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Holomorphic θ-Line Bundles over a Compact Riemann Surface of Genus 2

Hiromichi MATSUNAGA
Department of Mathematics, Shimane University, Matsue, Japan
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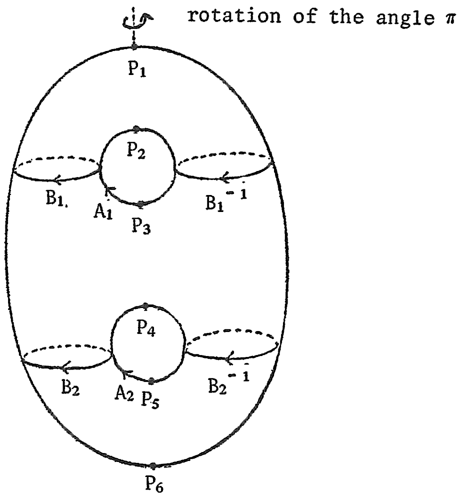
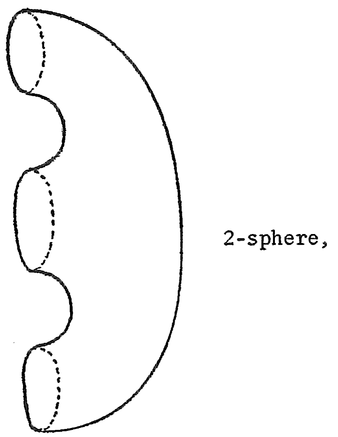
In this note, the auther determine the order of the group which consists of equivalence classes of holomorphic Z_2-line bundles over a compact Riemann surface of genus 2. The order is 32 for each degree. Main tools are an exact sequence due to A. Grothendieck and the representation of the Picard variety as the quotient variety of the first cohomology group with coefficient in the complex number field C.

Let M_2 be a compact Riemann surface of genus 2. By the theorem 25 and 28 (b), [2], the surface admits a holomorphic involution θ and the quotient space $M_2/(\theta)$ is the complex projective line. The involution θ has just 6 fixed points. Further by the corollary 1, p. 246 in the lecture note, the involution θ is unique. Then θ can be described topologically as follows:





where $p_1, \ldots, p_6$ are fixed points of θ. The involution induces the transformation of one dimensional homology basis, $A_1 \to -A_1$, $B_1 \to -B_1$, $A_2 \to -A_2$, $B_2 \to -B_2$. The main result of this note is

THEOREM. *The number of equivalence classes of holomorphic θ-line bundles* *over M_2 is 32 for each degree.*

We prove the theorem in the last part of this paper. Some preliminaries are required.

Let O^* be the sheaf of germs of nowhere vanishing holomorphic functions on M_2. For a germ f_x at x, define $\tilde{\theta}$ by $\tilde{\theta}(f_x)=(f\cdot\theta^{-1})_{\theta(x)}$. Then the sheaf O^* is a θ-sheaf. Let S be a θ-sheaf over M_2. For an open set U of M_2, we denote by $\Gamma(U, S)$ the abelian group of all sections over U. Define a homomorphism $\theta\colon \Gamma(U, S)\to\Gamma(\theta U, S)$ by $\theta(s)(y)=\tilde{\theta}(s\cdot\theta^{-1}(y))$ for $s\in\Gamma(U, S)$. We apply this homomorphism to each transition function $g_{ij}\colon U_i\cap U_j\to O^*$, then

$$(\theta g_{ij})(y)=\tilde{\theta}(g_{ij}\cdot\theta^{-1}(y))=(g_{ij}\cdot\theta^{-1})_y.$$

Hence the involution θ induces an automorphism $\theta\colon H^1(M_2, O^*)\to H^1(M_2, O^*)$.

