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On Regular Extensions of a Semigroup which is a Semilattice of Completely Simple Semigroups

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The structure of orthodox semigroups was described by Hall [2], Warne [4], [5] and the author [7], [8], [9] in terms of bands and inverse semigroups. In this paper, we introduce the concept of generalized orthodox semigroups and show that some analogues to the results given by the papers above for the class of orthodox semigroups are also fulfilled by the class of generalized orthodox semigroups. Further, we completely describe the structure of generalized orthodox semigroups in terms of Cliffordian semigroups (that is, semigroups which are unions of groups) and inverse semigroups. In the latter half of the paper, we introduce the concept of split extensions of Cliffordian semigroups by inverse semigroups, and next establish some necessary and sufficient conditions in order that a regular semigroup S be a split extension of a normal Cliffordian subsemigroup of S by an inverse semigroup. Any notation and terminology should be referred to [1], unless otherwise stated.

1. Generalized orthodox semigroups.

A regular semigroup is called a *Cliffordian semigroup* if it is a union of groups. It is well-known that any Cliffordian semigroup G is decomposed into a semilattice Γ of completely simple subsemigroups G_γ; that is, there exist a semilattice Γ and, for each $\gamma \in \Gamma$, a completely simple subsemigroup G such that (1) $G = \Sigma\{G_\gamma : \gamma \in \Gamma\}$ (Σ means *disjoint sum*) and (2) $G_\alpha C_\beta \subset G_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$ (see [1]). Further, the uniqueness of such a decomposition of G is also proved as follows: Let $\{G_\gamma : \gamma \in \Gamma\}$, $\{G'_\delta : \delta \in \Delta\}$ be decompositions of G into semilattices Γ, Δ of completely simple subsemigroups G_γ and G'_δ respectively. We next prove that for any G_α there exists G'_δ such that $G_\alpha \subset G'_\delta$. Put $G_\alpha \cap G'_\gamma = G^*_\gamma$ for each $\gamma \in \Delta$, and let $\Pi = \{\gamma \in \Delta : G^*_\gamma \neq \square\}$. Then Π is a subsemilattice of Δ. Now, define $\phi : G_\alpha \to \Pi$ by $a\phi = \gamma$ if $a \in G^*_\gamma$. Then it is obvious that ϕ is an epimorphism (that is, an onto-homomorphism). If Π is not simple, then there exists a proper ideal Λ of Π. Hence, $G^\natural_\alpha = \bigcup \{G^*_\rho : \rho \in \Lambda\}$ is a proper ideal of G_α. This contradicts to the simplicity of G_α. Thus, Π is simple and hence is a single element. Therefore, for any G_α there exists G'_δ such that $G_\alpha \subset G'_\delta$. Similarly, it is proved that for any G'_δ there exists G_α such that $G'_\delta \subset G_\alpha$. Hence, two decompositions $\{G_\gamma : \gamma \in \Gamma\}$, $\{G'_\delta : \delta \in \Delta\}$ are essentially same.

Hereafter, "a Cliffordian semigroup $G \equiv \Sigma\{G_\gamma : \gamma \in \Gamma\}$" means "a Cliffordian semigroup G which is a semilattice Γ of completely simple semigroups G_γ".

Let S be a regular semigroup, and E the set of idempotents of S. For each $e \in E$, let G_e be a subgroup containing e. If

(I) $M = \cup \{G_e : e \in E\}$ is a subsemigroup of S (accordingly, a Cliffordian subsemigroup) : $M \equiv \Sigma\{M_\lambda : \lambda \in \Lambda\}$ (Λ : a semilattice ; and M_λ : a completely simple subsemigroup),

(II) $aa^*, bb^* \in M_\lambda$, $a^*a, b^*b \in M_\xi$, $ab^*, b^*a \in M$ (where x^* means an inverse of x) and $a \in M$ imply $b \in M$,

(hereafter, for an element a of a regular semigroup the notation a^* will mean an inverse of a),

(III) $aa^*, bb^*, ab^*, ba^* \in M_\lambda$, $a^*a, b^*b, a^*b, b^*a \in M_\eta$ imply that for any $\tau \in \Lambda$ there exist M_δ, M_ξ such that

$aM_\tau a^*, aM_\tau b^*, bM_\tau a^*, bM_\tau b^* \subset M_\delta$ and $a^*M_\tau a, a^*M_\tau b, b^*M_\tau a, b^*M_\tau b \subset M_\xi$,

(IV) $aa^*, bb^*, ab^*, ba^* \in M_\lambda$, $a^*a, b^*b, a^*b, b^*a \in M_\eta$ imply that there exist $M_\xi, M_\delta, M_\varepsilon$ and M_τ such that for any $c, c^*, (bc)^*$ and $(cb)^*$,

$acc^*b^*, ac(bc)^* \in M_\xi$, $b^*c^*ca, (cb)^*ca \in M_\delta$,

$cab^*c^*, ca(cb)^* \in M_\varepsilon$ and $c^*b^*ac, (bc)^*ac \in M_\tau$,

then $M \equiv \Sigma\{M_\lambda : \lambda \in \Lambda\}$ is called a *normal Cliffordian subsemigroup* of S.

LEMMA 1. *If* $M \equiv \Sigma\{M_\lambda : \lambda \in \Lambda\}$ *is a normal Cliffordian subsemigroup of* S, *then* $a \in M$ *implies* $a^\# \in M$ *for all inverses* $a^\#$ *of* a.

Proof. Since M is a union of groups, a has an inverse a^* in M. Let $a^\#$ be any inverse of a, and let $aa^* \in M_\alpha$ and $aa^\# \in M_\beta$. Then $aa^\# = aa^*aa^\# \in M_{\alpha\beta}$. Hence $\beta = \alpha\beta$. Similarly, we have $\alpha = \alpha\beta$. Thus, $\alpha = \beta$. That is, $aa^*, aa^\#$ are contained in the same M_λ. Similarly, it is also proved that a^*a, $a^\#a$ are contained in the same M_ξ. Since a is an inverse of both a^* and $a^\#$, by (II) $a^*a, a^\#a \in M_\xi$, $aa^*, aa^\# \in M_\lambda$, $a^*a, aa^* \in M$ and $a^* \in M$ imply $a^\# \in M$.

Remark. The following is easily proved : If a regular semigroup S contains a normal Cliffordian subsemigroup, then the intersection of all normal Cliffordian subsemigroups of S is also a normal Cliffordian subsemigroup. Hence, S has the least normal Cliffordian subsemigroup.

Hereafter, a regular semigroup having a normal Cliffordian subsemigroup will be called a *generalized orthodox semigroup*. Of course, both a Cliffordian semigroup and an orthodox semigroup are generalized orthodox semigroups.

Let S be a generalized orthodox semigroup, and $M \equiv \Sigma\{M_\lambda : \lambda \in \Lambda\}$ a normal Cliffordian subsemigroup of S. Define a relation π'_M on S as follows :

(1.1) $a\ \pi'_M\ b$ if and only if there exist M_λ, M_ξ such that $aa^\#$, $bb^\#$, $ab^\#$, $ba^\# \in M_\lambda$ and $a^\# a$, $b^\# b$, $a^\# b$, $b^\# a \in M_\xi$ for all inverses $a^\#$, $b^\#$ of a, b.

It is obvious that this relation is reflexive and symmetric. Next define π_M as follows :

(1.2) $a\ \pi_M\ b$ if and only if there exist $x_0, x_1, .., x_n \in S$ such that $a = x_0\ \pi'_M\ x_1\ \pi'_M\ x_2 . . x_{n-1}\ \pi_M\ x_n = b$.

Then, it is easily proved by using (I)-(IV) and by simple calculation that this relation π_M on S is a congruence.

THEOREM 1. *S/π_M is an inverse semigroup. Let ρ be the decomposition of M into the semilattice Λ of the completely simple subsemigroups M_λ (that is, $M/\rho = \{M_\lambda : \lambda \in \Lambda\}$). Then, each ρ-class M_λ is a complete π_M-class and hence $\{M_\lambda : \lambda \in \Lambda\}$ is a normal system of subsets of S (in the sense of* [1]).

Proof. For any $x \in S$, let $\tilde{x}$ be the π_M-class containing x. It is well-known that every idempotent $\tilde{a}$ of S/π_M contains at least one idempotent of S (see [3]). Now, let $\tilde{a}, \tilde{b}$ be idempotents of S/π_M. Then, there exist idempotents e, f of S such that $\tilde{a} = \tilde{e}$ and $\tilde{b} = \tilde{f}$. By the definition of π_M, it is obvious that every ρ-class M_λ is contained in some π_M-class. For $x \in M$, let $\bar{x}$ be the ρ-class containing x. Since M/ρ is a semilattice, $\bar{e}\bar{f} = \bar{f}\bar{e}$. Hence $\overline{ef} = \overline{fe}$, and hence $\widetilde{ef} = \widetilde{fe}$. Thus, we obtain $\tilde{e}\tilde{f} = \tilde{f}\tilde{e}$, that is, $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$. Therefore, S/π_M is an inverse semigroup. Now, suppose that $a \in M_\lambda$ and $a\ \pi_M\ b$. By the definition of π_M, there exist $x_0, x_1, \ldots, x_n$ such that $a = x_0\ \pi'_M\ x_1 \ldots x_{n-1}\ \pi'_M\ x_n = b$. We shall next show that $x_i\ \pi'_M\ x_{i+1}$, $x_i \in M_\lambda$ imply $x_{i+1} \in M_\lambda$. Since $x_i\ \pi'_M\ x_{i+1}$, we have $x_i x_i^*$, $x_{i+1} x_{i+1}^*$, $x_i x_{i+1}^*$, $x_{i+1} x_i^* \in M_\xi$, $x_i^* x_i$, $x_{i+1}^* x_{i+1}$, $x_i^* x_{i+1}$, $x_{i+1}^* x_i \in M_\eta$ and $x_i \in M$. Therefore, $x_{i+1} \in M$ by (II). Let $x_{i+1} \in M_\beta$. Then, $\bar{x}_{i+1}\ \bar{x}_{i+1}^* = \bar{x}_{i+1}$ and hence $\beta = \xi$ (since $x_{i+1} x_{i+1}^* \in M_\xi$ and $x_{i+1}^* \in M_\beta$). Similarly, $\bar{x}_i \bar{x}_i^* = \bar{x}_i$ and hence $\lambda = \xi$. Therefore, $\beta = \lambda$. This implies that $x_{i+1} \in M_\lambda$. Thus, we can conclude that $a\ \pi_M\ b$, $a \in M_\lambda$ imply $b \in M_\lambda$. Hence, every π_M-class containing an element a of M just coincides with the ρ-class containing a. Hence, every ρ-class is a complete π_M-class. Since $\cup \{M_\lambda : \lambda \in \Lambda\}$ contains the set of idempotents of S, $\{M_\lambda : \lambda \in \Lambda\}$ is a normal system of subsets of S.

