(\gamma,\tau)}h)$.

This S is denoted by $C(\Gamma, K(\Lambda) ; \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$, and is called a complete regular product of $K(\Lambda)$ and $\Gamma(\Lambda)$. Conversely every regular extension of $K(\Lambda)$ by $\Gamma(\Lambda)$ can be obtained as a complete regular product of $K(\Lambda)$ and $\Gamma(\Lambda)$.

Every element x of K_λ is uniquely expressed in the form

$x = iu_\lambda j,\ i \in I_\lambda,\ j \in E_\lambda$ (see [3]).

Hence, $K = \{iu_\lambda j : \lambda \in \Lambda,\ i \in I_\lambda,\ j \in E_\lambda\}$. We shall call Δ above *a CR-factor set in* $K = \{iu_\lambda j : \lambda \in \Lambda,\ i \in I_\lambda,\ j \in E_\lambda\}$ *belonging to* $\Gamma(\Lambda)$. The semigroup S above will be sometimes called *the regular extension of* $K(\Lambda)$ *determined by* $\Gamma(\Lambda)$ *and* $\{\mathscr{I}, \mathscr{E}, \{u_\lambda\}, \Delta\}$.

As a special case, we next consider the case where each E_λ consists of a single element, say e_λ. If we denote each element $(i, \gamma, e_{\gamma^{-1}\gamma}) \in S = C(\Gamma, K(\Lambda) ; \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ by $[i, \gamma]$ and define $\alpha^*_{(\gamma,\tau)} : I_{\tau\tau^{-1}} \to I_{\gamma\tau(\gamma\tau)^{-1}}$ by $(e_{\gamma^{-1}\gamma}, k)\alpha_{(\gamma,\tau)} = k\,\alpha^*_{(\gamma,\tau)}$ (for each pair (γ, τ) of elements γ, τ of Γ), then the family $\Delta^* = \{\alpha^*_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ satisfies the following :

(0. 1)* (I)* $D(\alpha^*_{(\gamma,\tau)}) = I_{\tau\tau^{-1}}$; $R(\alpha^*_{(\gamma,\tau)}) \subset I_{\gamma\tau(\gamma\tau)^{-1}}$;

(II)* for $t \in I_{\tau\tau^{-1}}$ and $v \in I_{\delta\delta^{-1}}$,

$(t\alpha^*_{(\gamma,\tau)})(v\alpha^*_{(\gamma\tau,\delta)}) = (t(v\alpha^*_{(\tau,\delta)}))\alpha^*_{(\gamma,\tau\delta)}$;

(III)* for $\gamma \in \Gamma$, $p \in I_{\gamma\gamma^{-1}}$, there exists $k \in I_{\gamma^{-1}\gamma}$ such that

$p(k\alpha^*_{(\gamma,\tau)})(p\alpha^*_{(\gamma\gamma^{-1},\gamma)}) = p$.

Further, the family $\Omega^* = \{\alpha^*_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\}$ satisfies

(IV)* $u_\lambda e_\lambda ku_\delta = k\alpha^*_{(\lambda,\delta)} u_{\lambda\delta} e_{\lambda\delta}$ for $\lambda, \delta \in \Lambda$ $k \in I_\delta$.

Now, by using this family Δ^*, the semigroup S above can be expressed as follows :

(0. 3) $\begin{cases} S = \{[i, \gamma] : \gamma \in \Gamma,\ i \in I \quad \}. \\ \text{the multiplication in } S : [i, \gamma][k, \tau] = [i(k\alpha^*_{(\gamma,\tau)}), \gamma\tau]. \end{cases}$

Conversely, let $\Delta^* = \{\alpha^*_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ be a family of mappings satisfying (I)*, (II)*, (III)*, (IV)* of (0. 1)*. Define $\alpha_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \longrightarrow I_{(\gamma\tau)(\gamma\tau)^{-1}}$ and $\beta_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \longrightarrow E_{(\gamma\tau)^{-1}\gamma\tau}$ as follows :

$(e_{\gamma^{-1}\gamma}, k)\alpha_{(\gamma,\tau)} = k\alpha^*_{(\gamma,\tau)}$, $k \in I_{\tau\tau^{-1}}$ and $(e_{\gamma^{-1}\gamma}, k)\beta_{(\gamma,\tau)} = e_{(\gamma\tau)^{-1}\gamma\tau}$, $k \in I_{\tau\tau^{-1}}$.

Then, $\Delta = \{\alpha_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\} \cup \{\beta_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ satisfies (I), (II), (III), (IV) of (0. 1) and hence we can consider the complete regular product $C(\Gamma, K(\Lambda); \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$.

In this system, the product of two elements $(i, \gamma, e_{\gamma^{-1}\gamma}) = [i, \gamma]$ and $(k, \tau, e_{\tau^{-1}\tau}) = [k, \tau]$ is given as follows :

$$\begin{aligned}[i, \gamma][k, \tau] &= (i, \gamma, e_{\gamma^{-1}\gamma})(k, \tau, e_{\tau^{-1}\tau}) = (i(e_{\gamma^{-1}\gamma}, k)\alpha_{(\gamma,\tau)}, \gamma\tau, (e_{\gamma^{-1}\gamma}, k)\beta_{(\gamma,\tau)}\, e_{\tau^{-1}\tau}) \\ &= (i(k\alpha^*_{(\gamma,\tau)}), \gamma\tau, e_{(\gamma\tau)^{-1}\gamma\tau}\, e_{\tau^{-1}\tau}) = (i(k\alpha^*_{(\gamma,\tau)}), \gamma\tau, e_{(\gamma\tau)^{-1}\gamma\tau}) \\ &= [i(k\alpha^*_{(\gamma,\tau)}), \gamma\tau].\end{aligned}$$

Accordingly, when every E_λ consists of a single elements e_λ, a family $\Delta^* = \{\alpha^*_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ of mappings satisfying (I)* (II)*, (III)*, (IV)* of (0. 1)* will be called *a* CR-*factor set in* $K = \{iu_\lambda e_\lambda : i \in I_\lambda, \lambda \in \Lambda\}$ *belonging to* $\Gamma(\Lambda)$. Further, the semigroup S defined by (0. 3) is called *the regular extension of* $K(\Lambda)$ *determined by* $\Gamma(\Lambda)$ *and* $\{\mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \Delta^*\}$ (or *the complete regular product of* $K(\Lambda)$ *and* $\Gamma(\Lambda)$ *determined by* $\{\mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \Delta^*\}$. In this case, S is simply denoted by $C(\Gamma, K(\Lambda); \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$.

