Mem. Fac. Lit. & Sci., Shimane Univ.,
Nat. Sci., **8**, pp. 13-16, March 10, 1975

On Holomorphic G-vector Bundles

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(Received November 5, 1974)

In this note we apply the technic in [5] to a classification problem of holomorphic G-vector bundles over a normal complex space with a proper complex Lie transformation group G. In § 1, we deduce some properties about normal complex spaces from the basic works due to H. Holmann, [2], [3]. The main theorem is obtained in § 2, which is an analogous one to the theorem in [5].

§ 1. Normal complex spaces.

Let M be a complex space and G be a complex Lie transformation group of M. We denote by μ: $G\times M\to M$ the action. Suppose that G acts properly on M, then all isotropy group G_x is finite. By Satz 19 and Definition 15, [2], we have

PROPOSITION 1. *Every point $x\in M$ admits a neighborhood U_x, a neighborhood V_ε of the identity element ε of G such that the following conditions are satisfied:*

1. *The restriction μ: $V_\varepsilon\times S_x\to U_x$ is biholomorphic,*
2. *S_x is G_x-invariant and for $g\in G$, if $gS_x\cap S_x$ is non empty, then $g\in G_x$.*

By Satz 14 and its proof in [2], we have a biholomorphism $G\times_{G_x}S_x\to GU_x$. Thus S_x is a slice at x. If M is a normal complex space, then the quotient M/G is also normal (Satz 12, [2]). In this note we use very often the following

PROPOSITION 2, (Satz 23, [1]). *A topological holomorphic map τ: $M_1\to M_2$ from a complex space M_1 onto a normal complex space M_2 is biholomorphic.*

The next one is an elementary result about a complex Lie group.

PROPOSITION 3. *Let H be a finite subgroup of G, then its normalizer $N(H)$ in G is a complex Lie group.*

PROOF. If $N(H)$ is a discrete group, then it is a zero dimensional complex Lie group. Suppose that the dimension of $N(H)_\varepsilon$, the connected component of the identity, is positive. For each $g\in N(H)$, $gh_ig^{-1}=h_j$, where h_i, $h_j\in H$. Since H is finite group, $hgh^{-1}=g$ for all $g\in N(H)_\varepsilon$ and $h\in H$. We denote by $L(N(H))$ the Lie algebra of $N(H)$. For $h\in H$, $X\in L(N(H))$, $\mathrm{Ad}_h(X)=X$ and Ad_h is complex linear, then $\mathrm{Ad}_h(\mathrm{J}X)=\mathrm{J}\mathrm{Ad}_h(X)=\mathrm{J}X$, where J is an almost complex structure on $L(G)$. Thus $\mathrm{J}X\in L(N(H))$ and $N(H)$ is a complex Lie group.

We denote by M_H the set $\{x \in M;\ G_x = H\}$.

PROPOSITION 4. *M_H is an analytic set in M.*

PROOF. By the proposition 1, $\mu: V_\varepsilon \times S_x \to U_x$ is biholomorphic and the restriction $(V_\varepsilon \cap N(H)) \times S_x \to U_x \cap M_H$ is homeomorphism for each $x \in M_H$, where we assume that V_ε is connected. Since H is finite, $g \in V_\varepsilon \cap N(H)$ if and only if $ghg^{-1} = h$ for all $h \in H$, then $(V_\varepsilon \cap N(H))$ is an analytic set in V_ε and S_x is an analytic set in U_x, and the product $(V_\varepsilon \cap N(H)) \times S_x$ is an analytic set in $V_\varepsilon \times S_x$. Q.E.D.

Now we consider an action with one orbit type (H),

PROPOSITION 5. *The induced mapping, say, $\tilde{\mu}: G \times_{N(H)} M_H \to M$ is biholomorphic.*

PROOF. The restriction $\mu: G \times M_H \to M$ is holomorphic and $N(H)$-invariant, then $\tilde{\mu}$ is holomorphic, ((*f*), §2, [3]). $\tilde{\mu}$ is homeomorphic, then by the Proposition 2, it is biholomorphic.

We denote by $\Gamma(H)$ the quotient group $N(H)/H$, which is a complex Lie group and acts freely on M_H. By the same argument as in the proof of the proposition 5, the quotient $M_H/\Gamma(H)$ is biholomorphic to the quotient space M/G.

§2. Holomorphic *G*-vector bundles.