From the result above, the congruence π_M on S is uniquely determined by $\{M_\lambda : \lambda \in \Lambda\}$, accordingly by M. Hereafter, we shall denote S/π_M by S/M.

THEOREM 2. *Let S be a generalized orthodox semigroup, and $K \equiv \Sigma\ \{K_\delta : \delta \in \Delta\}$ the least normal Cliffordian subsemigroup of S. Then, S/K is the greatest inverse semigroup homomorphic image of S. That is, π_K is the least inverse semigroup congruence on S.*

Proof. Let σ be an inverse semigroup congruence such that $\sigma \leq \pi_K$. Let

Λ be the basic semilattice (that is, the semilattice of all idempotents) of $\Gamma = S/\sigma$ (see [6]). Let $\phi : S \to \Gamma$ be the natural homomorphism of S onto Γ, and put $S_\lambda = \gamma\phi^{-1}$ for each $\gamma \in \Lambda$. Then S_λ is a regular subsemigroup, and $\cup \{S_\gamma : \gamma \in \Lambda\} = T$ contains the set E of idempotents of S. Since S_γ is contained in some K_λ, each idempotent of S_γ is primitive in S_γ. Hence S_γ is a primitive regular semigroup, and hence S_γ is a completely simple semigroup. Therefore, T is a semilattice Λ of completely simple semigroups S_γ. Since $\{S_\gamma : \gamma \in \Lambda\}$ is the kernel of ϕ[1], $T = \cup \{S_\gamma : \gamma \in \Lambda\}$ is a normal Cliffordian subsemigroup of S. By the assumption, $K \equiv \Sigma \{K_\lambda : \lambda \in \Delta\}$ is the least normal Cliffordian subsemigroup of S and accordingly $K \subset T$. On the other hand, every S_γ is contained in some K_λ and hence $T \subset K$. Thus, we have $T = K$. Since T $(=K)$ is uniquely decomposed into a semilattice of completely simple semigroups, it follows that $\{S_\gamma : \gamma \in \Lambda\} = \{K_\lambda : \lambda \in \Delta\}$. Since π_K, σ are the congruences (on S) determined by $\{K_\lambda : \lambda \in \Delta\}$, $\{S_\gamma : \gamma \in \Lambda\}$ respectively, we have $\sigma = \pi_K$. Hence π_K is the least inverse semigroup congruence on S.

Remark. For any regular semigroup S, the existence of the least inverse semigroup congruence on S is easily proved.

COROLLARY. *Let σ be the least inverse semigroup congruence on a generalized orthodox semigroup S. Let $\phi : S \to S/\sigma$ be the natural homomorphism of S onto S/σ. Then, the sum of members of the kernel of ϕ is the least normal Cliffordian subsemigroup of S.*

Let $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ be a Cliffordian semigroup, and Γ an inverse semigroup. Suppose that a regular semigroup S contains K as its normal Cliffordian subsemigroup and $S/K \cong \Gamma$. Let Δ be the basic semilattice of Γ, and put $S/K = \Pi$. Let $\phi : S \to \Pi = S/K$ be the natural homomorphism, and put $K_\lambda \phi = \lambda'$ for all $\lambda \in \Lambda$. Then $\Lambda' = \{\lambda' : \lambda \in \Lambda\}$ is isomorphic to Λ and is the basic semilattice of Π. If $\psi : \Pi \to \Gamma$ is an isomorphism, then of course $\Lambda'\psi = \Delta$. Put $\lambda'\psi = \bar{\lambda}$. Then, K can be rewritten as $K \equiv \Sigma\{K_{\bar{\lambda}} : \bar{\lambda} \in \Delta\}$ where $K_{\bar{\lambda}} = K_\lambda$. Hence, we introduce the concept of regular extensions of a Cliffordian semigroup by an inverse semigroup as follows: Let Γ be an inverse semigroup, and Δ its basic semilattice. Let $K \equiv \Sigma\{K_\delta : \delta \in \Delta\}$ be a Cliffordian semigroup. Then, a regular semigroup S is called a *regular extension* of $K \equiv \Sigma\{K_\delta : \delta \in \Delta\}$ by $\Gamma(\Delta)$ if S satisfies the following conditions : (1) S contains K as a normal Cliffordian subsemigroup ; and (2) there exists an epimorphism $\phi : S \to \Gamma$ such that $\delta\phi^{-1} = K_\delta$ for each $\delta \in \Delta$.

Now, we have the following :

1) Let ψ be a homomorphism of a regular semigroup A into a regular semigroup B. Let E be the set of idempotents of $A\psi$. Put $\gamma\psi^{-1} = A_\gamma$ for all $\gamma \in E$. Then the set $\{A_\gamma : \gamma \in E\}$ is called the kernel of ψ.

THEOREM 3. (1) *Let S be a generalized orthodox semigroup, and $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ a normal Cliffordian subsemigroup of S. Then S is a regular extension of $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by some inverse semigroup $\Omega(\Lambda)$*[2]. *In this case, S/K can be taken as $\Omega(\Lambda)$.*

(2) *Let $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ be a Cliffordian semigroup, and $\Omega(\Lambda)$ an inverse semigroup having Λ as its basic semilattice. Then any regular extension of $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by $\Omega(\Lambda)$ is a generalized orthodox semigroup.*

Proof. The part (1) follows from the definition of regular extensions and the results above. Let S be a regular extension of $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by $\Omega(\Lambda)$. Then there exists an epimorphism $\phi : S \to \Omega(\Lambda)$ such that $\lambda\phi^{-1} = K_\lambda$ for all $\lambda \in \Lambda$. Hence, it is obvious that $\cup\{K_\lambda : \lambda \in \Lambda\}$ is a normal Cliffordian subsemigroup of S. Therefore, S is a generalized orthodox semigroup.

By the theorem above, the problem of describing all possible generalized orthodox semigroups is reduced to the following problem : Let $\Omega(\Lambda)$ be a given inverse semigroup having Λ as its basic semilattice, and $K \equiv \Sigma\{K \ : \lambda \in \Lambda\}$ a given Cliffordian semigroup. Construct all possible regular extensions of $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by $\Omega(\Lambda)$.

We shall investigate this problem in the following sections.

2. Elementary properties.

Let S be a generalized orthodox semigroup, and $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ a normal Cliffordian subsemigroup of S. Then, there exists the unique inverse semigroup congruence ρ determined by $\{K_\lambda : \lambda \in \Lambda\}$; that is, $S/\rho = \{S_\gamma : \gamma \in \Gamma\}$, where Γ is an inverse semigroup containing Λ as its basic semilattice, such that (1) $\Sigma\{S_\gamma : \gamma \in \Gamma\} = S$, (2) $S_\xi S_\eta \subset S_{\xi\eta}$ for all $\xi, \eta \in \Gamma$ and (3) $S_\lambda = K_\lambda$ for $\lambda \in \Lambda$. Take an $\mathscr{L}$-class I_λ and an $\mathscr{R}$-class J_λ from each K_λ, $\lambda \in \Lambda$.[3] Let $K_\lambda \cong \{(g)_{ij} : i \in U_\lambda, j \in V_\lambda, g \in G_\lambda\}$ be a Rees matrix representation of K_λ over a group G_λ. Let $[g_{ji}]_\lambda$ be the sandwich matrix in this representation. We can identify K_λ with $\{(g)_{ij}: i \in U_\lambda, j \in V_\lambda, g \in G_\lambda\}$.

For $(x)_{ij}, (y)_{ks}$ of K_λ, it is easy to see that (1) $(x)_{ij}\mathscr{L}(y)_{ks}$ if and only if $j=s$; and (2) $(x)_{ij}\ \mathscr{R}(y)_{ks}$ if and only if $i = k$. Hence, $I_\lambda = \{(x)_{kj} : k \in U_\lambda, x \in G_\lambda\}$ for some j and $J_\lambda = \{(x)_{is} : s \in V_\lambda, x \in G_\lambda\}$ for some i. Let E_λ be the set of idempotents of J_λ. Then, $E_\lambda = \{(g_{ti}^{-1})_{it} : t \in V_\lambda, g_{ti}$ is the (t, i)-element of $[g_{ji}]_\lambda\}$. By simple calculation, it is easily proved that J_λ is a regular semigroup

2) Hereafter, we sometimes use the symbol $\Gamma(\Lambda)$ to denote an inverse semigroup Γ having Λ as its basic semilattice.

3) $\mathscr{L}$, $\mathscr{R}$ denote the Green's L-relation and R-relation respectively.

and E_λ is a right zero semigroup. Hence, J_λ is *a* right group (see [6]). Similarly, I_λ is a left group. Now, let $\mathscr{I}=\Sigma\{I_\lambda : \lambda\in\Lambda\}$, $\mathscr{J}=\Sigma\{J_\lambda : \lambda\in\Lambda\}$ and $\mathscr{E}=\Sigma\{E_\lambda : \lambda\in\Lambda\}$. Warne [5] introduced the concept of lower [upper] partial chains of semigroups. We next show that $\mathscr{I}$ is a lower partial chain of left groups $\{I_\lambda : \lambda\in\Lambda\}$ and $\mathscr{J}$ is an upper partial chain of right groups $\{J_\lambda : \lambda\in\Lambda\}$. Let $x\in I_\lambda$, $y\in I_\mu$ and $\lambda\leq\mu$. Since $yx\in K_\lambda$, assume that $yx=(h)_{st}$ in K_λ. If $x=(g)_{ij}$ in I_λ, x^*x has a form $(u)_{pj}$ in K_λ (x^* means an inverse of x). Hence, $yx=yxx^*x=(h)_{st}(u)_{pj}=(v)_{sj}$ for some $v\in G_\lambda$. Since $(v)_{sj}$ has j as a column number, $yx\ \mathscr{L}\ x$ in K_λ. Thus, $yx\in I_\lambda$. That is, $\mathscr{I}$ is a lower partial chain of $\{I_\lambda : \lambda\in\Lambda\}$. By a similar method to the above, we can prove that $\mathscr{J}$ is an upper partial chain of right groups $\{J_\lambda : \lambda\in\Lambda\}$.