Next, let us consider the case where K is a left regular band (that is, a band satisfying the identity $xyx = xy$). In this case, for each $\lambda \in \Lambda$, $I_\lambda = K_\lambda$ and $(J_\lambda =) E_\lambda$ consists of a single element, say e_λ. Hence, every regular extension of $K(\Lambda)$ by $\Gamma(\Lambda)$ can be obtained as a complete regular product $C(\Gamma, K(\Lambda); \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$ of $K(\Lambda)$ and $\Gamma(\Lambda)$. On the other hand, it has been shown by [2] that every regular extension of $K(\Lambda)$ by $\Gamma(\Lambda)$ is obtained as follows :

Let ϕ be a mapping of $\Gamma(\Lambda)$ into the endomorphism semigroup End (K) on K such that the family $\{\sigma_\gamma : \gamma \in \Gamma\}$, where $\sigma_\gamma = \gamma\phi$, satisfies the following (0. 4) and (0. 5) :

(0. 4) Each σ_γ is an endomorphism on K such that $K_\lambda \sigma_\gamma \subset K_{\gamma\lambda(\gamma\lambda)^{-1}}$ for all $\lambda \in \Lambda$. In particular, for $\lambda \in \Lambda$, σ_λ is an inner endomorphism on K.[3)]

(0. 5) $\sigma_\beta \sigma_\alpha \delta_f \delta_e = \sigma_{\alpha\beta} \delta_f \delta_e$ for $e \in K_{\alpha\alpha^{-1}}$, $f \in K_{\alpha\beta(\alpha\beta)^{-1}}$, $\alpha, \beta \in \Gamma$ (where δ_e is the inner endomorphism on K induced by e (see [2])).

Consider the set $K \underset{\phi}{\times} \Gamma$ defined as follows : $K \underset{\phi}{\times} \Gamma = \{(e, \gamma) : \gamma \in \Gamma, e \in K_{\gamma\gamma^{-1}}\}$ and the multiplication in $K \underset{\phi}{\times} \Gamma$ is given by

(0. 6) $(e, \gamma)(f, \tau) = (ef^{\sigma_\gamma}, \gamma\tau)$, where $f^{\sigma_\gamma} = f\sigma_\gamma$.

This $K \underset{\phi}{\times} \Gamma$ is a regular extension of $K(\Lambda)$ by $\Gamma(\Lambda)$ and is called *the* L.H.D.-*product of* $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ *and* $\Gamma(\Lambda)$ *determined by* ϕ (hence, *by*

3) Let K be a regular band (that is, a band satisfying the identity $xyxzx = xyzx$). Let $e \in K$. Then the mapping $\varphi_e : K \to K$ defined by $z\varphi_e = eze$ is an endomorphism on K. Such a φ_e is called an inner endomorphism on K.

$\{\sigma_\gamma : \gamma \in \Gamma\}$). The set $\{\sigma_\gamma\ \gamma \in \Gamma\}$ is called *an* L. H. D.*-factor set in* $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ *belonging to* $\Gamma(\Lambda)$. Conversely, every regular extension of $K(\Lambda)$ by $\Gamma(\Lambda)$ can be obtained in this way.

Now, we have the following problem :

Problem 1. Let $K(\Lambda)$ be a left regular band. For any given complete regular product $C(\Gamma, K(\Lambda); \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\}) = \{[e, \gamma] : \gamma \in \Gamma, e \in K_{\gamma\gamma^{-1}}\}$ of $K(\Lambda)$ and $\Gamma(\Lambda)$, consider how we can construct the L. H. D. -product $K \underset{\phi}{\times} \Gamma = \{(e, \gamma) : \gamma \in \Gamma,\ e \in \dot{K}_{\gamma\gamma^{-1}}\}$ that coincides with $C(\Gamma, K(\Lambda); \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$ when each (e, γ) is identified with $[e, f]$ (i. e. $K \underset{\phi}{\times} \Gamma$ such that $\Phi : C(\Gamma, K(\Lambda); \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\}) \longrightarrow K \underset{\phi}{\times} \Gamma$ defined by $[e, \gamma]\Phi = (e, \gamma)$ gives an isomorphism). Consider also the converse.

In the next section, we shall discuss this problem.

Next, let us also consider the case where both K and Γ are groups. In this case, K is of course right Cliffordian and $\Lambda = \{1\}$, $I_1 = K$ and $E_1 = \{1\}$. Hence, every regular extension of K by Γ can be obtained as a complete regular product $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{\alpha^*_{(\gamma,\tau)}\})$ of $K(\{1\})$ and $\Gamma(\{1\})$. On the other hand, it is well-known from the group theory that every regular extension of K by Γ can be obtained as a Schreier extension of K by Γ. That is : Let $\varphi : \Gamma \longrightarrow \mathrm{Aut}(K)$ be a mapping of Γ into the group of automorphisms of K and $C(\gamma, \tau)$ an element of K for each pair (γ, τ) of elements of Γ, such that

(0. 7) $(a^{\tilde{\gamma}})^{\tilde{\tau}} = C(\tau, \gamma) a^{\widetilde{\tau\gamma}} C(\tau, \gamma)^{-1}$ and

(0. 8) $C(\gamma, \tau)C(\gamma\tau, \delta) = C(\tau, \delta)^{\tilde{\gamma}} C(\gamma, \tau\delta)$

where $\tilde{\gamma} = \gamma\varphi$ for $\gamma \in \Gamma$ and $a^{\tilde{\gamma}} = a\tilde{\gamma}$.

