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# Balanced Sets in Countably A-Ultrabarrelled Spaces

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**In this paper we define a countably $A$-ultrabarrelled space and $A$-$UDF$ space, and we study a countably A-ultrabarrelled space with an increasing sequence of balanced subsets.**

In this paper, we study a linear topological space which has an increasing sequence of balanced subsets.

Let $E$ be a linear topological space (l. t. s). Let $A\subset P(E)$ and $\bigcup_{A_\alpha\in A} A_\alpha=E$. Then a balanced set $U_0$ is said to be an $A$-suprabarrel in $E$ if there is a sequence $(U_n)$ of balanced subsets of $E$ satisfying $U_0\supset U_1+U_1$ and $U_n\supset U_{n+1}+U_{n+1}$ for all $n$ such that each $U_n$ absorbs all $A_\alpha\in A$. If in addition $U_0$ is closed, in which case each $U_n$ can be chosen to be closed, then $U_0$ is called an $A$-ultrabarrel in $E$. In any case, we say that $(U_n)$ is a defining sequence for $U_0$.

Definition 1. *An l.t.s $E[\mathfrak{T}]$ is said to be A-ultrabarrelled if every A-ultrabarrel in E is a neighbourhood of the origin.*

Let $E$ be an l. t. s and in special a family $A$ be contained in the family of all bounded sets in $E$. Then for each positive integer $n$, $(U_n^{(j)})_{j=0}^\infty$ be a sequence of closed balanced neighbourhoods in $E$ such that $U_n^{(j+1)}+U_n^{(j+1)}\subseteq U_n^{(j)}$ for all $j$. If $U^{(j)}=\bigcap_{n=1}^\infty U_n^{(j)}$ is an $A$-ultrabarrel, we call $U^{(0)}$ an $A$-ultrabarrel of type ($\alpha$).

Definition 2. *We say that an l.t.s $E[\mathfrak{T}]$ is a countably A-ultrabarrelled space if in E, every A-ultrabarrel of type ($\alpha$) is a neighbourhood of the origin.*

Theorem 1. *Let $(A_n)$ be an increasing sequence of balanced sets of a countably A-ultrabarrelled space $E[\mathfrak{T}]$ such that $E=\bigcup_{n=1}^\infty A_n$, $A_n+A_n\subset A_{n+1}$ for $n=1, 2,\dots$, and an arbitrary $A_\alpha\in A$ is absorbed by some $A_n$. Let $(V^{(n)})$ be a balanced sequence such that $V^{(0)}\supset V^{(1)}+V^{(1)},\dots, V^{(j)}\supset V^{(j+1)}+V^{(j+1)},\dots$. Then if $V^{(j)}\cap A_n$ is a neighbourhood of the origin in $A_n$ for any $j=0, 1, 2,\dots$ and any $n=1, 2,\dots$, $V^{(0)}$ is a neighbourhood of the origin in $E[\mathfrak{T}]$.*

Proof. By virtue of our assumption, there exists a balanced closed neighbourhood $U_n^{(0)}$ of the origin such that $A_{n+1}\cap U_n^{(0)}\subset A_{n+1}\cap V^{(1)}$. Then we have $A_n\cap U_n^{(0)}\subset A_n\cap V^{(1)}$. It is easy to see that there exists a defining sequence $(U_n^{(j)})_{j=1}^\infty$ of balanced closed neighbourhoods of the origin for each $U_n^{(0)}$ such that $A_n \cap U_n^{(j)} \subset A_n \cap V^{(j+1)}$ for $n, j \geq 1$. Since $A_{n-j} \subset A_n$ for $j<n$, we have $A_{n-j} \cap U_n^{(j)} \subset A_{n-j} \cap V^{(j+1)}$ for any $n \geq 1$ and any $0 \leq j < n$.