Next, let u_γ be a representative of S_γ for each $\gamma\in\Gamma$.

LEMMA 2. *For any $a\in S_\gamma$, there exist a unique p and a unique q such that $p\in I_{\gamma\gamma^{-1}}$, $q\in E_{\gamma^{-1}\gamma}$ and $pu_\gamma q=a$.*

Proof. First we prove that there exist x, y such that $x\in S_{\gamma\gamma^{-1}}$, $y\in S_{\gamma^{-1}\gamma}$ and $a=xu_\gamma y$. It is obvious that $a=aa^*aa^*a$ and a^*a, $u_\gamma^*u_\gamma\in S_{\gamma^{-1}\gamma}$. Since $S_{\gamma^{-1}\gamma}$ $(=K_{\gamma^{-1}\gamma})$ is $\{(g)_{ks} : k\in U_{\gamma^{-1}\gamma}, s\in V_{\gamma^{-1}\gamma}, g\in G_{\gamma^{-1}\gamma}\}$. Let $u_\gamma^*u_\gamma=(g_{ks}^{-1})_{sk}$ and $a^*a=(g_{tu}^{-1})_{ut}$. If $x=g_{ku}^{-1}g_{ks}g_{vs}^{-1}$, then $(x)_{uv}(g_{ks}^{-1})_{sk}(g_{tu}^{-1})_{ut}=(g_{tu}^{-1})_{ut}$. Hence, $a=aa^*a=a(g_{tu}^{-1})_{ut}=a(xg_{vs}g_{ks}^{-1}g_{ku}g_{tu}^{-1})_{ut}=(a(x)_{uv}u_\gamma^*)u_\gamma(a^*a)$. Since $a(x)_{uv}u_\gamma^*\in S_{\gamma\gamma^{-1}}$ and $a^*a\in S_{\gamma^{-1}\gamma}$, there exist x, y such that $x\in S_{\gamma\gamma^{-1}}$, $y\in S_{\gamma^{-1}\gamma}$ and $a=xu_\gamma y$. Now since $u_\gamma y=u_\gamma u_\gamma^*u_\gamma y$, $u_\gamma^*u_\gamma\in S_{\gamma^{-1}\gamma}$ and $y\in S_{\gamma^{-1}\gamma}$, if $y=(g)_{sk}$ and $u_\gamma^*u_\gamma=(h)_{tu}$ then $u_\gamma^*u_\gamma y=(h)_{tu}(g)_{sk}=(hg_{us}g)_{tk}$. Let $J_{\gamma^{-1}\gamma}=\{(g)_{is} : s\in V_{\gamma^{-1}\gamma}, g\in G_{\gamma^{-1}\gamma}\}$. Take n such that $ng_{ui}hg_{ui}g_{ki}^{-1}=hg_{us}g$, and put $e=(g_{ki}^{-1})_{ik}$ and $(n)_{tu}=z$. Then $zu_\gamma^*u_\gamma e=u_\gamma^*u_\gamma y$. Hence, $a=xu_\gamma zu_\gamma^*u_\gamma e$, $e\in E_{\gamma^{-1}\gamma}$ and $xu_\gamma zu_\gamma^*\in S_{\gamma\gamma^{-1}}$. Now, let $v=xu_\gamma zu_\gamma^*$. Then $a=vu_\gamma e$, $v\in S_{\gamma\gamma^{-1}}$ and $e\in E_{\gamma^{-1}\gamma}$. Now, let $I_{\gamma\gamma^{-1}}=\{(g)_{sj} : s\in U_{\gamma\gamma^{-1}}, g\in G_{\gamma\gamma^{-1}}\}$. $vu_\gamma=vu_\gamma u_\gamma^*u_\gamma$, $v\ni S_{\gamma\gamma^{-1}}$ and $u_\gamma u_\gamma^*\in S_{\gamma\gamma^{-1}}$. If $v=(g)_{kn}$ and if $u_\gamma u_\gamma^*=(h)_{st}$, then $vu_\gamma u_\gamma^*=(g)_{kn}(h)_{st}=(gg_{ns}h)_{kt}$. Take w such that $wg_{js}h=gg_{ns}h$. Then $vu_\gamma u_\gamma^*=(w)_{kj}u_\gamma u_\gamma^*$ and $(w)_{kj}\in I_{\gamma\gamma^{-1}}$. Putting $(w)_{kj}=p$, $e=q$, we have $a=pu_\gamma u_\gamma^*u_\gamma q=pu_\gamma q$, $p\in I_{\gamma\gamma^{-1}}$, $q\in E_{\gamma^{-1}\gamma}$. Next, we prove the uniqueness of such p, q. Assume that $a=xu_\gamma y=zu_\gamma v$, $x, z\in I_{\gamma\gamma^{-1}}$, y, $v\in E_{\gamma^{-1}\gamma}$. $xu_\gamma y=zu_\gamma v$ implies $xu_\gamma yy'=zu_\gamma vy$. Since $E_{\gamma^{-1}\gamma}$ is a right zero semigroup, $xu_\gamma y=zu_\gamma y$. Since $u_\gamma y\in S_\gamma$, we have $u_\gamma y(u_\gamma y)^*\in S_{\gamma\gamma^{-1}}$. Put $u_\gamma y(u_\gamma y)^*=(g)_{nk}$ and $x=(t)_{sj}$. Take c and p such that $gg_{kp}c=g_{jn}^{-1}$. Then $(g)_{nk}(c)_{pj}=(g_{jn}^{-1})_{nj}$. Hence, $xu_\gamma y=zu_\gamma y$ implies $x(u_\gamma y)(u_\gamma y)^*(c)_{pj}=z(u_\gamma y)(u_\gamma y)^*(c)_{pj}$ and hence implies $x(g_{jn}^{-1})_{nj}=z(g_{jn}^{-1})_{nj}$. Therefore, $x(x^*x)(g_{jn}^{-1})_{nj}=z(z^*z)(g_{jn}^{-1})_{nj}$ and x^*x, $z^*z\in I_{\gamma\gamma^{-1}}$. Since the idempotents of $I_{\gamma\gamma^{-1}}$ form a left zero semigroup,

$x^*x(g_{jn}^{-1})_{nj} = x^*x$ and $z^*z\,(g_{jn}^{-1})_{nj} = z^*z$. Thus we obtain $x = z$. Similarly, we also obtain $y = v$.

3. Regular products.

Let Γ be an inverse semigroup, and Λ the basic semilattice of Γ. Let $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ be a Cliffordian semigroup. Let I_λ, J_λ be an $\mathscr{L}$-class, an $\mathscr{R}$-class of K_λ respectively for each $\lambda \in \Lambda$, and put $\mathscr{I} = \Sigma\{I_\lambda : \lambda \in \Lambda\}$ and $\mathscr{J} = \Sigma\{J_\lambda : \lambda \in \Lambda\}$. Let E_λ be the right zero subsemigroup of idempotents of J_λ, and put $\mathscr{E} = \Sigma\{E_\lambda : \lambda \in \Lambda\}$. Then, it is easy to see from the section 2 that $\mathscr{I}$ and $\mathscr{J}$ are a lower partial chain of left groups $\{I_\lambda : \lambda \in \Lambda\}$ and an upper partial chain of right groups $\{J_\lambda : \lambda \in \Lambda\}$. Now, let $P(A, B)$ denote the set of partial transformations of A into B (see [5]). Suppose that $\phi : \Gamma^2 \to P(\mathscr{E} \times \mathscr{I} \times \mathscr{E}, \mathscr{I})$ and $\psi : \Gamma^2 \to P(\mathscr{E} \times \mathscr{I} \times \mathscr{E}, \mathscr{E})$ satisfy the following conditions (A), (B), and (C) : Put $(\gamma, \delta)\phi = \alpha_{(\lambda,\delta)}$ and $(\lambda, \delta)\psi = \beta_{(\lambda,\delta)}$ for $(\gamma, \delta) \in \Gamma^2$.

(A) $D(\alpha_{(\lambda,\delta)}) = D(\beta_{(\gamma,\tau)}) = E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \times E_{\tau^{-1}\tau}$: $R(\alpha_{(\gamma,\tau)}) \subset I_{\gamma\tau(\gamma\tau)^{-1}}$ and $R(\beta_{(\gamma,\tau)}) \subset E_{(\gamma\tau)^{-1}\gamma\tau}$, where $D(\xi)$, $R(\xi)$ denote the domain and the range of ξ respectively.

(B) If $q \in E_{\gamma^{-1}\gamma}$, $t \in I_{\tau\tau^{-1}}$, $h \in E_{\tau^{-1}\tau}$, $v \in I_{\delta\delta^{-1}}$ and $w \in E_{\delta^{-1}\delta}$, then
$(q, t\, h)\alpha_{(\gamma,\tau)}((q, t, h)\beta_{(\gamma,\tau)}, v, w)\alpha_{(\gamma\tau,\delta)}$
$= (q, t(h, v, w)\alpha_{(\tau,\delta)}, (h, v, w)\,\beta_{(\lambda,\delta)})\,\alpha_{(\gamma,\tau\delta)}$
and $((q, t, h)\beta_{(\gamma,\tau)}, v, w)\,\beta_{(\gamma\tau,\delta)}$
$= (q, t(h, v, w)\alpha_{(\tau,\delta)}, (h, v, w)\beta_{(\tau,\delta)})\beta_{(\gamma,\tau\delta)}$.