Then, $S = \{(a, \gamma) : \gamma \in \Gamma, a \in K\}$ become a regular extension of K by Γ under the multiplication defined by $(a, \gamma)(b, \tau) = (ab^{\tilde{\gamma}}C(\gamma, \tau), \gamma\tau)$. Further, every extension of K by Γ can be obtained in this way. The system $\{\tilde{\gamma}, C(\gamma, \tau)\}$ is called *a factor set in K belonging to* Γ, and the S above is called *the Schreier extension of K determined by* Γ *and* $\{\tilde{\gamma}, C(\gamma, \tau)\}$.

Now, we have another problem as follows :

Problem 2. Let K, Γ be groups. For any given complete regular product $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\}) = \{[e, \gamma] : \gamma \in \Gamma, e \in K\}$ consider how we can construct the Schreier extension $S = \{(e, \gamma) : \gamma \in \Gamma, e \in K\}$ of K by Γ that coincides with $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$ when each element (e, γ) of S is identified with $[e, \gamma]$ (i. e., S such that $\Phi : C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\}) \longrightarrow S$ defined by $[e, \gamma]\Phi = (e, \gamma)$ gives an isomorphism). Consider also the converse.

We shall investigate the two problems above in the following sections.

1. CR-factor sets and L. H. D. -factor sets.

In this section, we discuss the problem 1. Let $\Gamma(\Lambda)$ be an inverse semigroup having Λ as its basic scmilattice, and $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ a left regular band[4]. Let $C(\Gamma, K(\Lambda); \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$ be a complete regular product of $K(\Lambda)$ and $\Gamma(\Lambda)$. In this case, of course $\mathscr{I} = K = \Sigma\{K_\lambda : \lambda \in \Lambda\}$.

The following (1. 1) is obvious from (IV)* of (0. 1)* :

(1. 1) $u_\lambda k u_\tau = k\alpha^*_{(\lambda,\tau)} u_{\lambda\tau} = k\alpha^*_{(\lambda,\tau)}$ for $\lambda, \tau \in \Lambda$, $k \in K_\tau$.

For every $\gamma \in \Gamma$, define $\tilde{\gamma}$ as follows :

(1. 2) $e^{\tilde{\gamma}} = e\alpha^*_{(\gamma,\tau)}$ if $e \in K_{\tau\tau^{-1}}$.

Such $\tilde{\gamma}$ is well-defined : Suppose that $i \in K_{\tau_1\tau_1^{-1}} = K_{\tau_2\tau_2^{-1}}$ and $j \in K_{\tau_1^{-1}\tau_1}$. By (II)*, we have $(i\alpha^*_{(\gamma,\tau_1)})(j\alpha^*_{(\gamma\tau_1,\tau_1^{-1})}) = (i(j\alpha^*_{(\tau_1,\tau_1^{-1})}))\alpha^*_{(\gamma,\tau_1\tau_1^{-1})}$. Since $j\alpha^*_{(\gamma\tau_1,\tau_1^{-1})} \in K_{\gamma\tau_1(\gamma\tau_1^{-1})}$, the left side of the equation above is equal to $i\alpha^*_{(\gamma,\tau_1)}$. Since $j\alpha^*_{(\tau_1,\tau_1^{-1})} \in K_{\tau_1\tau_1^{-1}}$, the right side is equal to $i\alpha^*_{(\gamma,\tau_1\tau_1^{-1})}$. Hence $i\alpha^*_{(\gamma,\tau_1)} = i\alpha^*_{(\gamma,\tau_1\tau_1^{-1})})$. Similarly, we have $i\alpha^*_{(\gamma,\tau_2)} = i\alpha^*_{(\gamma,\tau_2\tau_2^{-1})}$. Since $\tau_1\tau_1^{-1} = \tau_2\tau_2^{-1}$, $i\alpha^*_{(\gamma,\tau_1)} = i\alpha^*_{(\gamma,\tau_2)}$. Thus, $\tilde{\gamma}$ is well-defined.

Lemma 1. 1. For $\lambda \in \Lambda$, $\tilde{\lambda}$ is an inner endomorphism on K, especially $f^{\tilde{\lambda}} = u_\lambda f$ for $f \in K$.

Proof. Let f be an element of K_τ, $\tau \in \Lambda$. Since K_λ is a left zero semigroup we have $f^{\tilde{\lambda}} = f\alpha^*_{(\lambda,\tau)} = u_\lambda f u_\tau$ (by (1. 1)) $= u_\lambda f$.

Lemma 1. 2. For $e \in K_{\gamma\gamma^{-1}}$, $f \in K_{\gamma\tau(\gamma\tau)^{-1}}$, where $\gamma, \tau \in \Gamma$, $ef = e(u_{\gamma^{-1}\gamma} f^{\widetilde{\gamma^{-1}}})^{\tilde{\gamma}}$.

Proof. $ef = eu_{\gamma\gamma^{-1}}f = ef^{\widetilde{\gamma\gamma^{-1}}} = e(f\alpha^*_{(\gamma\gamma^{-1},\gamma\tau)}) = e(u_{\gamma^{-1}\gamma}\alpha^*_{(\gamma,\gamma^{-1})})(f\alpha^*_{(\gamma\gamma^{-1},\gamma\tau)})$ (since $u_{\gamma^{-1}\gamma}\alpha^*_{(\gamma,\gamma^{-1})} \in K_{\gamma\gamma^{-1}}$) $= e((u_{\gamma^{-1}\gamma}(f\alpha^*_{(\gamma^{-1},\gamma\tau)}))\alpha^*_{(\gamma,\gamma^{-1}\gamma\tau)})$ (by (II)*) $= e(u_{\gamma^{-1}\gamma} f^{\widetilde{\gamma^{-1}}})^{\tilde{\gamma}}$.

Lemma 1. 3. Each $\tilde{\gamma}$ is an endomorphism on K.

