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Holomorphic θ-Line Budles over a Compact Riemann Surface of Genus 3

Hiromichi MATSUNAGA
Department of Mathematics, Shimane University, Matsue, Japan
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This paper is a continuation of the author's preceding note [5]. We want to study mainly about the group of equivalence classes of holomorphic Z_2-line bundles over a compact Riemann surface of genus three. To assure that an involution is holomorphic and to see explicitly an aspect of a ramification, we treat plane algebraic curves without singularity. § 1 contains reformulations of some known results in convenient forms, and these are used explicitly or implicitly in § 2 and Remark. Especially, a fundamental result due to A. Hurwitz is effectively used to see topological structures of surfaces. The exact sequence (3) in § 2 is one of our main results. In Remark, an example is given, and it is proved that there exists no holomorphic G-line bundle other than trivial bundle.

§ 1. Some generalities

This section describes some results which we use in the next section. These results are reformulations of known results. C denotes the field of complex numbers. Suppose $V\subset C^3$ is an irreducible analytic variety which admits a good $C^*=C-\{0\}$ action leaving V invariant, i.e.

$$\sigma(t, (z_0, z_1, z_2))=(t^{q_0}z_0, t^{q_1}z_1, t^{q_2}z_2), \qquad (1.2, [6]).$$

Let φ: $C^3\to C^3$ be defined by $\varphi(z_0, z_1, z_2)=(z_0^{q_0}, z_1^{q_1}, z_2^{q_2})$ and $V'=\varphi^{-1}(V)$ which is called the cone over V following P. Orlik and Wagreich. The quotient space X' of $V'-\{0\}$ by C^* is an analytic space, and the analogue is true for V, where we denote by X the quotient space of $V-\{0\}$. The holomorphic map φ induces the quotient map $\tilde{\varphi}$: $X'\to X$. It can be easily seen that C^* acts *separably* on $V'-\{0\}$ and $V-\{0\}$, (Definition 15, [2]). By Zusatz to Satz 12 in [2],

PROPOSITION 1. *The map $\tilde{\varphi}$ is holomorphic.*

The sheaf of germs of holomorphic functions or holomorphic differential forms of type (1, 0) on a compact Riemann surface M are denoted by O or $O^{1,0}$ respectively. We denotes by $\Gamma(M, O^{1,0})$ the set of all sections of the sheaf $O^{1,0}$, i.e. the space of abelian differentials on the surface M. The exact sequence

$$0\longrightarrow C\longrightarrow O\xrightarrow{d}O^{1,0}\longrightarrow 0 \qquad (1)$$

leads to the exact sequence

$$(2) \qquad 0 \longrightarrow \Gamma(M, O^{1,0}) \xrightarrow{\delta} H^1(M, C) \longrightarrow H^1(M, O) \longrightarrow 0.$$

By the fundamental result of Hodge theory and Theorem 10.2, [8], we have

PROPOSITION 2. $\quad H^1(M, C) \cong \delta\Gamma(M, O^{1,0}) + \overline{\delta\Gamma(M, O^{1,0})}$,
where the bar over the second term on the right denotes the complex conjugate. This proposition supplies an information about the dimension of the invariant subspace of the Picard variety. In fact, let H be a group of automorphisms of the space M and $\Gamma(M, O^{1,0})^H$ be the H-invariant subspace, then it is isomorphic to the space $\Gamma(M/H, O^{1,0})$, and so its dimension is equal to the genus of the quotient space M/H, ((b), Theorem 3, [4]).

The next proposition can be used as a criterion for the hyperellipticity of a compact Riemann surface M. We can find such an example at 10–10 in [8].