(C) For $\gamma \in \Gamma$, $p \in I_{\gamma\gamma^{-1}}$, $q \in E_{\gamma^{-1}\gamma}$, there exist $k \in I_{\gamma^{-1}\gamma}$ and $n \in E_{\gamma\gamma^{-1}}$ such that $p(q, k, n)\alpha_{(\gamma,\gamma^{-1})}\,((q, k, n)\beta_{(\gamma,\gamma^{-1})}), p, q)\alpha_{(\gamma\gamma^{-1},\gamma)} = p$
and $((q, k, n)\beta_{(\gamma,\gamma^{-1})}, p, q)\beta_{(\gamma\gamma^{-1},\gamma)} = q$.

In this case, for the set $(\Gamma, \mathscr{I}, \mathscr{E}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\}) = \{(i, \gamma, j) : \gamma \in \Gamma, i \in I_{\gamma\gamma^{-1}}, j \in E_{\gamma^{-1}\gamma}\}$ we have

LEMMA 3. $(\Gamma, \mathscr{I}, \mathscr{E}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ *is a regular semigroup under the multiplication defined by*

$(i, \gamma, j)\,(p, \tau, q) = (i(j, p, q)\alpha_{(\gamma,\tau)}, \gamma\tau, (j, p, q)\beta_{(\gamma,\tau)})$.

Proof. It is easily seen from the conditions (A), (B) and by simple calculation that $(\Gamma, \mathscr{I}, \mathscr{E}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ is a semigroup. Also, the regularity follows from the condition (C). In fact, for (i, γ, j), take $k \in I_{\gamma^{-1}\gamma}$ and $n \in E_{\gamma\gamma^{-1}}$ which satisfy the condition (C). Then (k, γ^{-1}, n) satisfies
$(i, \gamma, j)\,(k, \gamma^{-1}, n)\,(i, \gamma, j) = (i, \gamma, j)$.

Now, let $M \equiv \Sigma\{M_\lambda : \lambda \in \Omega\}$ be a Cliffordian semigroup. Let u_λ be a representative of M_λ for each $\lambda \in \Omega$, and L_λ, R_λ an $\mathscr{L}$-class, an $\mathscr{R}$-class of M_λ respectively. Let F_λ be the set of idempotents of R_λ. Then by Lemma 2, for any $a \in M_\lambda$ there exist a unique $p \in L_\lambda$ and a unique $q \in F_\lambda$ such that $a = pu_\lambda q$. Now, for $(\lambda, \tau) \in \Lambda^2$, define $\alpha_{(\lambda,\tau)} : F_\lambda \times L_\lambda \times F_\tau \to L_{\lambda\tau}$ and $\beta_{(\lambda,\tau)} : F_\lambda \times L_\tau \times F_\tau \to F_{\lambda\tau}$ as follows:

$(q, v, w)\alpha_{(\lambda,\tau)} = t$ and $(q, v, w)\beta_{(\lambda,\tau)} = h$ if $u_\lambda qvu_\lambda w = tu_\lambda h$,, where $q \in F_\lambda$, $v \in L_\tau$, $w \in F_\tau$, $t \in L_{\lambda\tau}$ and $h \in F_{\lambda\tau}$.

It is easy to see that such $\{\alpha_{(\lambda,\tau)}\}$ and $\{\beta_{(\lambda,\tau)}\}$ satisfy the conditions (A), (B), (C). Hence we can consider $(\Omega, L, F, \{\alpha_{(\lambda,\tau)}\}, \{\beta_{(\lambda,\tau)}\})$, where $L = \Sigma\{L_\lambda : \lambda \in \Omega\}$ and $F = \Sigma\{F_\lambda : \lambda \in \Omega\}$.

Now, we obtain

LENMA 4. *M is isomorphic to* $(\Omega, L, F, \{\alpha_{(\lambda,\tau)}\}, \{\beta_{(\lambda,\tau)}\})$.

Proof. By Lemma 2, $M = \{iu_\lambda j : \lambda \in \Omega, i \in L_\lambda, j \in F_\lambda\}$. Define $\phi : M \to (\Omega, L, F, \{\alpha_{(\lambda,\tau)}\}, \{\beta_{(\lambda,\tau)}\})$ by $(iu_\lambda j)\phi = (i, \lambda, j)$. Then ϕ is clearly surjective and injective. Further, $((iu_\lambda j)(ku_\lambda \mathrm{h}))\phi = (i(j, k, h)\alpha_{(\lambda,\tau)} u_{\gamma\tau}(j, k, h)\beta_{(\lambda,\tau)})\phi = (i(j, k,h)\alpha_{(\lambda,\tau)}, \lambda\tau, (j, k, h)\beta_{(\lambda,\tau)}) = (i, \lambda, j)(k, \tau, h) = ((iu_\lambda j)\phi)((ku_\lambda h)\phi)$.

In $M = \{iu_\lambda j : \lambda \in \Omega, i \in L_\lambda, j \in F_\lambda\}$, the multiplication is given by $(iu_\lambda j)(ku_\lambda h) = i(j, k, h)\alpha_{(\lambda,\tau)}u_{\lambda\tau}(j, k\ h)\beta_{(\lambda,\tau)}$. In general, for $M = \{iu_\lambda j : \lambda \in \Omega, i \in L_\lambda, j \in F_\lambda\}$, a system $\Delta = \{\alpha_{(\lambda,\tau)} : \lambda, \tau \in \Omega\} \cup \{\beta_{(\lambda,\tau)} : \lambda, \tau \in \Omega\}$ satisfying (A), (B), (C) and the following

(D) $u_\lambda jku_\lambda h = (j, k, h)\alpha_{(\lambda,\tau)}u_{\lambda\tau}(j, k, h)\beta_{(\lambda,\tau)}$ for $\lambda, \tau \in \Omega$, $j \in F_\lambda$, $k \in L_\lambda$, $h \in F_\tau$

is uniquely determined. We shall call this system Δ the *characteristic family* of $M = \{iu_\lambda j : \lambda \in \Omega, i \in L_\lambda, j \in F_\lambda\}$. If $\Delta = \{\alpha_{(\lambda,\tau)} : \lambda, \tau \in \Omega\} \cup \{\beta_{(\lambda,\tau)} : \lambda, \tau \in \Omega\}$ is the characteristic family of $M = \{iu_\lambda j : \lambda \in \Omega, i \in L_\lambda, j \in F_\lambda\}$, then of course the multiplication in M is given by

$$(iu_\lambda j)(ku_\tau h) = i(j\ k, h)\alpha_{(\lambda,\tau)}u_{\lambda\tau}(j, k, h)\beta_{(\lambda,\tau)},$$

Now, consider the regular semigroup $(\Gamma, \mathscr{I}, \mathscr{E}\ \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ above. Take a representative u_λ from each K_λ, $\lambda \in \Lambda$, and express K as follows: $K = \{iu_\lambda j : \lambda \in \Lambda, i \in I_\lambda, j \in E_\lambda\}$. Then, we have

LEMMA 5. *If* $\{\alpha_{(\lambda,\tau)} : \lambda, \tau \in \Lambda\} \cup \{\beta_{(\lambda,\tau)} : \lambda, \tau \in \Lambda\}$ *is the characteristic family of* $K = \{iu_\lambda j : \lambda \in \Lambda, i \in I_\lambda, j \in E_\lambda\}$, *then there exists a homomorphism* ϕ *of* $(\Gamma, \mathscr{I}, \mathscr{E}, \{\alpha_{(\lambda,\tau)}\}, \{\beta_{(\lambda,\tau)}\})$ *onto* Γ *such that* $\cup \ker \phi \cong K$ *(where* $\cup \ker \phi$ *means the union of members of the kernel of* ϕ[4]*)*.

Proof. Define ϕ as follows: $(i, \gamma, j)\phi = \gamma$, $(i, \gamma, j) \in (\Gamma, \mathscr{I}, \mathscr{E},$

4). If $\ker \Phi = \{K_\lambda : \lambda \in \Lambda\}$, then $\cup \ker \Phi$ means $\cup\{K_\lambda : \lambda \in \Lambda\}$.

$\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$). This mapping ϕ is clearly a homomorphism of (Γ, $\mathscr{I}$, $\mathscr{E}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) onto Γ. Now, it is obvious that $\bigcup$ ker $\phi = \{(i, \lambda, j) : \lambda \in \Lambda,\ i \in I_\lambda,\ j \in E_\lambda\}$. Define $\eta : K \to \bigcup$ ker ϕ by $(iu_\lambda j)\eta = (i, \lambda, j)$. Then by the definition of characteristic families, for $iu_\lambda j$, $ku_\tau h \in K$ (where $i \in I_\lambda$, $j \in E_\lambda$, $k \in I_\tau$, $h \in E_\tau$)

$((iu_\lambda j)(ku_\tau h))\eta = (i(j, k, h)\alpha_{(\lambda,\tau)}u_{\lambda\tau}(j, k, h)\beta_{(\lambda,\tau)})\eta =$

$(i(j, k, h)\alpha_{(\lambda,\tau)}, \lambda\tau, (j, k, h)\beta_{(\lambda,\tau)}) = (i, \lambda, j)(k, \tau, h) = ((iu_\lambda j)\eta)((ku_\tau h)\eta)$.

Since η is clearly injective, η is an isomorphism.

The semigroup (Γ, $\mathscr{I}$, $\mathscr{E}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) is particularly denoted by $R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) if there exists a set $\{u_\lambda : \lambda \in \Lambda\}$, where $u_\lambda \in K_\lambda$, such that $\{\alpha_{(\lambda,\tau)} : \lambda, \tau \in \Lambda\} \cup \{\beta_{(\lambda,\tau)} : \lambda, \tau \in \Lambda\}$ is the characteristic family of $K = \{iu_\lambda j : \lambda \in \Lambda,\ i \in I_\lambda,\ j \in E_\lambda\}$. We call this $R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) a *regular product* of $K(\Lambda)$ and $\Gamma(\Lambda)$.