Proof. Let $e \in K_\tau$ and $f \in K_\delta$. $(ef)^{\tilde{\gamma}} = (ef)\alpha^*_{(\gamma,\tau\delta)} = (eu_\tau f)\alpha^*_{(\gamma,\tau\delta)}$
$= (e(f\alpha^*_{(\tau,\delta)}))\alpha^*_{(\gamma\tau,\delta)} = (e\alpha^*_{(\gamma,\tau)})(f\alpha^*_{(\gamma\tau,\delta)})$ (by (II)*) $= e^{\tilde{\gamma}}(u_{\gamma\gamma^{-1}}\alpha^*_{(\gamma\tau(\gamma\tau)^{-1},\gamma)})(f\alpha^*_{(\gamma\tau,\delta)})$
(since $e^{\tilde{\gamma}}, u_{\gamma\gamma^{-1}}\alpha^*_{(\gamma\tau(\gamma\tau)^{-1},\gamma)} \in K_{\gamma\tau(\gamma\tau)^{-1}}$) $= e^{\tilde{\gamma}}((u_{\gamma\gamma^{-1}}(f\alpha^*_{(\gamma,\delta)}))\alpha^*_{(\gamma\tau(\gamma\tau)^{-1},\gamma\delta)})$
$= e^{\tilde{\gamma}}(u_{\gamma\gamma^{-1}}(f\alpha^*_{(\gamma,\delta)}))^{\widetilde{\gamma\tau(\gamma\tau)^{-1}}} = e^{\tilde{\gamma}}u_{\gamma\tau(\gamma\tau)^{-1}}(u_{\gamma\gamma^{-1}}(f\alpha^*_{(\gamma,\delta)}))$ (by Lemma 1. 1)
$= e^{\tilde{\gamma}}u_{\gamma\gamma^{-1}}f^{\tilde{\gamma}} = e^{\tilde{\gamma}}f^{\tilde{\gamma}}$ (since $e^{\tilde{\gamma}} \in K_{\gamma\tau(\gamma\tau)^{-1}}$ and hence $e^{\tilde{\gamma}}u_{\gamma\gamma^{-1}} \in K_{\gamma\tau(\gamma\tau)^{-1}}$).

Lemma 1. 4. For $e \in K_{\gamma\gamma^{-1}}$, $f \in K_{\gamma\tau(\gamma\tau)^{-1}}$,

$$\tilde{\tau}\tilde{\gamma}\delta_f\delta_e = \widetilde{\gamma\tau}\delta_f\delta_e$$

where δ_e means the left multiplication on K determined by e.

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4) In this case, each K_λ is a left zero semigroup.

Proof. Let $g \in K_\delta$. $g^{\widetilde{\gamma\tau}\delta f\delta e} = ef(g\alpha^*_{(\gamma\tau,\delta)}) = e(u_{\gamma^{-1}\gamma}f^{\tilde{\gamma}^{-1}})^{\tilde{\gamma}}(g\alpha^*_{(\gamma\tau,\delta)})$
$= e((u_{\gamma^{-1}\gamma}f^{\tilde{\gamma}^{-1}})\,\alpha^*_{(\gamma,\gamma^{-1}\gamma\tau)})\,(g\alpha^*_{(\gamma\tau,\delta)}) = e((u_{\gamma^{-1}\gamma}f^{\tilde{\gamma}^{-1}}(g\alpha^*_{(\gamma^{-1}\gamma\tau,\delta)}))\alpha^*_{(\gamma,\gamma^{-1}\gamma\tau\delta)})$ (by (II)*)
$= e(u_{\gamma^{-1}\gamma}f^{\tilde{\gamma}^{-1}}(g\alpha^*_{(\gamma^{-1}\gamma,\tau\delta)}))^{\tilde{\gamma}}$. If we can prove the equation $f^{\tilde{\gamma}^{-1}}(g\alpha^*_{(\gamma^{-1}\gamma\tau,\delta)})$
$= f^{\tilde{\gamma}^{-1}}g^{\tilde{\tau}}$, we can obtain the lemma. Hence, we next prove the equation above.
$f^{\tilde{\gamma}^{-1}}(g\alpha^*_{(\gamma^{-1}\gamma\tau,\delta)}) = f^{\tilde{\gamma}^{-1}}u_{\gamma^{-1}\gamma}(u_{\tau\tau^{-1}}\alpha^*_{(\gamma^{-1}\gamma,\tau)})(g\alpha^*_{(\gamma^{-1}\gamma\tau,\delta)})$
$= f^{\tilde{\gamma}^{-1}}u_{\gamma^{-1}\gamma}((u_{\tau\tau^{-1}}(g\alpha^*_{(\tau,\delta)}))\alpha_{(\gamma^{-1}\gamma,\tau\delta)})$ (by (II)*) $= f^{\tilde{\gamma}^{-1}}u_{\gamma^{-1}\gamma}(u_{\tau\tau^{-1}}(g\alpha^*_{(\tau,\delta)}))^{\widetilde{\gamma^{-1}\gamma}}$
$= f^{\tilde{\gamma}^{-1}}u_{\gamma^{-1}\gamma}u_{\gamma^{-1}\gamma}(u_{\tau\tau^{-1}}(g\alpha^*_{(\tau,\delta)})) = f^{\tilde{\gamma}^{-1}}g^{\tilde{\tau}}$ (since $f^{\tilde{\gamma}^{-1}}u_{\gamma^{-1}\gamma}u_{\gamma^{-1}\gamma}u_{\tau\tau^{-1}} = f^{\tilde{\gamma}^{-1}}$).

Let us define $\phi : \Gamma \longrightarrow \mathrm{End}\,(K)$ by $\gamma\phi = \tilde{\gamma}$. Then, the system $\{\tilde{\gamma} : \gamma \in \Gamma\}$ is an L. H. D. -factor set in $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ belonging to $\Gamma(\Lambda)$. Hence, we can consider the L. H. D. -product $K \underset{\phi}{\times} \Gamma = \{(e, \gamma) : \gamma \in \Gamma,\ e \in K_{\gamma\gamma^{-1}}\}$ of $K(\Lambda)$ and $\Gamma(\Lambda)$ determined by ϕ.