We shall construct a sequence $(W_n^{(0)})_{n=1}^{\infty}$ of balanced closed neighbourhoods of the origin such that each $W_n^{(0)}$ has a defining sequence $(W_n^{(j)})$ of balanced closed neighbourhoods of the origin, each $W^{(j)} = \bigcap_{n=1}^{\infty} W_n^{(j)}$ $(j=0, 1, 2, \dots)$ absorbs all $A_\alpha \in A$ and $W^{(0)} \subset V^{(0)}$. Then $W^{(0)}$ is a neighbourhood of the origin which is contained in $V^{(0)}$. If we set

$$W_n^{(j)} = \begin{cases} \overline{(A_{n-j} \cap V^{(j+1)}) + U_n^{(j+1)}} & \text{for } j=0, 1, \dots, n-1 \\ U_n^{(j+1)} & \text{for } j=n, n+1, \dots, \end{cases}$$

then each $W_n^{(j)}$ is a closed balanced neighbourhood of the origin and $(W_n^{(j)})_{j=1}^{\infty}$ is a defining sequence of $W_n^{(0)}$ for each $n$. In fact, we have for $1 \leq j < n$.

$$\begin{aligned} W_n^{(j)} + W_n^{(j)} &= \overline{(A_{n-j} \cap V^{(j+1)}) + U_n^{(j+1)}} + \overline{(A_{n-j} \cap V^{(j+1)}) + U_n^{(j+1)}} \\ &\subset \overline{(A_{n-j} \cap V^{(j+1)}) + (A_{n-j} \cap V^{(j+1)}) + U_n^{(j+1)} + U_n^{(j+1)}} \\ &\subset \overline{(A_{n-(j-1)} \cap V^{(j)}) + U_n^{(j)}} = W_n^{(j-1)}. \end{aligned}$$

For $j=n$, we have

$$W_n^{(n)} + W_n^{(n)} = U_n^{(n+1)} + U_n^{(n+1)} \subset U_n^{(n)} \subset W_n^{(n-1)}.$$

For $j>n$, since $(U_n^{(j)})$ is a defining sequence of $U_n^{(0)}$, the insistence is also true.

Now we show that each $W^{(j)} = \bigcap_{n=1}^{\infty} W_n^{(j)}$ absorbs all $A_\alpha \in A$. In fact for an arbitrarily fixed $j$ and an arbitrarily fixed $A_\alpha \in A$, there exist a positive real number $\rho_n$ and a positive integer $n_0 > j$ such that

$$\begin{aligned} A_\alpha &\subset \rho_n (A_{n-j} \cap U_n^{(j)}) \subset \rho_n (A_{n-j} \cap V^{(j+1)}) \\ &\subset \rho_n (A_{n-j+p} \cap V^{(j+1)}) \subset \rho_n W_{n+p}^{(j)} \\ &\qquad \text{for all } n \geq n_0 \text{ and all } p=1, 2, \dots . \end{aligned}$$

Hence $A_\alpha \subset \rho_n \bigcap_{p=0}^{\infty} W_{n+p}^{(j)}$. Since $W^{(j)} = (\bigcap_{m=1}^{n_0} W_m^{(j)}) \cap (\bigcap_{p=1}^{\infty} W_{n_0+p}^{(j)})$ and $(\bigcap_{m=1}^{n_0} W_m^{(j)})$ absorbs $A_\alpha$, $A_\alpha$ is absorbed by $W^{(j)}$. Therefore each $W^{(j)}$ absorbs all $A_\alpha \in A$.

Finally we shall show that $W^{(0)}$ is contained in $V^{(0)}$.

$$\begin{aligned} W_n^{(0)} = \overline{(A_n \cap V^{(1)}) + U_n^{(1)}} &\subset A_n \cap V^{(1)} + U_n^{(1)} + U_n^{(1)} \\ &\subset A_n \cap V^{(1)} + U_n^{(0)}. \end{aligned}$$

Therefore we have

$$\begin{aligned} A_n \cap W_n^{(0)} &\subset \{A_n \cap [(A_n \cap V^{(1)}) + U_n^{(0)}]\} \\ &\subset \{(A_n \cap V^{(1)}) + [(A_n + A_n) \cap U_n^{(0)}]\} \\ &\subset (A_{n+1} \cap V^{(1)}) + (A_{n+1} \cap U_n^{(0)}) \\ &\subset (A_{n+1} \cap V^{(1)}) + (A_{n+1} \cap V^{(1)}) \quad \text{for} \quad n=1, 2, \ldots. \end{aligned}$$

Consequently we have

$$\begin{aligned} W^{(0)} &= \bigcup_{n=1}^{\infty} (A_n \cap W^{(0)}) \\ &\subset \bigcup_{n=1}^{\infty} [(A_{n+1} \cap V^{(1)}) + (A_{n+1} \cap V^{(1)})] \\ &\subset \bigcup_{n=1}^{\infty} (A_{n+1} \cap V^{(1)}) + \bigcup_{n=1}^{\infty} (A_{n+1} \cap V^{(1)}) \\ &= V^{(1)} + V^{(1)} \subset V^{(0)} \end{aligned}$$

The proof is complete.