PROPOSITION 1. *Let $E\to M_2$ be a holomorphic θ-line bundle, then $\theta\{E\}=\{E\}$, where $\{E\}$ denotes the equivalence class containing E.*

PROOF. Since E is a holomorphic θ-line bundle, the involution θ admits an involution $\bar{\theta}\colon E\to E$ as a lift of θ. Denote by $\hat{\theta}_i$ the composition $\psi_{\theta(i)}\cdot\bar{\theta}\cdot\psi_i^{-1}$, where $\{\psi_i\}$ is a local triviality of E. The commutative diagram

$$\begin{array}{ccc}
(U_i\cap U_j)\times C & \xrightarrow{\hat{\theta}_i} & (\theta U_i\cap\theta U_j)\times C \\
\uparrow{\scriptstyle \psi_i} & & \uparrow{\scriptstyle \psi_{\theta(i)}} \\
E\,|\,U_i\cap U_j & \xrightarrow{\bar{\theta}} & E\,|\,\theta U_i\cap\theta U_j \\
\downarrow{\scriptstyle \psi_j} & & \downarrow{\scriptstyle \psi_{\theta(j)}} \\
(U_i\cap U_j)\times C & \xrightarrow{\hat{\theta}_j} & (\theta U_i\cap\theta U_j)\times C
\end{array}$$

gives an equality

$$g_{\theta(i)\theta(j)}=\hat{\theta}_j\cdot(g_{ij}\cdot\theta^{-1})\cdot\hat{\theta}_i^{-1},$$

then $(g_{\theta(i)\theta(j)})$ is equivalent to $(g_{ij}\cdot\theta^{-1})$.

PROPOSITION 2. *Each equivalence class of holomorphic line bundles over M_2 contains at most two equivalence classes of holomorphic θ-line bundles.*

PROOF. We describe the exact sequence in 143–03, [1], in the case of $\mathfrak{g}=O^*$ and $\pi=Z_2$ generated by θ, then

$$(1)\qquad e\longrightarrow H^1(Z_2, H^0(M_2, O^*))\xrightarrow{\ i\ }H^1(M_2, Z_2; O^*)\longrightarrow H^1(M_2, O^*)^{Z_2}.$$

Since M_2 is compact, every holomorphic function $M_2\to C^*=C-\{0\}$ is constant, then $H^0(M_2, O^*)=\Gamma(M_2, O^*)=C^*$, and for each $\{\varphi\}\in H^1(Z_2, H^0(M_2, O^*))$,

$$1=\varphi(\theta^2)=\varphi(\theta)\cdot\varphi(\theta) \quad \text{and so} \quad \varphi(\theta)=\pm 1 .$$

φ is equivalent to φ' if and only if there exists $m \in H^0(M_2, O^*)$ such that $\varphi'(\theta)=m\varphi(\theta)(\theta\cdot m)^{-1}=\varphi(\theta)$, thus $\varphi=\varphi'$. Hence $H^1(Z_2, H^0(M_2, O^*))=Z_2$, and the proposition is proved.

Now we are ready to prove the theorem. Let $P_0(M_2)=\{\xi \in H^1(M_2, O^*);\ c_1(\xi)=0\}$, where c_1 is the first Chern class. By the naturality of Chern classes, the involution θ induces an automorphism θ: $P_0(M_2)\to P_0(M_2)$. Let $O^{1,0}$ be the sheaf of germs of abelian differentials and δ be the coboundary in the exact sequence associated with the short exact sequence of sheaves $0\to C\to O\to O^{1,0}\to 0$, where O is the sheaf of germs of holomorphic functions on M_2 and C is the field of complex numbers. We have the natural isomorphism

$$P_0(M_2)\approx \frac{H^1(M_2, C)}{H^1(M_2, Z)+\delta\Gamma(M_2, O^{1,0})}, \quad \text{p. 134, [2]},$$

where Z is the additive group of integers and $\Gamma(M_2, O^{1,0})$ is the additive group of abelian differentials. Let a_1, b_1, a_2, b_2 be the Poincaré duals of the homology classes A_1, B_1, A_2, B_2 respectively, then $\theta(a_1)=-a_1$, $\theta(b_1)=-b_1$, $\theta(a_2)=-a_2$, $\theta(b_2)=-b_2$. We choose them as a basis of $H^1(M_2, Z)$ and $H^1(M_2, C)$. For an element ξ of $P_0(M_2)$, let (α_i) be a representative of ξ in $H^1(M_2, C)$ with respect to the basis a_1, b_1, a_2, b_2, (w_{i1}), (w_{i2}) be a basis of $\delta\Gamma(M_2, O^{1,0})$ and x_1, x_2 be complex numbers. Since the above isomorphism is θ-equivariant, $\theta(\xi)=\xi$ if and only if there exist integers n_i $(i=1, 2, 3, 4)$ such that