We denote by $Vect_G^\omega(M)$ the abelian semi-group of equivalence classes of all holomorphic G-vector bundles over a normal complex space M. The total space of any holomorphic vector bundle over M is also normal. For a holomorphic G-vector bundle $E \to M$, $E|M_H$ denotes the portion over M_H. Then we have an equivalence $G \times_{N(H)} (E|M_H) \to E$ of holomorphic G-vector bundles (cf. the proof of (1), §1, [4]) and we obtain

PROPOSITION 6. *We have an isomorphism of semi-groups*

$$\pi_*^{(1)}: Vect_G^\omega(G \times_{N(H)} M_H) \xrightarrow{\approx} Vect_{N(H)}^\omega(M_H).$$

Since $\Gamma(H) \to M_H \to M_H/\Gamma(H)$ is a complex analytic principal bundle, there exists an open covering $\{W_i\}$ of $M_H/\Gamma(H)$, and a $\Gamma(H)$-equivariant biholomorphism $\varphi_i: M_H | W_i \to W_i \times \Gamma(H)$ for each i. Now we consider the case which fulfills the condition: $N(H) \approx H \cdot \Gamma(H)$, the semi-direct product. Let $p: E \to M_H$ be a holomorphic $N(H)$-vector bundle and $E_i \to W_i \times \Gamma(H)$ be its portion over $W_i \times \Gamma(H)$.

We may suppose that $E_i | W_i \times (e)$ is holomorphically isomorphic to $W_i \times V_i$, where e is the identity element of $\Gamma(H)$ and $V_i = p^{-1}(x)$ for some $x \in W_i$, which is a complex H-module. We denote the isomorphism by Φ_i, which corresponds to a cross section

$s(\Phi_i)$: $W_i\times(e)\to\mathrm{Hom}(E_i|W_i\times(e), W_i\times V_i)$. Since $M_H/\Gamma(H)=M/G$ is locally compact, there exists a neighborhood U'_i of $x_i\in W_i$ such that $\bar{U}'_i$, the closure of U'_i, is compact and contained in W_i. Put the minimum $\mathrm{Min}\{|s(\Phi_i)(x)|;\ x\in\bar{U}'_i\}=\varepsilon$. On the other hand we may suppose that we have choosed the covering $\{W_i\}$ which admits isomorphisms Φ'_i: $E_i|W_i\times(e)\to W_i\times V_i$ of H-vector bundles (cf. Proof of Proposition (2.2), [6]) and that $s(\Phi'_i)(x_i)=s(\Phi_i)(x_i)$, where if it is necessary, we take $[s(\Phi'_i)(x_i)]\circ[s(\Phi_i)(x_i)]^{-1}\circ s(\Phi_i)(x)$ instead of $s(\Phi_i)(x)$. There exists a neighborhood U_i of x_i such that $\bar{U}_i$ is compact and $\bar{U}_i\subset U'_i$, further,

$$|s(\Phi'_i)(x)-s(\Phi'_i)(y)|<\varepsilon/3,\quad\text{and}\quad |s(\Phi_i)(x)-s(\Phi_i)(y)|<\varepsilon/3\quad\text{for}\quad x,y\in\bar{U}_i.$$

Then

$$|s(\Phi_i)(x)-s(\Phi'_i)(x)|<\frac{2}{3}\varepsilon<\varepsilon.$$

Thus $\frac{1}{|H|}\sum_{h\in H}h\cdot s(\Phi_i)(x)$, say $\Psi'_i(x)$, is a holomorphic H-equivalence, where $|H|$ denotes the order of H. Hence we have a holomorphic $N(H)$-equivalence Ψ_i: $U_i\times(V_i\times_H H\cdot\Gamma(H))\to E_i$ over $U_i\times\Gamma(H)$.

For $h\in H$ and $(x,(v,h\gamma))\in(U_i\cap U_j)\times V_i\times_H H\cdot\Gamma(H)$,

$$\begin{aligned}\Psi_j^{-1}\cdot\Psi_i(x,(v,h\gamma))&=(x,(g_{ji}(x)v,\gamma_{ji}(x)h\gamma))\\&=(x,(g_{ji}(x)(v))\cdot I(\gamma_{ji}(x))(h),\gamma_{ji}(x)\gamma),\end{aligned}$$

where (γ_{ji}) is the set of transition functions of the principal bundle $\Gamma(H)\to M_H\to M_H/\Gamma(H)$. We have the important relation (cf. § 3, [5]),

$$g_{ji}(x)(vh)=\{g_{ji}(x)(v)\}\cdot I(\gamma_{ji}(x))(h).$$

Hence we obtain our main result:

THEOREM. *We have an isomorphism of semi-groups*

$$\pi_*^{(2)}:\ Vect^{\omega}_{N(H)}(M_H)\to Vect^{\omega}_{H\gamma}(M/G),$$

where $Vect^{\omega}_{H\gamma}(M/G)$ denotes the semi-group of isomorphism classes of holomorphic local H-vector bundles, (cf. Theorem in § 3, [5]).

COROLLARY. *If $\Gamma(H)$ is connected, then*

$$\pi_*^{(2)}:\ Vect^{\omega}_{N(H)}(M_H)\cong Vect^{\omega}_{H}(M/G).$$

PROOF. Since $\Gamma(H)$ is connected and H is finite, $g_{ji}(x)(vh)=\{g_{ji}(x)(v)\}\cdot h$.

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