The following proposition follows from the above theorem.

PROPOSITION 1. *Let $\mathfrak{A}$ be a collection of linear mappings which map a countably A-ultrabarrelled space $E[\mathfrak{T}]$ into an l. t. s $F[\mathfrak{T}']$ and let $(A_n)$ be an increasing sequence of balanced sets such that $E = \bigcup_{n=1}^{\infty} A_n$, $A_n + A_n \subset A_{n+1}$ for $n=1, 2,\ldots$ and an arbitrary $A_\alpha \in A$ is absorbed by some $A_n$. Then $\mathfrak{A}$ is equicontinuous if and only if the restriction of $\mathfrak{A}$ on any $A_n (n=1, 2,\ldots)$ is equicontinuous at the origin. In special a linear mapping from $E[\mathfrak{T}]$ into $F[\mathfrak{T}']$ is continuous if and only if on any $A_n$ it is continuous at the origin.*

COROLLARY. *Let $(E_n)$ be an increasing sequence of linear subspaces of a countably A-ultrabarrelled space $E[\mathfrak{T}]$ such that $E = \bigcup_{n=1}^{\infty} E_n$ and an arbitrary $A_\alpha \in A$ is absorbed by some $E_n$, and let $\mathfrak{T}_n$ be the induced topology of $\mathfrak{T}$ on $E_n$. Then $E[\mathfrak{T}]$ is the strict inductive limit of $E_n[\mathfrak{T}_n]$.*

PROOF. Let $\mathfrak{T}'$ be the topology of the inductive limit on $E[\mathfrak{T}]$. Then $\mathfrak{T}'$ is finer than $\mathfrak{T}$. In fact, all canonical mapping $I_n$: $E_n[\mathfrak{T}_n] \to E[\mathfrak{T}]$ is continuous. Therefore it is sufficient to show that the identity mapping $I$ from $E[\mathfrak{T}]$ onto $E[\mathfrak{T}']$ is continuous. The mapping $I \circ I_n$ is restriction of $I$ on $E_n[\mathfrak{T}_n]$. Therefore $I$ is continuous on all $E_n[\mathfrak{T}_n]$ and by Proposition 1 on $E[\mathfrak{T}]$. Hence $E[\mathfrak{T}]$ is the inductive limit of $\{E_n[\mathfrak{T}_n]\}$. Since $\mathfrak{T}_{n+1}$ induces the topology $\mathfrak{T}_n$ on $E_n$, $E[\mathfrak{T}] = \bigcup_{n=1}^{\infty} E_n[\mathfrak{T}_n]$ is the strict inductive limit.

The proof is complete.

THEOREM 2. *Let $(A_n)$ be an increasing sequence of closed balanced sets of a countably A-ultrabarrelled space $E[\mathfrak{T}]$ such that $E=\bigcup_{n=1}^{\infty} A_n$, $A_n+A_n\subset A_{n+1}$ for $n=1, 2,\dots$ and each $A_\alpha\in A$ is absorbed by some $A_n$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of non-zero real numbers such that $\lim_{n\to\infty}\lambda_n=\infty$. Then if B is an arbitrarily given bounded set, there exists a positive integer $n_0$ such that $B\subset\lambda_{n_0}A_{n_0}$.*

PROOF. For each positive integer $n$, if $B$ is not contained in $\lambda_n A_n$, it will be possible to find an $x_n\in B$ such that $x_n\notin\lambda_n A_n$ and hence $y_n=\frac{1}{\lambda_n}x_n\notin A_n$. Since $A_n$ is closed it is possible to find balanced neighbourhoods $V_n$ of the origin such that $(y_n+V_n)\cap A_n=\phi$ $(n=1,2,\dots)$, and hence taking a sequence $(V_n^{(j)})_{j=1}^{\infty}$ of balanced neighbourhoods of the origin such that $V_n\supset V_n^{(1)}+V_n^{(1)},\dots, V_n^{(j)}\supset V_n^{(j+1)}+V_n^{(j+1)},\dots$ for any $n=1, 2,\dots$, we have $(y_n+V_n^{(1)})\cap(A_n+V_n^{(1)})=\phi$. Therefore we have $y_n\notin A_n+V_n^{(1)}$ for $n=1, 2,\dots$. The each set $A_n+V_n^{(1)}$ $(n=1, 2,\dots)$ is a balanced neighbourhood of the origin and hence $V=\bigcap_{n=1}^{\infty}(A_n+V_n^{(1)})$ is a balanced set. The sequence $(y_n)_{n=1}^{\infty}$ converges to 0 and all their terms are not contained in $V$. If we prove that $V$ is a neighbourhood of the origin, we have a contradiction and the theorem is proved. If we set

