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On Random Fields on Homogeneous Space

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In this paper, we define the homogeneous random field, the isotropic random field, w-homogeneous random field and w-isotropic random field on homogeneous space $S=G/K$. We give some representations of w-homogeneous field from the view point of $L^2(\Omega)$-theory.

§ 1. Introduction

Let G be a locally compact group and $S=G/K$ be the associated homogeneous space, or let G be a connected Lie group and (G, K) be the Riemannian symmetric pair with K compact subgroup and $S=G/K$ be the associated Riemannian symmetric space. Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space and $\{X(P), P\in S\}$ be a family of $\mathfrak{B}$-measurable, complex-valued functions. Such a family we call *a random field on* $S=G/K$.

DEFINITION 1. For any set of Borel measurable subsets of complex field $\mathbf{C}$, $E_1,\dots, E_n$, we define the function

$$F(E_1,\dots, E_n; P_1,\dots, P_n)=\mu\{X(P_1)\in E_1,\dots, X(P_n)\in E_n\}, \tag{1}$$

where $P_1,\dots, P_n$ are points of S.

The random field $X=\{X(P), P\in S\}$ is called *homogeneous* iff

$$F(E_1,\dots, E_n; gP_1,\dots, gP_n)=F(E_1,\dots, E_n; P_1,\dots, P_n) \qquad \text{for any } g \text{ of } G. \tag{2}$$

The random field X is called *isotropic* iff

$$F(E_1,\dots, E_n; kP_1,\dots, kP_n)=F(E_1,\dots, E_n; P_1,\dots, P_n) \qquad \text{for any } k \text{ of } K. \tag{3}$$

REMARK. In the case of S being the d-dimensional vector space E^d (see, p. 28, (A)), our homogeneous random field is called *homogeneous and isotoropic*, and our isotropic one is called *isotropic*.

DEFINITION 2. Let us suppose our random field X belongs to the class $L^2(\Omega)$, and we define the covariance function $C(P, Q)(P, Q\in S)$ as

$$C(P, Q)=E[X(P)-EX(P)]\overline{[X(Q)-EX(Q)]}. \tag{4}$$

The random field X is called homogeneous *in the wide sense* (*w-homogeneous*) iff

(5) $$C(gP, gQ)=C(P, Q) \qquad \text{for any } g \text{ of } G,$$

and is called isotropic *in the wide sense* (*w-isotropic*) iff

(6) $$C(kP, kQ)=C(P, Q) \qquad \text{for any } k \text{ of } K.$$

In the case of Gaussian random field on S, these two definitions are equivalent. Let P_0 be the fixed point of the isotropy subgroup K, then the covariance function C of the w-isotropic random field has the property $C(P_0, Q)=C(P_0, kQ)$, for any k of K, that is, it takes on each sphere through Q with center P_0 constant value.

Now, let us assume that the random field X belongs to the class $L^2(\Omega)$ and $L^2(\Omega)$-continuous. Under these assumptions the covariance function $C(P, Q)$ is a Hermitian symmetric positive definite kernel.

Thus, *in the case of S being compact*, we have the eigenfunction expansion

(7) $$C(P, Q)=\sum_{j=1}^{\infty}\lambda_j^2\psi_j(P)\overline{\psi_j(Q)},\ \lambda_j>0$$

by the Mercer's expansion theorem, where

(8) $$\lambda_j^2\psi_j(P)=\int C(P, Q)\psi_j(Q)dQ$$

$\lambda_1\geqslant\lambda_2\geqslant\ldots$ and ψ_j's are corresponding complete orthonormal eigenfunctions.

Then, let $\{X_j(P); j=1, 2,\ldots\}$ be random fields such that $EX_j(P)\equiv 0$, $EX_j(P)\overline{X_k(Q)}=\delta(j, k)\psi_j(P)\overline{\psi_k(Q)}$. Using these, we have $L^2(\Omega)$-decomposition of the random field X in the $L^2(\Omega)$-convergence.

(9) $$X(P)=\sum_{j=1}^{\infty}\lambda_j X_j(P).$$

However, in the general case, corresponding results are not so simple. In the following, we consider the w-homogeneous random field from another aspect.

§ 2. Representations of w-homogeneous random field

In the present section, we assume that w-homogeneous random field belongs to the class of $L^2(\Omega)$-continuous fields and $EX(P)\equiv 0$. The covariance function $C(P, Q)=C(gK, hK)$ satisfy an expression

(10) $$C(P, Q)=C(h^{-1}gK, eK)=EX(h^{-1}gK)\overline{X(eK)},$$

thus we could consider it as a function on G such that

(a) continuous on G

(b) $\sum_{i=1}^{n}\sum_{j=1}^{n} C(g_j^{-1}g_i)c_i\bar{c}_j \geqslant 0$ for any $g_1,\dots, g_n \in G$ and any $c_1,\dots, c_n \in \mathbf{C}$