PROPOSITION 3, (Theorem 6–11, [8]). *If $p_1(z)dz$ and $p_2(z)dz$ are two meromorphic differentials on a compact Riemann surface M, then the fraction $p_1(z)/p_2(z)$ defines a meromorphic function on M.*

Let M be a compact Riemann surface with a group of order n of effective automorphisms. Denote by $\{a_1,\ldots, a_k\}$ the set of branch points of the natural projection $p\colon M \rightarrow M/G$. Suppose the genus of the quotient space M/G is $\tilde{g}$. We choose an arbitrary point O on the surface M/G and $2\tilde{g}$ closed paths $A_1, B_1,\ldots, A_{\tilde{g}}, B_{\tilde{g}}$ with base point O, by which the surface M/G is developed in a simply connected surface with boundary, say $(M/G)'$. Then we draw simple paths $l_1,\ldots, l_k$ from the point O to the points $a_1,\ldots, a_k$ in the interior of $(M/G)'$ which do not intersect each other. Denote by $(M/G)''$ the arising surface. Here we collect a result due to A. Hurwitz (6. II Abschnitt, [3]). Let $\{S_1,\ldots, S_n\}$ be the elements of G. Further, let $T_1,\ldots, T_k, U_1, V_1,\ldots, U_{\tilde{g}}, V_{\tilde{g}}$ be any $k+2\tilde{g}$ elements of G, by which G is generated and the relation

$$T_1 \ldots T_k U_1 V_1 U_1^{-1} V_1^{-1} \ldots U_{\tilde{g}} V_{\tilde{g}} U_{\tilde{g}}^{-1} V_{\tilde{g}}^{-1} = 1$$

is satisfied. Denote by the same letters $S_1,\ldots, S_n$ n copies of $(M/G)''$.

PROPOSITION 4. *The surface M is obtained by a connection of these surfaces along the cuts l, A, B with suitably choiced T, U, V, that is as a scheme*

$$M = \begin{pmatrix} l_1,\ldots, l_k, A_1, B_1,\ldots, A_{\tilde{g}}, B_{\tilde{g}} \\ T_1,\ldots, T_k, U_1, V_1,\ldots, U_{\tilde{g}}, V_{\tilde{g}} \end{pmatrix}.$$

§2. Holomorphic G-vector bundles

Following the notations in §1 and Z to denote the group of integers, the group of complex line bundles of Chern class zero is isomorphic to

$$P_0(M)=\frac{H^1(M, C)}{H^1(M, Z)+\delta\Gamma(M, 0^{1,0})}.$$

As in [5], the exact sequence due to Grothendieck is

(1) $$e \longrightarrow H^1(G, H^0(M, O^*)) \longrightarrow H^1(M, G, O^*) \longrightarrow H^1(M, O^*)^G,$$

where O^* denotes the sheaf of germs of nowhere vanishing holomorphic functions on M. Suppose the natural projection $p: M \to M/G$ is an n-sheeted covering with branch points $a_1,\dots, a_k$, where the ramification order at a_i is m_i-1 for $i=1,\dots, k$, that is, the map p can be described locally as $w=z^{m_i}$ near the point a_i for each $i=1,\dots, k$. Then the Riemann-Hurwitz formula is

(2) $$g=\frac{1}{2}\sum_{i=1}^{k}(m_i-1)+n(\tilde{g}-1)+1.$$

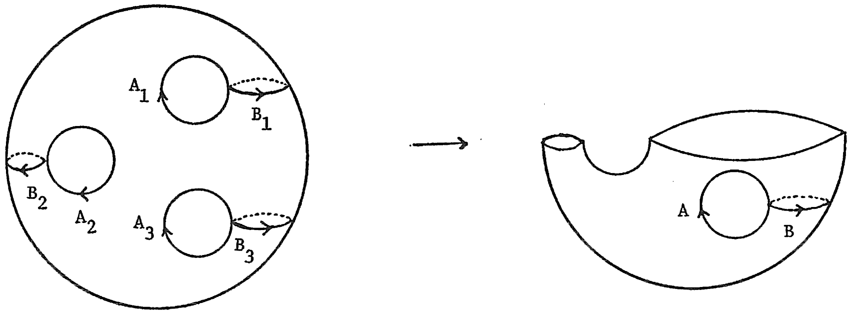
Let V' be a surface in C^3 with the only singularity (0, 0, 0), then by the C^*-action the quotient space is a plane algebraic curve. In this section we treat such a plane algebraic curve of genus three with an involution. Good examples are presented by the three classes of the first half in the six classes by Orlik-Wagreich (Definition 3.1.1, [6]). We denote by $[z_0, z_1, z_2]$ the homogeneous coordinate in the complex projective plane CP^2.