COROLLARY. *The regular product* $R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) *above is a generalized orthodox semigroup.*

Proof. Consider the mappings ϕ, η defined in the proof of Lemma 5. Then, $\bigcup$ ker $\phi \cong K$ by η. Since $\{iu_\lambda j : i \in I_\lambda, j \in E_\lambda\} = K_\lambda$ is clearly isomorphic to $\{(i, \lambda, j) : i \in I_\lambda, j \in E_\lambda\} = \lambda\phi^{-1}$ and since K is a Cliffordian semigroup, $\bigcup$ ker ϕ is a normal Cliffordian subsemigroup of $R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$). Therefore, $R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) is a generalized orthodox semigroup.

4. The structure of generalized orthodox semigroups

The structure of generalized orthodox semigroups can be described by slightly modifying the method given by Warne [5] for orthodox semigroups to describe their structure. Now, consider the generalized orthodox semigroup S in the section 2. Then, $S = \Sigma\{S_\gamma : \gamma \in \Gamma\} = \{iu_\gamma j : \gamma \in \Gamma,\ i \in I_{\gamma\gamma^{-1}},\ j \in E_{\gamma^{-1}\gamma}\}$. For each pair (γ, τ) of $\gamma, \tau \in \Gamma$, there exist a unique $\alpha_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \times E_{\tau^{-1}\tau} \to I_{\gamma\tau(\gamma\tau)^{-1}}$ and a unique $\beta_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \times E_{\tau^{-1}\tau} \to E_{(\gamma\tau)^{-1}\gamma\tau}$ such that for $iu_\gamma j$ and ku_τh of S (where $i \in I_{\gamma\gamma^{-1}}$, $j \in E_{\gamma^{-1}\gamma}$, $k \in I_{\tau\tau^{-1}}$, $h \in E_{\tau^{-1}\tau}$), $u_\gamma jku_\tau h = (j, k, h)\alpha_{(\gamma,\tau)}u_{\gamma\tau}(j, k, h)\beta_{(\gamma,\tau)}$(hence, $iu_\gamma jku_\tau h = i(j, k, h)\alpha_{(\gamma,\tau)}u_{\gamma\tau}(j, k, h)\beta_{(\gamma,\tau)}$). It is easy to see that the set $\{\alpha_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\} \cup \{\beta_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ satisfies the conditions (A), (B) and (C). In fact : The condition (A) obviously holds. Let $q \in E_{\gamma^{-1}\gamma}$, $t \in I_{\tau\tau^{-1}}$, $h \in E_{\tau^{-1}\tau}$, $v \in I_{\delta\delta^{-1}}$ and $w \in E_{\delta^{-1}\delta}$. Then,

$(u_\gamma qtu_\tau h)vu_\delta w = ((q, t, h)\alpha_{(\gamma,\tau)}u_{\gamma\tau}(q, t, h)\beta_{(\gamma,\tau)})vu_\delta w =$

$(q, t, h)\alpha_{(\gamma,\tau)}((q, t, h)\beta_{(\gamma,\tau)}, v, w)\alpha_{(\gamma\tau,\delta)}u_{\gamma\tau\delta}((q, t, h)\beta_{(\gamma,\tau)}, v, w)\beta_{(\gamma\tau,\delta)}$. On the

other hand, $u_\gamma qt(u_\tau hvu_\delta w) = u_\gamma q(t(h, v, w)\alpha_{(\tau,\delta)} u_{\tau\delta}(h, v, w)\beta_{(\tau,\delta)}) =$
$(q, t(h, v, w)\alpha_{(\tau,\delta)}, (h, v, w)\beta_{(\tau,\delta)})\alpha_{(\gamma,\tau\delta)} u_{\gamma\tau\delta}(q, t(h, v, w)\alpha_{(\tau,\delta)}, (h, v, w)\beta_{(\tau,\delta)})\beta_{(\gamma,\tau\delta)}$.

Hence, the condition (B) holds. Next, let $\gamma \in \Gamma$, $p \in I_{\gamma\gamma^{-1}}$ and $q \in E_{\gamma^{-1}\gamma}$. Since S is regular, there exists $ku_\xi n$ ($k \in I_{\xi\xi^{-1}}$ and $n \in E_{\xi^{-1}\xi}$) such that $pu_\gamma qku_\xi npu_\gamma q = pu_\gamma q$ and $ku_\xi n = ku_\xi npu_\gamma qku_\xi n$. It is easy to see that $\xi = \gamma^{-1}$. Hence $k \in I_{\gamma^{-1}\gamma}$ and $n \in E_{\gamma\gamma^{-1}}$. Now, $pu_\gamma qku_{\gamma^{-1}}npu_\gamma q = pu_\gamma q$ implies the condition (C). Further, $\{\alpha_{(\lambda,\tau)} : \lambda, \tau \in \Lambda\} \cup \{\beta_{(\lambda,\tau)} : \lambda, \tau \in \Lambda\}$ is clearly the characteristic family of $K = \{iu_\lambda j : \lambda \in \Lambda, i \in I_\lambda, j \in E_\lambda\}$. Hence $R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) (where $\mathscr{I} = \Sigma\{I_\lambda : \lambda \in \Lambda\}$ and $\mathscr{E} = \Sigma\{E_\lambda : \lambda \in \Lambda\}$) can be considered, and we have the following :

THEOREM 4, *S is isomorphic to* $R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$).

Proof. Let us define $\phi : S \to R(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\alpha_{(\gamma,\tau)}\}$, $\{\beta_{(\gamma,\tau)}\}$) by $(iu_\gamma j)\phi = (i, \gamma, j)$. Then, $((iu_\gamma j)(ku_\tau h))\phi = (i(j, k, h)\alpha_{(\gamma,\tau)} u_{\gamma\tau}(j, k, h)\beta_{(\gamma,\tau)})\phi$ $= (i(j, k, h)\alpha_{(\gamma,\tau)}, \gamma\tau, (j, k, h)\beta_{(\gamma,\tau)}) = (i, \gamma, j)(k, \tau, h) = ((iu_\gamma j)\phi)((ku_\tau h)\phi)$. Since ϕ is clearly surjective and injective, ϕ is an isomorphism.

By the theorem above and Corollary to Lemma 5, we have the following result :

COROLLARY. Any generalized orthodox semigroup is isomorphic to a regular product of a Cliffordian semigroup and an inverse semigroup. Conversely, any regular product of a Cliffordian semigroup and an inverse semigroup is a generalized orthodox semigroup.

5. Preorthodox semigroups.

Let S be a regular semigroup. If there exist an epimorphism (i. e., onto-homomorphism) $\phi : S \to \Gamma$ of S onto an inverse semigroup Γ and a homomorphism $\psi : \Gamma \to S$ such that

(1) $\psi\phi = 1$ (identity mapping) and

(2) $\cup$ ker ϕ = a Cliffordian subsemigroup $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ of Γ, and $K_\lambda = \lambda\phi^{-1}$ for all $\lambda \in \Lambda$,

then S is called a *split extension* of $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by $\Gamma(\Lambda)$.

The following results are obvious from the preceding sections : Let S be a split extension of a Cliffordian subsemigroup $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by an inverse semigroup $\Gamma(\Lambda)$. Then, there exist an epimorphism $\phi : S \to \Gamma$ and a homomorphism $\psi : \Gamma \to S$ such that ϕ and ψ satisfy the conditions (1), (2) above. Since K is clearly a normal Cliffordian subsemigroup of S, the decomposition of S

determined by the kernel $\{K_\lambda : \lambda \in \Lambda\}$ of ϕ is $S/K = \{S_\gamma : \gamma \in \Gamma\}$, where $S_\gamma = \gamma\phi^{-1}$ for $\gamma \in \Gamma$ and especially $S_\lambda = K_\lambda$ for $\lambda \in \Lambda$, and (i) $S_\alpha S_\beta \subset S_{\alpha\beta}$ for $\alpha, \beta \in \Gamma$; (ii) $S = \Sigma\{S_\gamma : \gamma \in \Gamma\}$. Further, let $\gamma\psi = u_\gamma$ for $\gamma \in \Gamma$. Then, (iii) $u_\gamma \in S_\gamma$ for all $\gamma \in \Gamma$ and the set $T = \{u_\gamma : \gamma \in \Gamma\}$ is isomorphic to Γ since ψ is a monomorphism (i. e., 1–1, into-homomorphism). Therefore, of course T is an inverse subsemigroup of S.

Now, let S be the above-mentioned split extension of $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by $\Gamma(\Lambda)$. For each $\lambda \in \Lambda$, *let* I_λ, J_λ be the $\mathscr{L}$-class, $\mathscr{R}$-class of K_λ respectively such that $I_\lambda \in u_\lambda$ and $j_\lambda \in u_\lambda$. Put $\Sigma\{I_\lambda : \lambda \in \Lambda\} = \mathscr{I}$ and $\Sigma\{J_\lambda : \lambda \in \Lambda\} = \mathscr{J}$. Then, it follows that $\mathscr{I}$ and $\mathscr{J}$ are a lower partial chain of left groups $\{I_\lambda : \lambda \in \Lambda\}$ and an upper partial chain of right groups $\{J_\lambda : \lambda \in \Lambda\}$ respectively. Let E_λ be the set of idempotents of J_λ, and put $\Sigma\{E_\lambda : \lambda \in \Lambda\} = \mathscr{E}$. Of course, each E_λ is a right zero semigroup. Further, we have the following :

LEMMA 6. $\mathscr{E} = \Sigma\{E_\lambda : \lambda \in \Lambda\}$ *is an upper partial chain of right zero semigroups* $\{E_\lambda : \lambda \in \Lambda\}$.

Proof. Let $\lambda \leq \mu$, $x \in E_\lambda$ and $y \in E_\mu$. We have $xyxy = xyu_\lambda xy = xyu_\mu u_\lambda xy = xu_\mu u_\lambda xy = xu_\lambda xy$ (since $T = \{u_\lambda : \lambda \in \Lambda\}$ is a semilattice and hence $u_\mu u_\lambda = u_\lambda$) $= x^2y = xy$. Hence, xy is an idempotent. Since $xy \in J_\lambda$, we have $xy \in E_\lambda$. Thus, $\mathscr{E}$ is an upper partal chain of $\{E_\lambda : \lambda \in \Lambda\}$.