Theorem 1. 1. $K \underset{\phi}{\times} \Gamma$ *coincides with* $C(\Gamma, K(\Lambda)\,; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}; \{\alpha^*_{(\gamma,\tau)}\})$ *if each element* (e, γ) *of* $K \underset{\phi}{\times} \Gamma$ *is identified with* $[e, \gamma] \in C(\Gamma, K(\Lambda)\,; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$.

Proof. Take two elements (e, γ), (f, τ) from $K \underset{\phi}{\times} \Gamma$. Assume that $(h, \xi) \equiv [h, \xi]$ for each $(h, \xi) \in K \underset{\phi}{\times} \Gamma$. Then, $(e, \gamma)\,(f, \tau)$ (in $K \underset{\phi}{\times} \Gamma$) $= (ef^{\tilde{\gamma}}, \gamma\tau) \equiv [ef^{\tilde{\gamma}}, \gamma\tau] = [e(f\alpha^*_{(\gamma,\tau)}), \gamma\tau] = [e, \gamma]\,[f, \tau]$ (in $C(\Gamma, K(\Lambda)\,; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$).

Conversely, let $K \underset{\phi}{\times} \Gamma$ be a given L. H. D. -product of $K(\Lambda)$ and $\Gamma(\Lambda)$ (where ϕ is a mapping of Γ into End (K) which satisfies (0. 4), (0. 5)). Put $\gamma\phi = \tilde{\gamma}$. Define $\alpha^*_{(\gamma,\tau)} : K_{\tau\tau^{-1}} \longrightarrow K_{\gamma\tau(\gamma\tau)^{-1}}$ by $e\alpha^*_{(\gamma,\tau)} = e^{\tilde{\gamma}}$. Then,

Lemma 1. 5. $\Delta^* = \{\alpha^*_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ *satisfies* (I)*, (II)*, (III)* *and* (IV)* *with respect to a system* $\{u_\lambda : \lambda \in \Lambda\}$ *of elements of* K_λ*'s such that* $e^{\tilde{\lambda}} = u_\lambda e u_\lambda$, $e \in K$, $\lambda \in \Lambda$.[5]

Proof. It is obvious that Δ^* satisfies (I)* and (III)*. Next, we show that Δ^* satisfies (II)*. For $t \in K_{\tau\tau^{-1}}$ and $v \in K_{\delta\delta^{-1}}$, $(t\alpha^*_{(\gamma,\tau)})\,(v\alpha^*_{(\gamma\tau,\delta)}) = t^{\tilde{\gamma}}v^{\widetilde{\gamma\tau}} = t^{\tilde{\gamma}}e_{\gamma\tau(\gamma\tau)^{-1}}e_{\gamma\gamma^{-1}}v^{\widetilde{\gamma\tau}}$ (where e_λ is a fixed element of K_λ for each $\lambda \in \Lambda$) $= t^{\tilde{\gamma}}e_{\gamma\tau(\gamma\tau)^{-1}}e_{\gamma\gamma^{-1}}(v^{\tilde{\tau}})^{\tilde{\gamma}}$ (by (0. 5)) $= t^{\tilde{\gamma}}(v^{\tilde{\tau}})^{\tilde{\gamma}} = (tv^{\tilde{\tau}})^{\tilde{\gamma}}$ (since $\tilde{\gamma}$ is an endomorphism) $= (t(v\alpha^*_{(\tau,\delta)}))\alpha^*_{(\gamma,\tau\delta)}$. Finally, we prove that Δ^* satisfies (IV)*. Since $\tilde{\lambda}$ is an inner endomorphism for each $\lambda \in \Lambda$, there exists $u_\lambda \in K_\lambda$ such that $e^{\tilde{\lambda}} = u_\lambda e u_\lambda = u_\lambda e$ for all $e \in K$. We need only to show that $u_\lambda k u_\tau = k\alpha^*_{(\lambda,\tau)}$ for all $\lambda, \tau \in \Lambda$ and for all $k \in K_\tau$. Now, $k\alpha^*_{(\lambda,\tau)} = k^{\tilde{\lambda}} = u_\lambda k u_\lambda = u_\lambda k = u_\lambda k u_\tau$.

By the lemmas above, we can consider the complete regular product $C(\Gamma, K(\Lambda)\,; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\}) = \{[e, \gamma] : \gamma \in \Gamma,\ e \in K_{\gamma\gamma^{-1}}\}$.

Theorem 1. 2. $C(\Gamma, K(\Lambda)\,; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$ *coincides with the given*

5) By the definition of $K \underset{\phi}{\times} \Gamma$, $\tilde{\lambda}$ is an inner endomorphism for each $\lambda \in \Lambda$. Hence, there exists $u_\lambda \in K_\lambda$ such that $e^{\tilde{\lambda}} = u_\lambda e u_\lambda$ for all $e \in K$ (see also [2]).

$K \underset{\phi}{\times} \Gamma = \{(e, \gamma) : \gamma \in \Gamma, e \in K_{\gamma\gamma^{-1}}\}$ *if each element* $[e, \gamma]$ *of* $C(\Gamma, K(\Lambda) ; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$ *is identified with* $(e, \gamma) \in K \underset{\phi}{\times} \Gamma$.

Proof. Take two elements $[e, \gamma]$, $[f, \tau]$ from $C(\Gamma, K(\Lambda) ; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$. Identify every $[h, \delta] \in C(\Gamma, K(\Lambda) ; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$ with $(h, \delta) \in K \underset{\phi}{\times} \Gamma$. $[e, \gamma][f, \tau]$ (in $C(\Gamma, K(\Lambda) ; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\})$) $= [e(f\alpha^*_{(\gamma,\tau)}), \gamma\tau] \equiv (e(f\alpha^*_{(\gamma,\tau)}), \gamma\tau) = (ef^{\bar{\gamma}}, \gamma\tau) = (e, \gamma)(f, \tau)$ (in $K \underset{\phi}{\times} \Gamma$).