$$\left\{\begin{matrix} 2\alpha_1-x_1w_{11}-x_2w_{12} \\ 2\alpha_2-x_1w_{21}-x_2w_{22} \\ 2\alpha_3-x_1w_{31}-x_2w_{32} \\ 2\alpha_4-x_1w_{41}-x_2w_{42} \end{matrix}\right\} = \left\{\begin{matrix} n_1a_1 \\ n_2b_1 \\ n_3a_2 \\ n_4b_2 \end{matrix}\right\}, \text{ and so} \left\{\begin{matrix} \alpha_1\equiv(n_1/2)a_1 \\ \alpha_2\equiv(n_2/2)b_1 \\ \alpha_3\equiv(n_3/2)a_2 \\ \alpha_4\equiv(n_4/2)b_2 \end{matrix}\right\} \bmod \delta\Gamma(M_2, O^{1,0}).$$

Then the variety $P_0(M_2)$ admits at most $2^4=16$ fixed points. If a class $\alpha_1a_1+\alpha_2b_1+\alpha_3a_2+\alpha_4b_2$ belongs to $\delta\Gamma(M_2, O^{1,0})$ and the coefficients α_1, α_2, α_3, α_4 are real, then the class is zero (Lemma 18, [2]). Hence, by the proposition 2 and its proof, there exists just 32 classes of holomorphic θ-line bundles with $c_1=0$. Next let ξ, η be holomorphic θ-line bundles with $c_1(\xi)=c_1(\eta)$, then $c_1(\xi\cdot\eta^{-1})=0$. Thus the Theorem is proved.

Remark. We attend to the formula in p. 36, [4] in case of $r=r'=1$, $\Theta'=1$. About each fixed point p_i, $i=1,\dots, 6$, we can choose a local coordinate $(U(p_i), z)$ such that the projection mapping f: $M_2\to M_2/(\theta)$ has the form $w=f(z)=z^2$ and $\theta z=-z$ in a neighborhood of p_i. A meromorphic function $d(z)$ near the point p_i is a local divisor of degree 1 if and only if $d(-z)=v(z)\cdot d(z)$, where $v(z)$ is a holomorphic function,

p. 54 and p. 52 in [4]. If $d(z)$ has a pole of degree 1 at p_i, then $v(0)=-1$. Thus $i(d(z))=-\frac{1}{2}$ (p. 52 in [4]). Define a divisor $\Theta^{(i)}$ of degree 1 over $M_2/(\theta)$ by

$$\Theta_p^{(i)}=\begin{cases} d(z) & \text{if } p=p_i, \\ I_p & \text{if } p\neq p_i. \end{cases}$$

Let

$N=$the dimension $\{\Phi$: meromorphic function on $M_2/(\theta)$ such that $\Theta\Phi$ is holomorphic$\}$,

$\sigma=$the dimension $\{dI$ differential on $M_2/(\theta)$ such that $(dI/dz)\Theta^{-1}$ is holomorphic$\}$,

$I(\Theta^{(i)})=\sum_p i(\Theta_p^{(i)})=i(d(z))=-\frac{1}{2}$.

The divisor $\Theta^{(i)}$, say Θ, can be described more simply as follows:

$$\Theta_p=\frac{1}{z}=\Delta\Theta_0,\text{ where } \Delta=z,\ \Theta_0=\frac{1}{z^2}=\frac{1}{w}.$$

Then, for a meromorphic function Φ on $M_2/(\theta)$, $\Theta\Phi$ is a holomorphic function of z if and only if $\Theta_0\Phi$ is a holomorphic function of w, and for a differential dI on $M_2/(\theta)$, $(dI/dz)\Theta^{-1}$ is a holomorphic function of z if and only if $(dI/dw)\Theta_0^{-1}$ is a holomorphic function of w. Now the classical Riemann-Roch theorem establishes

$$N-\sigma-1=d(\Theta_0)=-1=-c_1(f_*(\zeta_{p_i})),$$

where ζ_{p_i} is the point bundle given by the local divisor Θ, p. 114 in [2], and $f^*(\zeta_{p_i})$ is the image bundle, p. 49 in [3]. Thus the Riemann-Roch-Weil theorem takes the form

$$I(\Theta)=-\frac{1}{2}=-c_1(f_*(\zeta_{p_i}))+\frac{1}{2}.$$

It seems to be impossible to formulate the theorem in a global form, c.f. the formula in 143–13, [1].

References

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