A Cliffordiian semigroup $M \equiv \Sigma\{M_\lambda : \lambda \in \Lambda\}$ is called a *left* [*right*] *Cliffordian semigroup* if there exists a system $\{I_\lambda : \lambda \in \Lambda\}$ [$\{J_\lambda : \lambda \in \Lambda\}$] of $\mathscr{L}$-classes I_λ of M_λ's [$\mathscr{R}$-classes J_λ of M_λ's] (each I_λ [J_λ] is an $\mathscr{L}$-class [$\mathscr{R}$-class] of M_λ) such that $\Sigma\{F_\lambda : \lambda \in \Lambda\} = \mathscr{E}$, where F_λ is the set of idempotents of I_λ [J_λ], is a lower [upper] partial chain of $\{F_\lambda : \lambda \in \Lambda\}$. A generalized orthodox semigroup G is called a *preorthodox semigroup* if there exists a normal left Cliffordian subsemigroup or a normal right Cliffordian subsemigroup in G. Therefore, the split extension S of $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by $\Gamma(\Lambda)$ above is of course a preorthodox semigroup. In this case, it is easily seen from the dual result of Lemma 6 with respect to "left and right" that K should be necessarily also left Cliffordian. Further, it is also easily seen that any orthodox union of groups is both left and right Cliffordian and hence any orthodox semigroup is a preorthodox semigroup. Hence, we have

LEMMA 7. *A split extension of a Cliffordian semigroup* $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ *by an inverse semigroup* $\Gamma(\Lambda)$ *is a preorthodox semigroup. In this case,* K *should be necessarily left and right Cliffordian.*

6. Semidirect products

Let S be a preorthodox semigroup. Then, there exists a normal right or left Cliffordian subsemigroup $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$. We assume without loss of generality that K is a normal right Cliffordian subsemigroup. Let $\{I_\lambda : \lambda \in \Lambda\}$, $\{J_\lambda : \lambda \in \Lambda\}$ be two systems such that (i) each I_λ is an $\mathscr{L}$-class of K_λ and each J_λ is an $\mathscr{R}$-class of K_λ and (ii) $\mathscr{E} = \Sigma\{E_\lambda : \lambda \in \Lambda\}$, where each E_λ is the set of idempotents of J_λ, is an upper partial chain of $\{E_\lambda : \lambda \in \Lambda\}$. Since $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ is a normal Cliffordian subsemigroup of S, there exists the congruence π_k determined by the normal system $\{K_\lambda : \lambda \in \Lambda\}$ of subsets of S[5]; that is, $S/K = S/\pi_k$. There exists an inverse semigroup Γ having Λ as its basic semilattice such that (i) $S/K = \{S_\gamma : \gamma \in \Gamma\}$, (ii) $S_\lambda = K_\lambda$ for $\lambda \in \Lambda$ and (iii) $S = \Sigma\{S_\gamma : \gamma \in \Gamma\}$ and $S_\alpha S_\beta \subset S_{\alpha\beta}$ for $\alpha, \beta \in \Gamma$.

The following results is obvious from the preceding sections : Take a representative u_γ from each S_γ. Then, each element x can be uniquely expressed in the form $x = iu_\gamma j$, where $\gamma \in \Gamma$, $i \in I_{\gamma\gamma^{-1}}$ and $j \in E_{\gamma^{-1}\gamma}$. Hence, $S = \{iu_\gamma j : \gamma \in \Gamma,\ i \in I_{\gamma\gamma^{-1}},\ j \in E_{\gamma^{-1}\gamma}\}$, and $iu_\gamma j = ku_\tau h$, where $i \in I_{\gamma\gamma^{-1}}$, $j \in E_{\gamma^{-1}\gamma}$, $k \in I_{\tau\tau^{-1}}$ and $h \in E_{\tau^{-1}\tau}$, implies $\gamma = \tau$, $i = k$ and $j = h$. For $\gamma, \tau \in \Gamma$, define $\alpha_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \times E_{\tau^{-1}\tau} \to I_{\gamma\tau(\gamma\tau)^{-1}}$ and $\beta_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \times E_{\tau^{-1}\tau} \to E_{(\gamma\tau)^{-1}\gamma\tau}$ as follows : For $j \in E_{\gamma^{-1}\gamma}$, $k \in I_{\tau\tau^{-1}}$ and $h \in E_{\tau^{-1}\tau}$, $(j, k, h)\alpha_{(\gamma,\tau)} = t$ and $(j, k, h)\beta_{(\gamma,\tau)} = v$ if $u_\gamma jku_\tau h = tu_{\gamma\tau}v$ where $t \in I_{\gamma\tau(\gamma\tau)^{-1}}$ and $v \in E_{(\gamma\tau)^{-1}\gamma\tau}$. Then, the system $\Delta = \{\alpha_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\} \cup \{\beta_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ satisfies A, B, C of the section 3.

Further, the system $\{\alpha_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\} \cup \{\beta_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\}$ is the characteristic family of $K = \{iu_\lambda j : \lambda \in \Lambda,\ i \in I_\lambda,\ j \in E_\lambda\}$. Therefore we can consider the regular product $R(\Gamma, K(\Lambda) ; \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ (of $K(\Lambda)$ and $\Gamma(\Lambda)$), where $\mathscr{I} = \Sigma\{I_\lambda : \lambda \in \Lambda\}$, and the mapping $\phi : S \to R(\Gamma, K(\Lambda) ; \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ defined by $(iu_\gamma j)\phi = (i, \gamma, j)$ gives an isomorphism.

Now, define other mappings $\bar{\alpha}_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \to I_{\gamma\tau(\gamma\tau)^{-1}}$ and $\bar{\beta}_{(\gamma,\tau)} : E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}} \to E_{(\gamma\tau)^{-1}\gamma\tau}$ for each pair (γ, τ) of $\gamma, \tau \in \Gamma$ as follows :

For $j \in E_{\gamma^{-1}\gamma}$ and $k \in I_{\tau\tau^{-1}}$,

$(j, k)\,\bar{\alpha}_{(\gamma,\tau)} = t$ and $(j, k)\,\bar{\beta}_{(\gamma,\tau)} = v$ *if* $u_\gamma jku_\tau = tu_{\gamma\tau}v$ *where* $t \in I_{\gamma\tau(\gamma\tau)^{-1}}$ and $v \in E_{(\gamma\tau)^{-1}\gamma\tau}$.

Let $\bar{\Delta} = \{\bar{\alpha}_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\} \cup \{\bar{\beta}_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$. Then, this system $\bar{\Delta}$ satisfies the following $\bar{A}$, $\bar{B}$, $\bar{C}$:

$\bar{A}$. $D(\bar{\alpha}_{(\gamma,\tau)}) = D(\bar{\beta}_{(\gamma,\tau)}) = E_{\gamma^{-1}\gamma} \times I_{\tau\tau^{-1}}$; $R(\bar{\alpha}_{(\gamma,\tau)}) \subset I_{\gamma\tau(\gamma\tau)^{-1}}$, $R(\bar{\beta}_{(\gamma,\tau)}) \subset E_{(\gamma\tau)^{-1}\gamma\tau}$;

5) If $\{R_\lambda : \lambda \in \Lambda\}$ is a normal system of subsets of a regular semigroup R, then there exists a unigue congruence ρ on R such that each R_λ is a complete ρ-class. This congruence ρ is called *the congruence on R determined by* $\{R_\lambda : \lambda \in \Lambda_\lambda\}$. (see also [1].)

$\bar{B}$. for $q \in E_{\gamma^{-1}\gamma}$, $t \in I_{\tau\tau^{-1}}$, $h \in E_{\tau^{-1}\tau}$ and $v \in I_{\delta\delta^{-1}}$,
$(q, t)\bar{\alpha}_{(\gamma,\tau)}((q, t)\bar{\beta}_{(\gamma,\tau)}h, v)\,\bar{\alpha}_{(\gamma\tau,\delta)} = (q, t\,(h, v)\bar{\alpha}_{(\tau,\delta)})\bar{\alpha}_{(\gamma,\tau\delta)}$ and
$(q, t\,(h, v)\bar{\alpha}_{(\tau,\delta)})\bar{\beta}_{(\gamma,\tau\delta)}(h, v)\bar{\beta}_{(\tau,\delta)} = ((q, t)\,\bar{\beta}_{(\gamma,\tau)}h, v)\bar{\beta}_{(\gamma\tau,\delta)}$;
$\bar{C}$. for $\gamma \in \Gamma$, $p \in I_{\gamma\gamma^{-1}}$ and $q \in E_{\gamma^{-1}\gamma}$, there exist $k \in I_{\gamma^{-1}\gamma}$ and $n \in E_{\gamma\gamma^{-1}}$ such that

$$p(q, k)\bar{\alpha}_{(\gamma,\gamma^{-1})}((q, k)\bar{\beta}_{(\gamma,\gamma^{-1})}n, p)\bar{\alpha}_{(\gamma\gamma^{-1},\gamma)} = p.$$

Further, $\{\bar{\alpha}_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\} \cup \{\bar{\beta}_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\}$ satisfies
($\bar{D}$) $u_\lambda jku_\delta = (j, k)\bar{\alpha}_{(\lambda,\delta)}u_{\lambda\delta}(j, k)\bar{\beta}_{(\lambda,\delta)}$ for $\lambda, \delta \in \Lambda$, $j \in E_\lambda$ and $k \in I_\delta$.

Now, for $iu_\gamma j$, $ku_\tau h \in S$ (where $i \in I_{\gamma\gamma^{-1}}$, $j \in E_{\gamma^{-1}\gamma}$, $k \in I_{\tau\tau^{-1}}$ and $h \in E_{\tau^{-1}\tau}$)
$u_\gamma jku_\tau h = (j, k, h)\alpha_{(\gamma,\tau)}u_{\gamma\tau}(j, k, h)\beta_{(\gamma,\tau)}$.
On the other hand, $u_\gamma jku_\tau h = (j, k)\bar{\alpha}_{(\gamma,\tau)}u_{\gamma\tau}(j, k)\bar{\beta}_{(\gamma,\tau)}h$.
Hence,
(6. 1) $(j, k, h)\alpha_{(\gamma,\tau)} = (j, k)\bar{\alpha}_{(\gamma,\tau)}$ and $(j, k, h)\beta_{(\gamma,\tau)} = (j, k)\bar{\beta}_{(\gamma,\tau)}h$.
Accordingly, the multiplication in $R(\Gamma, K(\Lambda)\,;\ \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ is given by
(6. 2) $(i, \gamma, j)(k, \tau, h) = (i(j, k, h)\alpha_{(\gamma,\tau)}, \gamma\tau, (j, k, h)\beta_{(\gamma,\tau)})$
$= (i(j, k)\bar{\alpha}_{(\gamma,\tau)}, \gamma\tau, (j, k)\bar{\beta}_{(\gamma,\tau)}h)$.