Corollary. Let $\Gamma(\Lambda)$ *be an inverse semigroup having* Λ *as its basic semilattice, and* $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ *a left regular band (hence, each* K_λ *is a left zero semigroup). Let* $C(\Gamma, K(\Lambda) ; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\}) = \{[e, \gamma] : e \in K_{\gamma\gamma^{-1}}, \gamma \in \Gamma\}$ *be a complete regular product, and* $K \underset{\phi}{\times} \Gamma = \{(e, \gamma) : e \in K_{\gamma\gamma^{-1}}, \gamma \in \Gamma\}$ *an L. H. D.-product of* $K(\Lambda)$ *and* $\Gamma(\Lambda)$. *Let* $\bar{\gamma} = \gamma\phi$ *for each* $\gamma \in \Gamma$. *If* $\{\alpha^*_{(\gamma,\tau)} ; \gamma, \tau \in \Gamma\}$ *and* $\{\bar{\gamma} : \gamma \in \Gamma\}$ *satisfy*

(1. 3) $k^{\bar{\gamma}} = k\alpha^*_{(\gamma,\tau)}$ *for all* $\gamma, \tau \in \Gamma$ *and all* $k \in K_{\tau\tau^{-1}}$,

then $C(\Gamma, K(\Lambda) ; \mathscr{I}, \{e_\lambda\}, \{u_\lambda\}, \{\alpha^*_{(\gamma,\tau)}\}$ *and* $K \underset{\phi}{\times} \Gamma$ *are the same system if each* $[e, \gamma]$ *is identified with* (e, γ).

2. Group extensions.

In this section, we investigate the second problem. Let Γ, K be groups. The basic semilattice of Γ consists of only one element 1 (the identity of Γ). Further, it is easy to see that in this case, $K = K_1 = I_1$ (an $\mathscr{L}$-elass of K_1) and E_1 (the set of idempotents of an $\mathscr{R}$-class of $K = K_1$) $= 1$ (the identity of K). Now, let $C(\Gamma, K(\{1\}) ; \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\}) = \{[e, \gamma] : \gamma \in \Gamma, e \in K\}$ be a complete regular product of $K(\{1\})$ and $\Gamma(\{1\})$. Put $1\alpha^*_{(\gamma,\tau)} = C(\gamma, \tau)$. First, we have

Lemma 2. 1. $x\alpha^*_{(1,\delta)} = u_1 x$ *for all* $\delta \in \Gamma$ *and for* $x \in K$.

Proof. For $t \in K$, $u_1 t u_1 = t\alpha^*_{(1,1)} u_1$ (by (IV)*). Hence $u_1 t = t\alpha^*_{(1,1)}$. By (II)*,

$$(t\alpha^*_{(1,1)})(v\alpha^*_{(1,\delta)}) = (t(v\alpha^*_{(1,\delta)}))\alpha^*_{(1,\delta)}.$$

Hence, $u_1 t(v\alpha^*_{(1,\delta)}) = (t(v\alpha^*_{(1,\delta)}))\alpha^*_{(1,\delta)}$. Since $\{t(v\alpha^*_{(1,\delta)}) : t, v \in K\} = K$, $u_1 x = x\alpha^*_{(1,\delta)}$ for $x \in K$.

Putting $\tau = 1$, $v = 1$ in (II)*, we have

$$(t\alpha^*_{(\gamma,1)})(1\alpha^*_{(\gamma,\delta)}) = (t(1\alpha^*_{(1,\delta)}))\alpha^*_{(\gamma,\delta)}.$$

Hence, $t\alpha^*_{(\gamma,1)} C(\gamma, \delta) = (tu_1)\alpha^*_{(\gamma,\delta)}$. Let $t = yu_1^{-1}$. Then, $(yu_1^{-1})\alpha^*_{(\gamma,1)} C(\gamma, \delta) = y\alpha^*_{(\gamma,\delta)}$. Therefore, for any $a \in K$,

$$a\alpha^*_{(\gamma,\delta)} C(\gamma, \delta)^{-1} = (au_1^{-1})\alpha^*_{(\gamma,1)} C(\gamma, \delta) C(\gamma, \delta)^{-1} = (au_1^{-1})\alpha^*_{(\gamma,1)}.$$

Thus, $a\alpha^*_{(\gamma,\delta)} C(\gamma, \delta)^{-1}$ does not depend on the selection of δ. Hence, we can define $\bar{\gamma} : K \longrightarrow K$ as follows :

(2. 1) $a^{\tilde{\gamma}} = a\alpha^*_{(\gamma,\delta)}C(\gamma, \delta)^{-1} = (au_1^{-1})\alpha^*_{(\gamma,1)}$, $a \in K$.

Lemma 2. 2. For $\gamma \in \Gamma$, $\tilde{\gamma}$ is an automorphism on K.

Proof. Put $t = au_1^{-1}$, $v = bu_1^{-1}$ and $\tau = \delta = 1$ in (II)*. Then,

$$(au_1^{-1})\alpha^*_{(\gamma,1)}(bu_1^{-1})\alpha^*_{(\gamma,1)} = ((au_1^{-1})((bu_1^{-1})\alpha^*_{(1,1)}))\alpha^*_{(\gamma,1)}.$$

Since $(bu_1^{-1})\alpha^*_{(1,1)} = u_1bu_1^{-1}$ by Lemma 2. 1,

$$((au_1^{-1})\alpha^*_{(\gamma,1)})((bu_1^{-1})\alpha^*_{(\gamma,1)}) = (abu_1^{-1})\alpha^*_{(\gamma,1)}.$$

That is, $a^{\tilde{\gamma}}b^{\tilde{\gamma}} = (ab)^{\tilde{\gamma}}$. Next, let y be an element of K.

