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On the Quotient Topological Ordered Spaces (II)

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In § 1, for a topological ordered space X and an equivalence relation R on X, we shall study some sufficient conditions for X/R to be T_2-ordered. In § 2, we shall investigate the necessary and sufficient condition of R for X/R to be T_2-ordered, which is an analogue of a known fact in general topology. ([1] § 8 Proposition 8 or [2] Chapter 3 Theorem 11)

§ 1. We use the terms and notations in our previous paper [3]. Consider the following condition.

(C) $i_X(F)$ and $d_X(F)$ are closed for each closed set F of X. (In [3], we wrote this condition by (C. II).) We proposed a question in [3] which states

(Q) If X is a Hausdorff space satisfying (C), then is it T_2-ordered?

If X is regular, the next result was obtained in [3] (Proposition 2).

PROPOSITION 1. *Let X be a regular space satisfying* (C). *Then X is T_2-ordered.*

If X satisfies the first countability axiom, then the answer to (Q) is affirmative.

PROPOSITION 2. *Let X be a Hausdorff space satisfying* (C) *and the first countability axiom. Then X is T_2-ordered.*

PROOF. Since X satisfies the first countability axiom, we may replace "net" by "sequence" for convergence. ([2] Chapter 2 Theorem 8) Let $G(\leqq)$ be the graph of order $\leqq$ in X. X is T_2-ordered if and only if $G(\leqq)$ is closed in X^2. ([4] Chapter 1, Proposition 1 and the definition of T_2-ordered in [3]) Therefore we shall show that $G(\leqq)$ is closed in X^2. Let $\{(x_n, y_n)\}$ be a sequence in $G(\leqq)$ such that it converges to (x, y). Then, since $A=\{z: z=x_i$ or $z=x\}$ is closed in X, so $i_X(A)$ is closed in X by (C). On the other hand, since $\{y_1, y_2,\dots\}\subseteq i_X(A)$ and the sequence $\{y_n\}$ converges to y, therefore $y\in i_X(A)$. Hence $x\leqq y$ or there exists n such that $x_n\leqq y$. Let $B=\{(x_n, y_n): x_n\leqq y\}$. First, if $B=\phi$ or if B is a finite set, without loss of generality, we may assume $B=\phi$ from the beginning. Thus $x\leqq y$, hence $(x, y)\in G(\leqq)$. Next, if B is an infinite set, the sequence $\{x_n: (x_n, y_n)\in B\}$ converges to x. Moreover since $\{x_n: (x_n, y_n)\in B\}\subseteq d_X(y)$ from the construction of B and $d_X(y)$ is closed by (C), we see that $x\in d_X(y)$, i.e. $(x, y)\in G(\leqq)$. Thus, $G(\leqq)$ is closed and the proof is complete.

Q.E.D.

Hereafter, we show main theorems of this section. First, the following theorem was obtained in [3] (Theorem 2) by making use of Proposition 1 in the above.

THEOREM 1. *Let X be a regular space satisfying* (C). *Assume that p is a proper mapping. If X/R is a topological ordered space, then X/R is a regular space satisfying* (C). *Therefore, X/R is T_2-ordered by Proposition* 1.

Using the Proposition 2 in the above, we obtain the following theorems.

THEOREM 2. *Let X be a Hausdorff space satisfying* (C) *and the second countability axiom. Assume that p is a proper mapping. If X/R is a topological ordered space, then X/R is a Hausdorff space satisfying* (C) *and the second countability axiom. Therefore, X/R is T_2-ordered by Proposition* 2.

We use the following topological lemma in the proof of this theorem.

LEMMA. *Let X be a topological space satisfying the second countability axiom. If the natural projection from X to X/R is a proper mapping, then X/R also satisfies the second countability axiom.*

This lemma is clear by [2] Chapter 5 Theorem 20, [2] Chapter 3 Theorem 12 and [1] § 10 Theorem 1.

PROOF (of Theorem 2). First, X/R is a Hausdorff space by [1] § 10 Corollary 2 of Proposition 5, and by the lemma above X/R satisfies the second countability axiom. Next, X/R satisfies (C) by the fact that for any closed set F of X/R $i_{X/R}(F)=p(i_X(p^{-1}(F)))$, $d_{X/R}(F)=p(d_X(p^{-1}(F)))$, hence these sets are closed in X/R. *Q.E.D.*

THEOREM 3. *Let X be a Hausdorff space satisfying* (C) *and the first countability axiom. Assume that p is a proper and open mapping. If X/R is a topological ordered space, then X/R is a Hausdorff space satisfying* (C) *and the first countability axiom. Therefore, X/R is T_2-ordered by Proposition* 2.