$$W_n^{(j)}=\begin{cases} A_{n-j}+V_n^{(j+1)} & \text{for } j=0, 1,\dots, n-1 \\ V_n^{(j+1)} & \text{for } j=n, n+1,\dots, \end{cases}$$

$(W_n^{(j)})_{j=1}^{\infty}$ is a defining sequence of $W_n^{(0)}$. In fact, for $1\leq j<n$ we have

$$\begin{aligned} W_n^{(j)}+W_n^{(j)} &= A_{n-j}+V_n^{(j+1)}+A_{n-j}+V_n^{(j+1)} \\ &\subset A_{n-j}+A_{n-j}+V_n^{(j+1)}+V_n^{(j+1)} \\ &\subset A_{n-(j-1)}+V_n^{(j)}=W_n^{(j-1)} \end{aligned}$$

For $j=n$, we have

$$W_n^{(n)}+W_n^{(n)}=V_n^{(n+1)}+V_n^{(n+1)}\subset V_n^{(n)}\subset W_n^{(n-1)}.$$

For $j>n$, we have

$$W_n^{(j)}+W_n^{(j)}=V_n^{(j+1)}+V_n^{(j+1)}\subset V_n^{(j)}=W_n^{(j-1)}.$$

Here $V=\bigcap_{n=1}^{\infty}W_n^{(0)}$. If we set $W^{(j)}=\bigcap_{n=1}^{\infty}W_n^{(j)}$, each $W^{(j)}\cap A_n$ is a neighbourhood of the origin in $A_n$. Therefore by Theorem 1 $W^{(0)}=V$ is a neighbourhood of the origin. The proof is complete.

PROPOSITION 2. *Let $(A_n)$ be an increasing sequence of closed balanced sets of a* *countably A-ultrabarrelled space* $E[\mathfrak{T}]$ *such that* $E=\bigcup_{n=1}^{\infty} A_n$, $A_n+A_n\subset A_{n+1}$ $(n=1, 2,\dots)$ *and each* $A_\alpha\in A$ *is absorbed by some* $A_n$, *and let each* $U\cap A_n$ $(n=1, 2,\dots)$ *be sequentially closed in* $A_n$. *Then* $U$ *is sequentially closed in* $E[\mathfrak{T}]$.

PROOF. Let $\{x_n\}$ be elements of $U$ such that it converges to $x$ in $E[\mathfrak{T}]$. Then a set $\{x, x_1, x_2,\dots\}$ is bounded. Therefore by Theorem 2 it is contained in some $A_n$. Since $U\cap A_n$ is sequentially closed in $A_n$, $x$ is contained in $U$. The proof is complete.

PROPOSITION 3. *Let* $(A_n)$ *be an increasing sequence of closed balanced sets of a countably A-ultrabarrelled space* $E[\mathfrak{T}]$ *such that* $E=\bigcup_{n=1}^{\infty} A_n$, $A_n+A_n\subset A_{n+1}$ $(n=1, 2,\dots)$ *and each* $A_\alpha\in A$ *is absorbed by some* $A_n$. *Then if for every sequence* $(x_n)\subset E$ *there is a sequence* $(\rho_n)$ *of positive real numbers such that the sequence* $(\rho_n x_n)$ *is bounded in* $E[\mathfrak{T}]$, *then* $A_n$ *is absorbent for some* $n$.

PROOF. If none of the sets $A_n$ is absorbent, there is a sequence $(x_n)$ in $E$ such that $x_n$ is not absorbed by $A_n$. By assumption there is a sequence $(\rho_n)$ of positive real numbers such that $(\rho_n x_n\colon n\in N)$ is bounded. By Theorem 2, this set is absorbed by $A_m$ for some $m$. This contradicts the choice of $x_m$. The proof is complete.