Put $u_1^{-1}C(\gamma, \gamma^{-1})^{-1}y^{-1} = p$. By (III)*, there exists $k \in K$ such that $p(k\alpha^*_{(\gamma,\gamma^{-1})})(p\alpha^*_{(1,\gamma)}) = p$. Now,

$$\begin{aligned} k^{\tilde{\gamma}} = (ku_1^{-1})\alpha^*_{(\gamma,1)} &= k\alpha^*_{(\gamma,\gamma^{-1})}C(\gamma, \gamma^{-1})^{-1} \\ &= (p\alpha^*_{(1,\gamma)})^{-1}C(\gamma,\gamma^{-1})^{-1} = (C(\gamma, \gamma^{-1})p\alpha^*_{(1,\gamma)})^{-1} \\ &= (C(\gamma, \gamma^{-1})u_1p)^{-1} = (y^{-1})^{-1} = y. \end{aligned}$$

Hence, $\tilde{\gamma}$ is an onto-mapping. Next, assume $a^{\tilde{\gamma}} = b^{\tilde{\gamma}}$ for elements $a, b \in K$.

Then, $(au_1^{-1})\alpha^*_{(\gamma,1)} = (bu_1^{-1})\alpha^*_{(\gamma,1)}$. By (II)*, we have

$$(1\alpha^*_{(\gamma^{-1},\gamma)})(v\alpha^*_{(1,1)}) = (v\alpha^*_{(\gamma,1)})\alpha^*_{(\gamma^{-1},\gamma)}.$$

Hence, we have

$$(1\alpha^*_{(\gamma^{-1},\gamma)})((au_1^{-1})\alpha^*_{(1,1)}) = (1\alpha^*_{(\gamma^{-1},\gamma)})((bu_1^{-1})\alpha^*_{(1,1)})$$

and hence $(au_1^{-1})\alpha^*_{(1,1)} = (bu_1^{-1})\alpha^*_{(1,1)}$. Therefore, $u_1au_1^{-1} = u_1bu_1^{-1}$.

Consequently, $a = b$. Thus, $\tilde{\gamma}$ is 1—1.

Lemma 2. 3. (1) $(a^{\tilde{\sigma}})^{\tilde{\tau}} = C(\tau, \sigma)a^{\widetilde{\tau\sigma}}C(\tau, \sigma)^{-1}$.

(2) $C(\sigma, \tau)C(\sigma\tau, \rho) = C(\tau, \rho)^{\tilde{\sigma}}C(\sigma, \tau\rho)$.

Proof. By (II)*,

$$\begin{aligned} (1\alpha^*_{(\tau,\sigma)})((au_1^{-1})\alpha^*_{(\tau\sigma,1)}) &= ((au_1^{-1})\alpha^*_{(\sigma,1)})\alpha^*_{(\tau,\sigma)} \\ &= (((au_1^{-1})\alpha^*_{(\sigma,1)}u_1^{-1})\alpha^*_{(\tau,1)})(1\alpha^*_{(\tau,\sigma)}). \end{aligned}$$

Hence,
$$\begin{aligned} (a^{\tilde{\sigma}})^{\tilde{\tau}} = ((au_1^{-1})\alpha^*_{(\sigma,1)})^{\tilde{\tau}} &= ((au_1^{-1})\alpha^*_{(\sigma,1)}u_1^{-1})\alpha^*_{(\tau,1)} \\ &= (1\alpha^*_{(\tau,\sigma)})((au_1^{-1})\alpha^*_{(\tau\sigma,1)})(1\alpha^*_{(\tau,\sigma)})^{-1} \\ &= C(\tau, \sigma)a^{\widetilde{\tau\sigma}}C(\tau, \sigma)^{-1}. \end{aligned}$$

Thus, we obtained (1). Next, we prove (2).

By (II)*, $(1\alpha^*_{(\sigma,\tau)})(1\alpha^*_{(\sigma\tau,\rho)}) = (1\alpha^*_{(\tau,\rho)})\alpha^*_{(\sigma,\tau\rho)}$.

Hence,
$$\begin{aligned} C(\sigma, \tau)C(\sigma\tau, \rho) &= (1\alpha^*_{(\sigma,\tau)})(1\alpha^*_{(\sigma\tau,\rho)}) \\ &= (1\alpha^*_{(\tau,\rho)})\alpha^*_{(\sigma,\tau\rho)} = (C(\tau, \rho))\alpha^*_{(\sigma,\tau\rho)} \\ &= (C(\tau, \rho))\alpha^*_{(\sigma,\tau\rho)}C(\sigma, \tau\rho)^{-1}C(\sigma, \tau\rho) \\ &= (C(\tau, \rho))^{\tilde{\sigma}}C(\sigma, \tau\rho). \end{aligned}$$

By the lemma above, the system $\{\tilde{\sigma}, C(\sigma, \tau)\}$ is a factor set in K belonging to Γ. Hence, we can obtain the Schreier extension

$S=\{(a,\gamma): a\in K, \gamma\in\Gamma\}$ of K by Γ and $\{\tilde{\sigma}, C(\sigma,\tau)\}$.

Theorem 2. 1 *S coincides with the given $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$ if each element (a,γ) of S is identified with $[a,\gamma]$.*

Proof. Let (a,γ), (b,τ) be elements of S.

$$
\begin{aligned}
(a,\gamma)(b,\tau)\ (\text{in } S) &= (ab^{\tilde{\gamma}}C(\gamma,\tau), \gamma\tau) \equiv [ab^{\tilde{\gamma}}C(\gamma,\tau), \gamma\tau] \\
&= [a(b\alpha^*_{(\gamma,\tau)}), \gamma\tau] = [a,\gamma][b,\tau] \\
&(in\ C(\Gamma, K(\{1\}; \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})).
\end{aligned}
$$

Conversely, let $\{\tilde{\sigma}, C(\sigma,\tau)\}$ be a factor set in K belonging to Γ and $S=\{(a,\gamma): \gamma\in\Gamma, a\in K\}$ the Schreier extension of K determined by Γ and $\{\tilde{\sigma}, C(\sigma,\tau)\}$. Put $C(1,1)=u_1$, and define $\alpha^*_{(\gamma,\tau)}: K\longrightarrow K$ for each pair (γ,τ) of elements of Γ as follows : $a\alpha^*_{(\gamma,\tau)}=a^{\tilde{\gamma}}C(\gamma,\tau)$, $a\in K$. Then,

Lemma 2. 4. The system $\{\{\alpha^*_{(\gamma,\tau)}: \gamma,\tau\in\Gamma\}, \{u_1\}\}$ satisfies (I)* $\sim$ (IV)*.