PROOF. In general, if a topological space T satisfies the first countability axiom, then $f(T)$ satisfies the same axiom for any open continuous mapping f from T to the other space. Hence, we can prove this theorem in the same way of Theorem 2. *Q.E.D.*

REMARK 1. In connection with the above propositions and theorems, the topological assumptions of Proposition 1 (Theorem 1) and Proposition 2 (Theorem 2 or 3) are independent, respectively. In fact, [5] Example 103 shows the fact that there is a regular space not satisfying the first countability axiom, and [5] Example 74 shows the fact that there is a Hausdorff space, but not a regular space, satisfying the second countability axiom. Moreover, [5] Example 72 shows the existence of a Hausdorff, but not regular, space satisfying the first countability axiom which does not satisfy the second countability axiom.

REMARK 2. In Theorem 3, the condition that p is open is essential. In fact, an example of [2] Chapter 5 Problem N (a) shows that if p is not open, X/R does not satisfy the first countability axiom even if X satisfies the second countability axiom. Moreover, this example shows that in the above lemma the assumption that p is proper cannot be replaced by the one that p is closed.

§ **2.** In this section, we shall study an analogue of the following well known theorem in the theory of general topology. *For a topological space X, and an equivalence relation R on X, if the quotient space X/R is Hausdorff, then R is closed in the product space X^2. If the projection p of a space X onto the quotient space X/R is open and R is closed in X^2, then X/R is a Hausdorff space.* We begin with the following definitions.

DEFINITION 1. Let X be a topological (ordered) space, and R be an equivalence relation on X. Then a subset A of X is a *saturated* set (with respect to R) if and only if a saturation of A by R is also equal to A.

DEFINITION 2. Let X be a topological ordered space and $G(R)$ be the graph of an equivalence relation R on X. Then $G(R)$ is said to be *saturated order closed* (or *s. o. closed*) in X^2 if for $(x, y) \notin G(R)$ such that $p(x) \not\leq p(y)$, there exist a saturated increasing neighbourhood U of x and a saturated decreasing neighbourhood V of y such that $U \times V \cap G(R) = \phi$. (For an increasing or a decreasing set, see [3] Definition 1.)

Now, we obtain the following theorem.

THEOREM 4. *Let X be a topological ordered space. Assume that X/R is a topological ordered space. If X/R is T_2-ordered, then $G(R)$ is s. o. closed in X^2. If the natural projection p of X onto X/R is open and $G(R)$ is s. o. closed in X^2, then X/R is T_2-ordered.*

PROOF. The first part is shown in the following manner. We assume $p(x) \not\leq p(y)$ for $(x, y) \notin G(R)$. Since X/R is T_2-ordered, there exist an increasing neighbourhood U of $p(x)$ and a decreasing neighbourhood V of $p(y)$ such that $U \cap V = \phi$. Then, we easily see that $p^{-1}(U)$ is a saturated increasing neighbourhood of x and $p^{-1}(V)$ is a saturated decreasing neighbourhood of y such that $p^{-1}(U) \times p^{-1}(V) \cap G(R) = \phi$. Hence $G(R)$ is s. o. closed in X^2.

Next, we shall show the second part. We assume that $p(x) \not\leq p(y)$ for $p(x), p(y) \in X/R$. Then since $(x, y) \notin G(R)$, $p(x) \not\leq p(y)$, using the assumption that $G(R)$ is s. o. closed in X^2, there exist a saturated increasing neighbourhood U of x and a saturated decreasing neighbourhood V of y such that $U \times V \cap G(R) = \phi$. It now is easy to show that $p(U)$ is an increasing neighbourhood of $p(x)$, $p(V)$ is a decreasing neighbourhood of $p(y)$ and $p(U) \cap p(V) = \phi$. Therefore, X/R is T_2-ordered. *Q.E.D.*

REMARK 3. In the second part of this theorem, the assumption that p is open is essential. For this, see next example.

EXAMPLE. Let X be a closed interval [0, 1] in real line. We define the topology on X as follows: the neighbourhood system of 0 is $\{V_m: m=1, 2,\ldots\}$ where $V_m=\left\{x: 0\leqq x\leqq\frac{1}{m}, x\neq\frac{1}{m+1}, \frac{1}{m+2},\ldots\right\}$ and the neighbourhoods of other points as usual. Next, we introduce the discrete order as the partial order in X, and we define an equivalence relation R on X as follows: xRy if and only if $x, y\in\left\{1, \frac{1}{2}, \frac{1}{3},\ldots\right\}$ or $x=y$. Then, by this topology and order X is a Hausdorff space but is not a regular space, and X/R is not Hausdorff (T_2-ordered). In this example, it is easily seen that $G(R)$ is s. o. closed in X^2 and the natural projection p is not open.

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