DEFINITION 3. *Let* $A$ *be a family of bounded subsets of an l.t.s* $E[\mathfrak{T}]$ *and let* $\bigcup_{A_\alpha\in A} A_\alpha=E$. *If* $E[\mathfrak{T}]$ *satisfies the following condition, then we call* $E[\mathfrak{T}]$ *an A-UDF space*:

(a) *A has a fundamental sequence* $(A_n)$ *of balanced closed sets.*

(b) *Let* $(U_n^{(0)})_{n=1}^{\infty}$ *be a sequence of balanced closed neighbourhoods and let* $(U_n^{(j)})_{j=1}^{\infty}$ *be a defining sequence of balanced closed neighbourhoods of* $U_n^{(0)}$. *Then if each set* $U^{(j)}=\bigcap_{n=1}^{\infty} U_n^{(j)}$ *absorbs all sets of fundamental sequence* $(A_n)$, $U^{(0)}$ *is a neighbourhood of the origin in* $E[\mathfrak{T}]$.

THEOREM 3. *If a ultrabarrelled space* $E[\mathfrak{T}]$ *has a numerable family* $\{A_n\}$ *of balanced sets in* $A$ *such that* $E=\bigcup_{n=1}^{\infty} A_n$ *and for* $B, C\in A$, $B\cup C$, $B+C$ *and* $\bar{B}$ *are contained in* $A$, *then* $E[\mathfrak{T}]$ *is an A-UDF space.*

PROOF. We construct the following sets

$$\tilde{A}_1=\overline{A_1},\quad \tilde{A}_2=\overline{A_1\cup A_2+A_1\cup A_2},\dots,\quad \tilde{A}_n=\overline{\underbrace{(A_1\cup\dots\cup A_n)+\dots+(A_1\cup\dots\cup A_n)}_{2^{n-1}}},\dots.$$

Then by Theorem 2 any $A_\alpha\in A$ is contained in some $\tilde{A}_m$. Therefore $(\tilde{A}_n)_{n=1}^{\infty}$ is a fundamental sequence of $A$. The proof is complete.

LEMMA 1. *Let* $(A_n)_{n=1}^{\infty}$ *be an increasing sequence of balanced sets of a countably A-ultrabarrelled space* $E[\mathfrak{T}]$ *such that* $E=\bigcup_{n=1}^{\infty} A_n$, $A_n+A_n\subset A_{n+1}$ $(n=1, 2,\dots)$ *and* *each $A_\alpha \in A$ is absorbed by some $A_n$. Let $\mathscr{F}$ be Canchy filter in $E[\mathfrak{T}]$ and $\mathscr{G}$ is the filter which basis is constituted by all set of the form $M+U$, where $M$ varis in $\mathscr{F}$ and $U$ in filter of the neighbourhood of the origin in $E[\mathfrak{T}]$. Then there exists a positive integer $n_0$ such that $\mathscr{G}$ induces a filter in $A_{n_0}$.*

Proof. If the lemma is false there exists a decreasing sequence $(V_n)_{n=1}^\infty$ of closed balanced neighbourhoods of the origin and a sequence $(M_n)_{n=1}^\infty$ of elements of $\mathscr{F}$ such that $(M_n+V_n)\cap A_{2n+3}=\phi$ for $n=1, 2,\dots$. If we set

$$W_n^{(j)}=\begin{cases}\overline{(A_{n-j}\cap V^{(j+1)})+V_{n+2}^{(j+1)}} & \text{for } j=0, 1,\dots, n-1\\ V_{n+2}^{(j+1)} & \text{for } j=n, n+1,\dots\end{cases}$$

where we assume that $V_1\supset V_2+V_2,\dots, V_n\supset V_{n+1}+V_{n+1},\dots$, that $(V_n^{(j)})$ is a sequence of closed balanced neighbourhoods of the origin such that $V_n\supset V_n^{(1)}+V_n^{(1)},\dots, V_n^{(j)}\supset V_n^{(j+1)}+V_n^{(j+1)},\dots$ and that $V_{n+1}^{(j)}\subset V_n^{(j)}$ for $n, j=1, 2,\dots$.
Then the sequence $(W_n^{(j)})_{j=1}^\infty$ is a defining sequence of $W_n^{(0)}$. In fact we have for $1\leq j<n$

$$\begin{aligned}W_n^{(j)}+W_n^{(j)}&=\overline{(A_{n-j}\cap V^{(j+1)})+V_{n+2}^{(j+1)}}+\overline{(A_{n-j}\cap V^{(j+1)})+V_{n+2}^{(j+1)}}\\&\subset\overline{(A_{n-j}\cap V^{(j+1)})+(A_{n-j}\cap V^{(j+1)})+V_{n+2}^{(j+1)}+V_{n+2}^{(j+1)}}\\&\subset\overline{(A_{n-(j-1)}\cap V^{(j)})+V_{n+2}^{(j)}}=W_n^{(j-1)}.\end{aligned}$$