Next, we shall introduce the concept of a complete regular product of a right [left] Cliffordian semigroup and an inverse semigroup. Let Γ be an inverse semigroup, and Λ its basic semilattice. Let $K_\lambda \equiv \Sigma\{K_\lambda : \lambda \in \Delta\}$ be a right Cliffordian semigroup. Let I_λ, J_λ be an $\mathscr{L}$-class, an $\mathscr{R}$-class of K_λ respectively such that $\mathscr{E} = \Sigma\{E_\lambda : \lambda \in \Lambda\}$, where E_λ is the set of idempotents of J_λ, is an upper partial chain of $\{E_\lambda : \lambda \in \Lambda\}$. Suppose that a system $\bar{\Delta} = \{\bar{\alpha}_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\} \cup \{\bar{\beta}_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ satisfies $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\{\bar{\alpha}_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\} \cup \{\bar{\beta}_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\}$ satisfies ($\bar{D}$) with respect to a set $\{u_\lambda : \lambda \in \Lambda\}$, where u_λ is a representative of K_λ. Then, the set $G = \{(i, \gamma, j) : \gamma \in \Gamma, i \in I_{\gamma\gamma^{-1}}, j \in E_{\gamma^{-1}\gamma}\}$ becomes a regular extension of $K(\Lambda)$ by $\Gamma(\Lambda)$, accordingly to a preorthodox semigroup, under the multiplication defined by (6. 2).

In fact : If we define $\alpha_{(\gamma,\tau)}$, $\beta_{(\gamma,\tau)}$ by using $\bar{\alpha}_{(\gamma,\tau)}\,\bar{\beta}_{(\gamma,\tau)}$ and (6. 1) then the system $\Delta = \{\alpha_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\} \cup \{\beta_{(\gamma,\tau)} : \gamma, \tau \in \Gamma\}$ satisfies A, B, C and also $\{\alpha_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\} \cup \{\beta_{(\lambda,\delta)} : \lambda, \delta \in \Lambda\}$ becomes the characteristic family of $K = \{iu_\lambda j : \lambda \in \Lambda, i \in I_\lambda, j \in E_\lambda\}$. Hence, a regular product $R(\Gamma, K(\Lambda)\,;\ \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})^-$ (where $\mathscr{I} = \Sigma\{I_\lambda : \lambda \in \Lambda\}$) of $K(\Lambda)$ and $\Gamma(\Lambda)$ can be considered and coincides with the regular semigroup G above. Since $^-R(\Gamma, K(\Lambda)\,;\ ^-\mathscr{I}, ^-\mathscr{E}, \{u_\lambda\}, \{\alpha_{(\gamma,\tau)}\}, \{\beta_{(\gamma,\tau)}\})$ is clearly a preorthodox semigroup, G is also a preorthodox semigroup. We shall call such a G a *complete regular product* of $K(\Lambda)$ and $\Gamma(\Lambda)$, and denote G by $C(\Gamma, K(\Lambda);\ \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\bar{\alpha}_{(\gamma,\tau)}\}, \{\bar{\beta}_{(\gamma,\tau)}\})$.

By the results above, we have

THEOREM 5. *A regular semigroup S is a preorthodox semigroup if and only if S is isomorphic to a complete regular product of a right or left Cliffordian semigroup $K(\Lambda)$ and an inverse semigroup $\Gamma(\Lambda)$.*

Proof. Obvious.

Remark. A complete regular product of a left Cliffordian semigroup and an inverse semigroup can be defined by the dual method concerning "left and right".

Consider a complete regular product $C(\Gamma, K(\Lambda);\ \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\bar{\alpha}_{(\gamma,\tau)}\}, \{\bar{\beta}_{(\gamma,\tau)}\})$ as above. If there exists a system $\{i_\gamma : \gamma \in \Gamma\} \cup \{j_\gamma : \gamma \in \Gamma\}$, where $i_\gamma \in I_{\gamma\gamma^{-1}}$ and $j_\gamma \in E_{\gamma^{-1}\gamma}$, such that

(6. 3) $i_\gamma (j_\gamma, i_\tau)\bar{\alpha}_{(\gamma,\tau)} = i_{\gamma\tau}$ and $(j_\gamma, i_\tau)\bar{\beta}_{(\gamma,\tau)} j_\tau = j_{\gamma\tau}$ for $\gamma, \tau \in \Gamma$, then we shall call such special complete regular product a *semidirect product* of $K(\Lambda)$ and $\Gamma(\Lambda)$, and denote it especially by $S(\Gamma, K(\Lambda);\ \mathscr{I}, \mathscr{E}, \{u_\lambda\}, \{\bar{\alpha}_{(\gamma,\tau)}\}, \{\bar{\beta}_{(\gamma,\tau)}\})$.

When $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ is a left Cliffordian semigroup, we can also define the concept of "semidirect products of $K(\Lambda)$ and $\Gamma(\Lambda)$" by the dual method concerning "left and right".

7. Split extensions.

In this section, we establish necessary and sufficient conditions for a regular semigroup to be a split extension of a normal right Cliffordian subsemigroup by an inverse semigroup. If S is a split extension of a normal right Cliffordian subsemigroup $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by an inverse semigroup $\Gamma(\Lambda)$, then it follows from Lemma 7 that $K(\Lambda)$ is necessarily also left Cliffordian. Hence, S is a split extension of a normal left Cliffordian subsemigroup by an inverse semigroup.

THEOREM 6. *For a regular semigroup S, the following three conditions are equivalent* :

(1) *S is a split extension of a normal right Cliffordian subsemigroup by an inverse semigroup.*

(2) *S contains a normal right Cliffordian subsemigroup $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ and an inverse subsemigroup N such that*

(i) $K_\lambda \cap N =$ *a single element* u_λ *for all* $\lambda \in \Lambda$, *and*

(ii) $S = K \circ N \circ K = \bigcup \{K(uu^{-1})uK(u^{-1}u) : u \in N\}$, *where* $K(x)$ *means the class* K_λ *containing* x *and* u^{-1} *means the inverse of* u *in* N.

(3) *S is isomorphic to a semidirect product of a right Cliffordian semigroup and an inverse semigroup.*

Proof. (1) $\Rightarrow$ (2) : Assume that S is a split extension of a normal right Cliffordian subsemigroup $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ by an inverse semigroup $\Gamma(\Lambda)$. By the definition of split extensions, there exist an epimorphism $\phi : S \rightarrow \Gamma(\Lambda)$ and a homomorphism $\psi : \Gamma(\Lambda) \rightarrow S$ such that $\lambda\phi^{-1} = K_\lambda$ for all $\lambda \in \Lambda$ and $\psi\phi = 1$. Put $\{\gamma\psi : \gamma \in \Gamma\} = N$. Then, ψ is an isomorphism of $\Gamma(\Lambda)$ onto N. Hence, N is an inverse subsemigroup of S. Put $\gamma\phi^{-1} = S_\gamma$ for all $\gamma \in \Gamma$ (hence $K_\lambda = S_\lambda$ for $\lambda \in \Lambda$). If $K_\lambda \cap N \ni a$, then $a\phi = \lambda$. Let γ be an element of Γ such that $a = \gamma\psi$. Then, $\gamma\psi\phi = \lambda$. Since $\psi\phi = 1$, this implies $\gamma = \lambda$. Therefore γ is uniquely determined, and hence a is also unique. That is, $K_\lambda \cap N$ consists of a single element, say u_λ. Now, for each $\lambda \in \Lambda$, let I_λ, J_λ be the $\mathscr{L}$-class, the $\mathscr{R}$-class of K_λ such that $I_\lambda \ni u_\lambda$ and $J_\lambda \ni u_\lambda$ respectively. Let E_λ be the set of idempotents of J_λ. Since S is clearly a generalized orthodox semigroup and $u_\gamma \in S_\gamma$, any x of S_γ can be uniquely expressed in the form $x = iu_\gamma j$ where $i \in I_{\gamma\gamma^{-1}}$ and $j \in E_{\gamma^{-1}\gamma}$. Since $u_\gamma u_\gamma^{-1} \in K_{\gamma\gamma^{-1}}$, $u_\gamma^{-1}u_\gamma \in K_{\gamma^{-1}\gamma}$, it follows that $x \in K(u_\gamma u_\gamma^{-1})u_\gamma K(u_\gamma^{-1}u_\gamma)$. Thus, (i) and (ii) are satisfied.