Class I. The equation of V' is given by $z_0^4+z_1^4+z_2^4=0$, and the map θ: $[z_0, z_1, z_2] \to [z_0, z_1, -z_2]$ defines an involution of the quotient space X' in § 1. The set of fixed points is $[1, \varepsilon_4^j, 0]$, $j=1, 2, 3, 4$, where $\varepsilon_4=\exp \pi i/4$. The quotient space by the involution is a compact Riemann surface of genus one by (2). We have $X=\{\{z_0, z_1, z_2\}; z_0^4+z_1^4+z_2^2=0\}$, where $\{tz_0, tz_1, t^2z_2\}=\{z_0, z_1, z_2\}$ for each $t \in C^*$. Here we remark that the Riemann surface X' is not hyperelliptic. For the space X' is the surface of the algebraic function field which is given by the equation $f^4=-(1+z^4)$, then we can discuss analogously as in 10–10 in [8] by using Proposition 3 in § 1. Since the covering $\tilde{\varphi}$: $X' \to X$ is two sheeted with four branch points, then the scheme in Proposition 4 in § 1 is

$$X'=\begin{pmatrix} l_1, & l_2, & l_3, & l_4, & A, & B \\ T, & T, & T, & T, & I, & I \end{pmatrix},$$

where T is the transposition and I is the identity map. Therefore the map $\tilde{\varphi}$ can be described topologically as in the figure in p. 16.

The involution θ induces the transformation of the one dimensional homology basis, $A_1 \to -A_3$, $B_1 \to -B_3$, $A_2 \to -A_2$, $B_2 \to -B_2$. Let a_1, b_1, a_2, b_2, a_3, b_3 be the Poincaré duals of the homology classes A_1, B_1, A_2, B_2, A_3, B_3 respectively, then for arbitrary complex numbers $x_1, y_1, x_2, y_2, x_3, y_3$, $\theta(x_1a_1+y_1b_1+x_2a_2+y_2b_2+x_3a_3+y_3b_3) = -x_1a_3-y_1b_3-x_2a_2-y_2b_2-x_3a_1-y_3b_1$. Thus the cohomology class is θ-invariant



if and only if the class $(x_1+x_3)a_1+(y_1+y_3)b_1+2x_2a_2+2y_2b_2+(x_3+x_1)a_3+(y_3+y_1)b_3$ belongs to modulus $H^1(X', Z)+\delta\Gamma(X', O^{1,0})$, that is x_1+x_3, y_1+y_3, $2x_2$, $2y_2$, are integers modulo $\delta\Gamma(X', O^{1,0})$. Since the group $\delta\Gamma(M, O^{1,0})$ is generated by classes $a_k+y_{k,1}b_1+y_{k,2}b_2+y_{k,3}b_3$, $k=1, 2, 3$ for some $y_{k,j}\in C$, $j=1, 2, 3$, then by the exact sequence (1), we have an exact sequence

$$e \longrightarrow Z_2 \longrightarrow H^1(X', Z_2; 0^*) \longrightarrow \left(\frac{C}{Z\times Z}\oplus Z_2\right)\times Z. \tag{3}$$

Now we advance to the another two classes. As far as we concern holomorphic θ-line bundles, the aspect is quite similar to the class I.

Class II. The equation of V' is given by $z_0^4+z_1^4+z_1z_2^3=0$, and the map θ: $[z_0, z_1, z_2]\to[-z_0, z_1, z_2]$ defines an involution of the quotient space X'. The set of fixed points is $[0, 1, \varepsilon_3^j]$, $j=1, 2, 3$, and $[0, 0, 1]$, where $\varepsilon_3=\exp\pi i/3$. The quotient space X is of genus one and can be described as

$$X=\{\{z_0, z_1, z_2\};\ z_0^2+z_1^4+z_1z_2^3=0\},\ \text{where}\ \{t^2z_0, tz_1, tz_2\}$$
$$=\{z_0, z_1, z_2\}\quad \text{for each}\quad t\in C^*.$$