We have for $j=n$

$$W_n^{(n)}+W_n^{(n)}=V_{n+2}^{(n)}+V_{n+2}^{(n)}\subset V_{n+2}^{(n-1)}\subset W_n^{(n-1)}.$$

Since $(V_{n+2}^{(j)})_{j=1}^\infty$ is a defining sequence of $V_{n+2}$, the insistence is also true. Since $W^{(j)}=\bigcap_{n=1}^\infty W_n^{(j)}$ is an $A$-suprabarrel, $W^{(0)}$ is a neighbourhood of the origin in $E[\mathfrak{T}]$.
If we set

$$S=W^{(0)}=(A_1\cap V^{(1)}+V_3^{(1)})\cap(A_2\cap V^{(1)}+V_4^{(1)})\cap\cdots\cdots\cdots,$$

$$S_1=A_1\cap V^{(1)}+V_3^{(1)},$$

$$S_2=(A_2\cap V^{(1)}+V_4^{(1)})+(A_3\cap V^{(2)}+V_6^{(2)}),$$

$$\vdots$$

$$S_n=(A_n\cap V^{(1)}+V_{n+2}^{(1)})+(A_{n+1}\cap V^{(2)}+V_{n+4}^{(2)})+\cdots+(A_{2n-1}\cap V^{(n)}+V_{3n}^{(n)}),$$

$$\vdots$$

then each $S_n$ is a neighbourhood of the origin and $S\subset S_n$ $(n=1, 2,\dots)$.

Here if we take $P_n\in\mathscr{F}$ such that $P_n-P_n\subset S_n$, we have $(P_n+S_n)\cap A_{2n+2}=\phi$ $(n=1, 2,\dots)$. In fact if $x_0\in P_n\cap M_n\in\mathscr{F}$ and $y\in P_n$, then $y-x_0\in P_n-P_n\subset S_n$, which implies $y-x_0=a_n+a_{n+1}+\cdots+a_{2n-1}+v_n$, where $a_i\in A_i\cap V^{(i-n+1)}$ for $i=n,\dots, 2n-1$

and $v_n \in V^{(1)}_{n+2} + V^{(2)}_{n+4} + \cdots + V^{(n)}_{3n}$. If $z \in P_n + S_n$, we can write $z = y + b_n + c_n$, where $b_n \in S_n$ and $c_n \in S_n$. Therefore we can write $z = x_0 + a_n + \cdots + a_{2n-1} + v_n + e_n + e_{n+1} + \cdots + e_{2n-1} + u_n + f_n + \cdots + f_{2n-1} + t_n$, where $f_i$, $e_i \in A_i \cap V^{(i-n+1)}$ for $i = n, \ldots, 2n-1$ and $u_n$, $t_n \in V^{(1)}_{n+2} + V^{(2)}_{n+4} + \cdots + V^{(n)}_{3n}$. If we set $g = a_n + \cdots + a_{2n-1} + e_n + e_{n+1} + \cdots + e_{2n-1} + f_n + \cdots + f_{2n-1}$, we have $g \in A_{2n+2}$. Here we have $x_0 + v_n + u_n + t_n \in M_n + V_n$. If we set $q = v_n + u_n + t_n$, by our assumption we have $x_0 + q \notin A_{2n+3}$. If $z \in A_{2n+2}$, $x_0 + q = z - g \in A_{2n+2} + A_{2n+2} \subset A_{2n+3}$, which is a contradiction. Hence $(P_n + S_n) \cap A_{2n+2} = \phi$.