(2) $\Rightarrow$ (3) : Assume the condition (2). Since $\{K_\lambda : \lambda \in \Lambda\}$ constitutes a normal system of subsets of S, there exists the inverse semigroup congruence ρ determined by $\{K_\lambda : \lambda \in \Lambda\}$; that is, $S/\rho = \{S_\gamma : \gamma \in \Gamma\}$ (where Γ is an inverse semigroup containing Λ as its basic semilattice) and $K_\lambda = S_\lambda$ for $\lambda \in \Lambda$. For any $\gamma \in \Gamma$ and for any $x \in S_\gamma$, x can be expressed in the form $x = zuw$, where $u \in N$, $z \in K(uu^{-1})$ and $w \in K(u^{-1}u)$. Since $zuw \in S(uu^{-1})S(u)S(u^{-1}u)$, where $S(x)$ means the ρ-class containing x, we have $zuw \in S(u)$. Hence $u \in S_\gamma$. Therefore, S_γ contains at least one element of N. Suppose that $S_\gamma \cap N \ni u_1, u_2$. There exist K_δ, K_ξ such that $S(u_1u_1^{-1}) = S(u_2u_2^{-1}) = K_\delta$ and $S(u_2^{-1}u_1) = S(u_2^{-1}u_2) = K_\xi$. Since each of K_δ, K_ξ contains only one element of N, it follows that $u_1u_1^{-1} = u_2u_2^{-1}$ and $u_2^{-1}u_1 = u_2^{-1}u_2$. Hence, $u_1 = u_1u_1^{-1}u_1 = u_2u_2^{-1}u_1 = u_2u_2^{-1}u_2 = u_2$. Thus, $S_\gamma \cap N$ consists of a single element, say u_γ, for all $\gamma \in \Gamma$. Hence, $\psi : \Gamma \rightarrow N$ defined by $\gamma\psi = u_\gamma$ is an isomorphism. For each $\lambda \in \Lambda$, let I_{u_λ} be the $\mathscr{L}$-class of K_λ that contains u_λ, J_{u_λ} the $\mathscr{R}$-class of K_λ that contains u_λ, and E_{u_λ} the set of all idempotents of J_{u_λ}. Then, of course $S = \{iu_\gamma j : u_\gamma \in N, i \in I_{u_\gamma u_\gamma^{-1}}, j\ E_{u_\gamma^{-1}u_\gamma}\}$. Let ϕ be the natural homomorphism of S onto S/ρ. Then, $\psi\phi = 1$ and hence S is a split extension of $K \equiv \Sigma\{K_\lambda :$

$\lambda \in \Lambda\}$ by $\Gamma(\Lambda)$. Hence $\mathscr{E} = \Sigma\{E_{u_\lambda} : u_\lambda \in T\}$, where T is the basic semilattice of N, is an upper partial chain of $\{E_{u_\lambda} : u_\lambda \in T\}$. Now, define $\bar{\alpha}_{(u_\gamma, u_\tau)}$: $E_{u_\gamma^{-1}u_\gamma} \times I_{u_\tau u_\tau^{-1}} \to I_{u_{\gamma\tau}u_{\gamma\tau}^{-1}}$, $\bar{\beta}_{(u_\gamma, u_\tau)}$: $E_{u_\gamma^{-1}u_\gamma} \times I_{u_\tau u_\tau^{-1}} \to E_{u_{\gamma\tau}^{-1}u_{\gamma\tau}}$ by

$u_\gamma jku_\tau = (j, k)\bar{\alpha}_{(u_\gamma, u_\tau)} u_{\gamma\tau}(j, k)\bar{\beta}_{(u_\gamma, u_\tau)}$ for $j \in E_{u_\gamma^{-1}u_\gamma}$ and $k \in I_{u_\tau u_\tau^{-1}}$.

Then, as was shown above, $\{\bar{\alpha}_{(u_\gamma, u_\tau)} : u_\gamma, u_\tau \in N\} \cup \{\bar{\beta}_{(u_\gamma, u_\tau)} : u_\gamma, u_\tau \in N\}$ satisfies $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\{\bar{\alpha}_{(u_\lambda, u_\delta)} : u_\lambda, u_\delta \in T\} \cup \{\bar{\beta}_{(u_\lambda, u_\delta)} : u_\lambda, u_\delta \in T\}$ satisfies $(\bar{D})$. Hence, S is isomorphic to the complete regular product $C(N, K(T)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\bar{\alpha}_{(u_\gamma, u_\tau)}\}$, $\{\bar{\beta}_{(u_\gamma, u_\tau)}\}$) (where $\mathscr{I} = \Sigma\{I_{u_\lambda} : u_\lambda \in T\}$) under the following mapping ϕ : $(i, u_\gamma, j)\phi = iu_\gamma j$.

Now, put $u_\gamma u_\gamma^{-1} = i_{u_\gamma}$, $u_\gamma^{-1}u_\gamma = j_{u_\gamma}$. Then, $u_\gamma = i_{u_\gamma}u_\gamma j_{u_\gamma}$, $i_{u_\gamma} \in I_{u_\gamma u_\gamma^{-1}}$, $j_{u_\gamma} \in E_{u_\gamma^{-1}u_\gamma}$; $u_\tau = i_{u_\tau}u_\tau j_{u_\tau}$, $i_{u_\tau} \in I_{u_\tau u_\tau^{-1}}$, $j_{u_\tau} \in E_{u_\tau^{-1}u_\tau}$; and $u_{\gamma\tau} = i_{u_{\gamma\tau}}u_{\gamma\tau}j_{u_{\gamma\tau}}$, $i_{u_{\gamma\tau}} \in I_{u_{\gamma\tau}u_{\gamma\tau}^{-1}}$, $j_{u_{\gamma\tau}} \in E_{u_{\gamma\tau}^{-1}u_{\gamma\tau}}$. Hence $i_{u_{\gamma\tau}}u_{\gamma\tau}j_{u_{\gamma\tau}} = i_{u_\gamma}(j_{u_\gamma}, i_{u_\tau})\bar{\alpha}_{(u_\gamma, u_\tau)}u_{\gamma\tau}(j_{u_\gamma}, i_{u_\tau})\bar{\beta}_{(u_\gamma, u_\tau)}j_{u_\tau}$, and accordingly $i_{u_\gamma}(j_{u_\gamma}, i_{u_\tau})\bar{\alpha}_{(u_\gamma, u_\tau)} = i_{u_{\gamma\tau}} = i_{u_\gamma u_\tau}$ and $(j_{u_\gamma}, i_{u_\tau})\bar{\beta}_{(u_\gamma, u_\tau)}j_{u_\tau} = j_{u_{\gamma\tau}} = j_{u_\gamma u_\tau}$. Thus $C(N, K(T)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\bar{\alpha}_{(u_\gamma, u_\tau)}\}$, $\{\bar{\beta}_{(u_\gamma, u_\tau)}\}$) is a semidirect product of $K(T)$ and $N(T)$. Consequently, S is isomorphic to $S(N, K(T)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\bar{\alpha}(u_\gamma, u_\tau)\}$, $\{\bar{\beta}(u_\gamma, u_\tau)\}$).

(3) $\Rightarrow$ (1) : Let $K \equiv \Sigma\{K_\lambda : \lambda \in \Lambda\}$ be a right Cliffordian semigroup, and $\Gamma(\Lambda)$ an inverse semigroup having Λ as its basic semilattice. Let I_λ be an $\mathscr{L}$-class of K_λ for each $\lambda \in \Lambda$, and J_λ an $\mathscr{R}$-class of K_λ for each $\lambda \in \Lambda$ such that $\mathscr{E} = \Sigma\{E_\lambda : \lambda \in \Lambda\}$, where E_λ is the set of idempotents of J_λ, is an upper partial chain of $\{E_\lambda : \lambda \in \Lambda\}$. Let $S(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\bar{\alpha}_{(\gamma,\tau)}\}$, $\{\bar{\beta}_{(\gamma,\tau)}\}$) be the semidirect product of $K(\Lambda)$ and $\Gamma(\Lambda)$ determined by $\mathscr{I} = \Sigma\{I_\lambda : \lambda \in \Lambda\}$ and $\mathscr{E}$ and by a system $\{\{u_\lambda\}, \{\bar{\alpha}_{(\gamma,\tau)}\}, \{\bar{\beta}_{(\gamma,\tau)}\}\}$ satisfying $\bar{A}$, $\bar{B}$, $\bar{C}$, $(\bar{D})$ and (6. 3). By the definition of semidirect products, there exist a system $\{i_\gamma : \gamma \in \Gamma\} \cup \{j_\gamma : \gamma \in \Gamma\}$, where $i_\gamma \in I_{\gamma\gamma^{-1}}$ and $j_\gamma \in E_{\gamma^{-1}\gamma}$, such that $i_\gamma(j_\gamma, i_\tau)\bar{\alpha}_{(\gamma,\tau)} = i_{\gamma\tau}$ and $(j_\gamma, i_\tau)\bar{\beta}_{(\gamma,\tau)}j_\tau = j_{\gamma\tau}$. Put $\{(i_\gamma, \gamma, j_\gamma) : \gamma \in \Gamma\} = N$, and define ψ : $\Gamma \to N$ and ϕ : $S(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\bar{\alpha}_{(\gamma,\tau)}\}$, $\{\bar{\beta}_{(\gamma,\tau)}\}) \to \Gamma$ as follows : $\gamma\psi = (i_\gamma, \gamma, j_\gamma)$ and $(i, \tau, j)\phi = \tau$. Then, ψ and ϕ are an isomorphism and an epimorphism respectively, and satisfy $\psi\phi = 1$. Let $M_\lambda = \{(k, \lambda, h) : k \in I_\lambda, h \in E_\lambda\}$ for each $\lambda \in \Lambda$. Then, $\ker \phi = \{M_\lambda : \lambda \in \Lambda\}$. Since $M_\lambda \cong K_\lambda$ (see [4]), M_λ is a completely simple semigroup. Hence, $S(\Gamma, K(\Lambda)$; $\mathscr{I}$, $\mathscr{E}$, $\{u_\lambda\}$, $\{\bar{\alpha}_{(\gamma,\tau)}\}$, $\{\bar{\beta}_{(\gamma,\tau)}\}$) is a split extension of $M \equiv \Sigma\{M_\lambda : \lambda \in \Lambda\}$ by $\Gamma(\Lambda)$.

Remark. If a group G is a split extension (in the sense of this paper) of a normal right Cliffordian subsemigroup K and an inverse semigroup Γ, then K and Γ should be necessarily groups. Also, if G is isomorphic to a semidirect product (in the sense of this paper) of a right Cliffordian semigroup K' and an inverse semigroup Γ' then K' and Γ' should be necessarily groups. Moreover, it is easily prove that for the class of groups, the concepts of split extensions and semidirect products in the sense of this paper completely coincide with those in the group theory respectively. Further, if we restrict S to groups then $K(\Lambda)$, N in the theorem above are also groups, especially N is a normal subgroup of S, and $K\circ N\circ K = KN$. Hence as a special case, if we restrict S to groups then the theorem above means the following well-known result in the group theory : For a group S, the following three conditions are equivalent.

(1) S is a split extension of a normal subgroup H by a group G.

(2) There exist a normal subgroup H of S and a subgroup N of S such that (i) $S=HN$ and (ii) $H\cap N=1$.

(3) S is isomorphic to a semidirect product of a group M and a group G.

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