Proof. It is obvious that the system satisfies (I)*.

(II)* : For $t, v\in K$,

$$(t\alpha^*_{(\gamma,\tau)})(v\alpha^*_{(\gamma\tau,\delta)}) = t^{\tilde{\gamma}}C(\gamma,\tau)v^{\widetilde{\gamma\tau}}C(\gamma\tau,\delta).$$

On the other hand,

$$
\begin{aligned}
(t(v\alpha^*_{(\tau,\delta)}))\alpha^*_{(\gamma,\tau\delta)} &= (t(v^{\tilde{\tau}}C(\tau,\delta)))^{\tilde{\gamma}}C(\gamma,\tau\delta) \\
&= t^{\tilde{\gamma}}(v^{\tilde{\tau}})^{\tilde{\gamma}}C(\tau,\delta)^{\tilde{\gamma}}C(\gamma,\tau\delta) \\
&= t^{\tilde{\gamma}}C(\gamma,\tau)v^{\widetilde{\gamma\tau}}C(\gamma,\tau)^{-1}C(\tau,\delta)^{\tilde{\gamma}}C(\gamma,\tau\delta).
\end{aligned}
$$

Since $C(\tau,\delta)^{\tilde{\gamma}}C(\gamma,\tau\delta)=C(\gamma,\tau)C(\gamma\tau,\delta)$, we have $C(\tau,\delta)^{\tilde{\gamma}}=C(\gamma,\tau)C(\gamma\tau,\delta)C(\gamma,\tau\delta)^{-1}$. Hence, $(t\alpha^*_{(\gamma,\tau)})(v\alpha^*_{(\gamma\tau,\delta)})=(t(v\alpha^*_{(\tau,\delta)}))\alpha^*_{(\gamma,\tau\delta)}$.

(III)* : Let $\gamma\in\Gamma$ and $p\in K$. Since $\tilde{\gamma}$ is an onto-mapping, there exists k such that $k^{\tilde{\gamma}}C(\gamma,\gamma^{-1})p^{\tilde{1}}C(1,\gamma)=1$. Hence, $(k\alpha^*_{(\gamma,\gamma^{-1})})(p\alpha^*_{(1,\gamma)})=1$. That is, $p(k\alpha^*_{(\gamma,\gamma^{-1})})(p\alpha^*_{(1,\gamma)})=p$.

(IV)* : Let a $\in K$. In the equation $(a^{\tilde{\gamma}})^{\tilde{\tau}}=C(\tau,\gamma)a^{\widetilde{\tau\gamma}}C(\tau,\gamma)^{-1}$, put $\tau=1$ and $\gamma=1$. Then, $a=C(1,1)aC(1,1)^{-1}$. Hence, $aC(1,1)=C(1,1)a$. Now, for $k\in K$, $u_1ku_1=C(1,1)ku_1=kC(1,1)u_1=k^{\tilde{1}}C(1,1)u_1=k\alpha^*_{(1,1)}u_1$. Thus, (IV)* is satisfied.

By the lemma above, we can consider $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$.

Theorem 2. 2. $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})=\{[a,\gamma]: a\in K, \gamma\in\Gamma\}$ coincides with the given $S=\{(a,\gamma): a\in K, \gamma\in\Gamma\}$ if each element $[a,\gamma]$ of $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$ is identified with $(a,\gamma)\in S$.

Proof. Take $[e,\gamma]$, $[h,\tau]$ from $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$.

Then, $[e,\gamma][h,\tau]$ (in $C(\Gamma, K(\{1\}); \mathscr{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$)

$$= [e(h\alpha^*_{(\gamma,\tau)}), \gamma\tau] = [eh^{\tilde{\gamma}}C(\gamma,\tau), \gamma\tau] \equiv (eh^{\tilde{\gamma}}C(\gamma,\tau), \gamma\tau)$$

$= (e, \gamma)(h, \tau)$ (in S).

Corollary. Let K, Γ *be groups. Let* $C(\Gamma, K(\{1\}); \mathcal{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\}) = \{[e, \gamma] : e \in K, \gamma \in \Gamma\}$ *be a complete regular product of* $K(\{1\})$ *and* $\Gamma(\{1\})$. *Let* $\{\tilde{\sigma}, C(\sigma, \tau)\}$ *be a factor set of* Γ *with respect to* K, *and* $S = \{(e, \gamma) : e \in K, \gamma \in \Gamma\}$ *the Schreier extension of* K *determined by* Γ *and* $\{\tilde{\sigma}, C(\sigma, \tau)\}$. *If*

(2.2) $u_1 = C(1, 1)$ *and* $a\alpha^*_{(\gamma,\tau)} = a^{\tilde{\gamma}} C(\gamma, \tau)$ *for all* $a \in K$ *and all* $\gamma, \tau \in \Gamma$,

then $C(\Gamma, K(\{1\}); \mathcal{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$ *and* S *are the same system if each* $[e, \gamma] \in C(\Gamma, K(\{1\}); \mathcal{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\})$ *is identified with* $(e, \gamma) \in S$; *that is,* $\varphi : C(\Gamma, K(\{1\}); \mathcal{I}, \{1\}, \{u_1\}, \{\alpha^*_{(\gamma,\tau)}\}) \longrightarrow S$ *defined by* $[e, \gamma]\varphi = (e, \gamma)$ *gives an isomorphism.*

References

1. Warne, R. J. : The structure of a class of regular semigroups, II, Kyungpook Math. J. 11 (1971), 165-167.
2. Yamada, M. : Orthodox semigroups whose idempotents satisfy a certain identity, Semigroup Forum 6(1973), 113-128.
3. Yamada, M. : On regular extensions of a semigroup which is a semilattice of completely simple semigroups, Mem. Fac. Lit. & Sci. Shimane Univ. Nat. Sci., 7(1974) ; 1-17.