Assume that $P - P \subset S$. If $z_0 \in P$, there exists a positive integer $n_0$ such that $z_0 \in A_{2n_0+2}$. If $w$ is an arbitrary element of $P_{n_0}$, then $z_0 \notin w + S_{n_0}$, that is, $z_0 - w \notin S_{n_0}$. Since $S \subset S_{n_0}$, we obtain $z_0 - w \notin S$. Furthermore $P - P \subset S$ and $z_0 \in P$ and hence we deduce $w \notin P$, that is, $P \cap P_{n_0} = \phi$, which is clearly non true and the lemma has been proved.

THEOREM 4. *Let $(A_n)_{n=1}^{\infty}$ be an increasing sequence of balanced sets of a countably A-ultrabrraelled space $E[\mathfrak{T}]$ such that $E = \bigcup_{n=1}^{\infty} A_n$, $A_n + A_n \subset A_{n+1}$ $(n = 1, 2, \ldots)$ and each $A_\alpha \in A$ is absorbed by some $A_n$. Then if each $A_n$ is complete, $E$ is complete.*

PROOF. Let $\mathscr{F}$ be a Cauchy filter of $E$. Then there exists a positive integer $n_0$ such that the filter $\mathscr{G}$ described in this lemma, induces in $A_{n_0}$ a filter $\mathscr{G}'$. Obviously $\mathscr{G}'$ is a Cauchy filter and therefore there exists an $x \in A_{n_0}$ such that $\mathscr{G}'$ converges to $x$ and also $\mathscr{F}$ converges to $x$. The proof is complete.

THEOREM 5. *Let $(A_n)$ be an increasing sequence of closed balanced sets of a countably A-ultrabarrelled space $E[\mathfrak{T}]$ such that $E = \bigcup_{n=1}^{\infty} A_n$, $A_n + A_n \subset A_{n+1}$ $(n = 1, 2, \ldots)$ and each $A_\alpha \in A$ is absorbed by some $A_n$. Then if completion $\hat{E}[\hat{\mathfrak{T}}]$ is a Baire set, there exists an $A_n$ which have an inner point.*

PROOF. Let $x$ be an arbitrary element of $\hat{E}[\hat{\mathfrak{T}}]$ and $\mathscr{F}$ be a filter in $E$ converging to $x$. Since $\mathscr{F}$ is a Cauchy filter, there exists a positive integer $n_0$ such that the filter $\mathscr{G}$ described in Lemma 1, induces a Cauchy filter $\mathscr{G}'$ in $A_{n_0}$, converging to $x$. Hence if $\overline{A_{n_0}}$ is the closure of $A_{n_0}$ in $\hat{E}[\hat{\mathfrak{T}}]$, then $x \in \overline{A_{n_0}}$ and therefore $\hat{E}[\hat{\mathfrak{T}}] = \bigcup_{n=1}^{\infty} \overline{A_n}$. Since $\hat{E}[\hat{\mathfrak{T}}]$ is a Baire set, there exists a positive number $n_1$ such that $\overline{A_{n_1}}$ has an inner point in $\hat{E}[\hat{\mathfrak{T}}]$. Hence $\overline{A_{n_1}} \cap E = A_{n_1}$ has an inner point in $E$. The proof is complete.

COROLLARY 1. *Let $E[\mathfrak{T}]$ be a countably A-ultrabarrelled space whose completion $\hat{E}[\hat{\mathfrak{T}}]$ is a Baire space. Let us assume that there exists a numerable sequence $(B_n)$ of balanced bounded sets such that $\bigcup_{n=1}^{\infty} B_n = E$ and each $A_\alpha \in A$ is absorbed by some $B_n$. Then $E[\mathfrak{T}]$ is a quasinormed space.*

PROOF. Let us put $A_1 = \overline{B_1}$, $A_2 = \overline{B_1 + B_2 + B_1 + B_2}, \ldots$

$$A_n=\underbrace{\overline{(B_1+\cdots+B_n)+\cdots+(B_1+\cdots+B_n)}}_{2^{n-1}},\ldots.$$

Then $\bigcup_{n=1}^{\infty} A_n=E$. Therefore by Theorem 5, there exists an $A_{n_0}$ which has an inner point. Consequently $E[\mathfrak{T}]$ is quasinormed. The proof is complete.