(c) K-biinvariant,

that is, the function C on G is a K-biinvariant positive definite function on G. Thus, the covariance function C of the w-homogeneous random field X defines a unitary representation of the group G. Since the function C is a positive definite function, it has the property $|C(g)| \leqslant C(e)$ for any g of G. Gel'fand and Raĭkov [2] proved that there exists a complete system of irreducible unitary representations of the group G and these irreducible unitary representations correspond to elementary positive definite functions. Consider the set $\mathfrak{P}$ of positive definite functions on G such that $\phi(e) \leqslant C(e)$, the set $\mathfrak{P}$ is the smallest weakly-closed convex set containing elementary positive definite functions with $\phi(e)=C(e)$ and $\phi\equiv 0$. These elementary positive definite functions and $\phi\equiv 0$ exhaust extreme points of the set $\mathfrak{P}$ which form the closed set, and the corresponding representations are irreducible. Thus, from the Choque-Deny integral representation theorem [1], there exists the unique maximal positive Radon measure $F=F(C)$, supported by the set of extreme points of $\mathfrak{P}$, $\mathfrak{E}(\mathfrak{P})$, such that

(11) $$C=\int_{\mathfrak{E}(\mathfrak{P})\setminus\{o\}} \phi_\lambda dF(\lambda),$$

where ϕ_λ's are K-biinvariant normalized elementary postive definite functions (zonal spherical functions on G).

The irreducible unitary representations of the group G corresponding to ϕ_λ's are *the class* 1 *representations.*

We call the expression (11) *the spectral representation of the function* C and the measure F *the spectral measure corresponding to* C.

REMARK. In the case of G being the locally compact abelian group, ϕ_λ's are characters of G [5] and then let ν be a random measure such that

(12) $$E d\nu(\lambda)\overline{d\nu(\lambda')}=\delta(\lambda, \lambda')dF(\lambda)$$

and we have a spectral representation of the w-homogeneous field X,

(13) $$X(P)=\int \phi_\lambda(P)d\nu(\lambda).$$

Let us now assume that (G, K) is a symmetric Riemannian pair with K compact subgroup. Then, ϕ_λ's are positive definite zonal spherical functions on G, and they satisfy the differential equations $D\phi_\lambda=c_\lambda(D)\phi_\lambda$, $c_\lambda(D)\in\mathbf{C}$, for any invariant differential operator D on G/K and the integral equations

$$\int_K \phi_\lambda(gkh)dk = \phi_\lambda(g)\phi_\lambda(h).$$

REMARK. In the case of G/K being one-dimensional Euclidean space, we have the spectral representation (13), which is well-known. However, in general, when the w-homogeneous random field X is a field on the space $K\backslash G/K$, we have the spectral representation (13). These ϕ_λ functions are specified as the following [3]:

(A) THE EUCLIDEAN TYPE

Let S be a Euclidean space of dimension d and let G stand for $I_0(S)$, the largest connected group of isometries of S, and let K denote the compact subgroup of G leaving the origin $0 \in S$ fixed. The subgroup of all translations of S will be denoted also by S. Then, our function ϕ_λ's are the functions

$$\phi_\lambda(x) = \int_K \exp(\lambda(kxk^{-1}))dk, \qquad x \in S, \tag{14}$$

where λ is an arbitrary purely-imaginary valued linear function on S.

(B) THE COMPACT TYPE

Let (G, K) be a Riemannian symmetric pair, G compact. Then, our function φ_λ's are the functions of the form

$$\phi_\lambda(g) = \int_K \lambda(g^{-1}k)dk, \tag{15}$$

where λ is the character of a finite-dimensional representation T of G of class 1.

(C) THE NONCOMPACT TYPE

Let (G, K) be a Riemannian symmetric pair of the noncompact type. We assume that G has finite center so K is compact. Let $G = KA_{\mathfrak{P}}N$ be the Iwasawa decomposition of the group G. If x belong to G, let $H(x)$ denote the unique element in $\mathfrak{H}_{\mathfrak{P}_0}$ for which $x = k\exp H(x)n$, where $k \in K$, $n \in N$. Let ρ be the half of the sum of positive roots. Then, our function ϕ_λs are the functions

$$\phi_\lambda(x) = \int_K e^{(\sqrt{-1}\lambda-\rho)(H(xk))}dk, \qquad x \in G, \tag{16}$$

where λ is a real-valued linear function on $\mathfrak{H}_{\mathfrak{P}_0}$. These two functions are identical iff $\mu = s\lambda$ for some s in the Weyl group.

Now, let X_λ be a field on S such that $EX_\lambda \equiv 0$ and $EX_\lambda(P)\overline{X_\mu(Q)} = \delta(\lambda, \mu)\phi_\lambda(h^{-1}g)$, ν be a Radon measure (complex) such that $|d\nu(\lambda)|^2 = dF(\lambda)$, then we have a (formal) integral representation of the w-homogeneous field X,

$$X(P)=\int X_\lambda(P)d\nu(\lambda). \tag{17}$$

We call these random fields X_λ's *elementary components* of the field X, and we note that these component fields have some regularity properties of their paths owing to properties of the function ϕ_λ's.

References

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