Corollary 2. *Let $E[\mathfrak{T}]$ be a countably A-ultrabarrelled space, let each $A_\alpha \in A$ be finite dimension and let $\hat{E}[\hat{\mathfrak{T}}]$ be a Baire space. Then if $E_1$ is a closed subspace of E with at most numerable codimension, this codimension is finite.*

Proof. If the codimension of $E_1$ is infinite, there exists a sequence $(x_n)_{n=1}^{\infty}$ of linearly independent vectors such that the space generated by them is the algebraic complementary of $E_1$. Let $E_n$ be the space generated by $E_1 \cup \{x_1,\ldots, x_{n-1}\}$, for $n=2, 3,\ldots$. Since $E_1$ is closed, $E_2$ is also closed and in general $E_n$, $n=1, 2,\ldots$ are closed subspaces. Each of the set of the sequence $(E_n)_{n=1}^{\infty}$ is closed, balanced and without inner point. Theorem 5 contradicts the fact that $E=\bigcup_{n=1}^{\infty} E_n$.

Definition 4. *We denote by $\mathfrak{T}^{A^*}$ the topology on $E[\mathfrak{T}]$ which has all $\mathfrak{T}$-closed A-suprabarrel as a basis of neighbourhoods of the origin.*

Lemma 2. *Let A be a family of bounded sets and let $E[\mathfrak{T}]$ be a countably A-ultrabarrelled space. Then a set B is $\mathfrak{T}$-precompact if and only if it is $\mathfrak{T}^{A^*}$-precompact.*

This can be shown similarly as in I, (1) in [1].

Proposition 3. *Let $(A_n)$ be an increasing sequence of balanced sets of a countably A-ultrabarrelled space $E[\mathfrak{T}]$ such that $E=\bigcup_{n=1}^{\infty} A_n$, $A_n+A_n\subset A_{n+1}$ $(n=1, 2,\ldots)$ and each $A_\alpha\in A$ is absorbed by some $A_n$. Then if $(A_n)_{n=1}^{\infty}$ is a sequence of metrizable sets in $E[\mathfrak{T}]$, $E[\mathfrak{T}]$ is an A-ultrabarrelled space.*

Proof. A sequence $(x_n)_{n=1}^{\infty}$ converges in $E[\mathfrak{T}]$ if and only if it converges in $E[\mathfrak{T}^{A^*}]$. In fact since $\mathfrak{T}\subset\mathfrak{T}^{A^*}$, $x_n\to x$ in $E[\mathfrak{T}^{A^*}]$ implies $x_n\to x$ in $E[\mathfrak{T}]$. Conversely if $x_n\to x$ in $E[\mathfrak{T}]$, a set $\{x, x_1, x_2,\ldots, x_n,\ldots\}$ is a $\mathfrak{T}$-compact set. Therefore it is a $\mathfrak{T}^{A^*}$-precompact set and hence the topologies induced by $E[\mathfrak{T}]$ and $E[\mathfrak{T}^{A^*}]$ are equivalent on $\{x, x_1, x_2,\ldots\}$. Consequently $x_n\to x$ in $E[\mathfrak{T}^{A^*}]$.

Let $V_0$ be an $A$-ultrabarrel with a defining sequence $(V_j)_{j=1}^{\infty}$ of closed balanced sets and let $(x_n)$ be a sequence of $A_m$ such that $x_n\to 0$ in $E[\mathfrak{T}]$. Then $x_n\to 0$ in $E[\mathfrak{T}^{A^*}]$. Therefore there exists a positive integer $n_i$ such that $x_n\in V_i\cap A_m$ for any $n\geq n_i$. Since $A_m$ is metrizable, each $V_i$ is a neighbourhood of the origin on $A_m$ for any $m=1, 2,\ldots$. Consequently an $A$-ultrabarrel $V_0$ is a neighbourhood of the origin in $E[\mathfrak{T}]$. The proof is complete.

## References

[1] N. ADASCH und B. ERNST, Ultra-DF-Räume mit relativ kompakten becshränkten Teilmengen, Math. Ann. **206** (1973), 79–87.

[2] B. ERNST, Ultra-(DF)-Räume, J. reine angew. Math. **258** (1973), 87–102.

[3] S. O. IYAHEN, Linear topological spaces with fundamental sequences of compact sets, Math. Ann. **200** (1973), 179–183.

[4] M. VALDIVIA, Absolutely convex sets in barreled spaces, Ann. Inst. Fourier, Grenoble **21**, 2 (1971), 3–13.

[5] M. DE WILDE and C. HOUET, On increasing sequences of absolutely convex sets in locally convex spaces, Math. Ann, **192** (1971), 257–261.