Class III. The equation of V' is given by $z_0^4+z_1^3z_2+z_2^3z_1=0$, and the map θ: $[z_0, z_1, z_2]\to[-z_0, z_1, z_2]$ defines an involution of the quotient space X'. The set of fixed points is $\{[0, 1, \pm\sqrt{-1}]$, $[0, 0, 1]$, $[0, 1, 0]$. The quotient space X is of genus one and can be described as

$$X=\{\{z_0, z_1, z_2\};\ z_0^2+z_1^3z_2+z_2^3z_1=0\},\ \text{where}\ \{t^2z_0, tz_1, tz_2\}$$
$$=\{z_0, z_1, z_2\}\quad \text{for each}\quad t\in C^*.$$

Next we consider again the class I. Define a $Z_2\times Z_2$-action on X' by θ_1: $[z_0, z_1, z_2]\to[z_0, z_1, -z_2]$ which is the involution θ and θ_2: $[z_0, z_1, z_2]\to[z_0, -z_1, z_2]$. The action is without fixed point, but admits eight branch points $[1, \varepsilon_4^j, 0]$, $[1, 0, \varepsilon_4^j]$, $j=1, 2, 3, 4$, where $\varepsilon_4=\exp\pi i/4$. The involution θ_2 induces the transformation of one dimensional homology basis, $A_1 \to -A_2$, $B_1 \to -B_2$, $A_3 \to -A_3$, $B_3 \to -B_3$. Then for a cohomology class,

$$\theta_2(x_1a_1+y_1b_1+x_2a_2+y_2b_2+x_3a_3+y_3b_3) = -x_1a_2-y_1b_2-x_2a_1-y_2b_1 -x_3a_3-y_3b_3.$$

Thus the cohomology class is θ_2-invariant if and only if the class $(x_1+x_2)a_1+(y_1+y_2)b_1+(x_2+x_1)a_2+(y_2+y_1)b_2+2x_3a_3+2y_3b_3$ belongs to the modulus $H^1(X', Z)+\delta\Gamma(X', O^{1,0})$, that is x_1+x_2, y_1+y_2, $2x_3$, $2y_3$ are integers modulo $\delta\Gamma(X', O^{1,0})$. Hence the class $x_1a_1+y_1b_1+x_2a_2+y_2b_2+x_3a_3+y_3b_3$ is $Z_2 \times Z_2$-invariant if and only if $2x_2$, $2y_2$, $2x_3$, $2y_3$, (x_1+x_2), (y_1+y_2) are integers modulo $\delta\Gamma(X', O^{1,0})$. The sequence (1) is

$$e \longrightarrow H^1(Z_2 \oplus Z_2, H^0(X', O^*)) \longrightarrow H^1(X', Z_2 \oplus Z_2; O^*) \longrightarrow H^1(X', O^*)^{Z_2 \oplus Z_2}.$$

By the proof of Proposition 2 in [5], $H^1(Z_2 \oplus Z_2, H^0(X', O^*)) = Z_2 \oplus Z_2$. Hence the order of the group $H^1(X', Z_2 \oplus Z_2; O^*)$ is equal to $4 \times 2 \times 2 \times 2 \times 2 = 64$ for each degree. The quotient space is a compact Riemann surface of genus zero and so the projective line which is given by the equation $z_0^4+z_1^2+z_2^2=0$ with the identification $\{tz_0, t^2z_1, t^2z_2\} = \{z_0, z_1, z_2\}$ for each $t \in C^*$.

REMARK.

I. Let S be Klein's compact Riemann surface of genus three admitting a simple group Γ_{168} of 168 automorphisms, which is generated by U, V, T and $U^2=V^3=T^7=I$, the unit, [7]. For the canonical form of the surface given in 3, [7], we can put $A_1=A$, $B_1=B$, $A_2=C$, $B_2=D$, $A_3=E$, $B_3=F$. We want to see the structure of the group $H^1(S, O^*)^G$, where $G=\Gamma_{168}$. By the formula (1), [7], a cohomology class $x_1a_1+y_1b_1+x_2a_2+y_2b_2+x_3a_3+y_3b_3$ is T-invariant if and only if

1) $x_1 - y_1 + y_2 - y_3 \equiv 0,$

2) $x_2 - y_2 + y_3 \equiv 0,$

3) $x_2 + 2x_3 \equiv 0,$

4) $x_1 + x_2 + x_3 \equiv 0,$

5) $x_1 + x_2 - y_1 + 2y_2 - y_3 \equiv 0,$

6) $-y_1 + y_2 + y_3 \equiv 0,$

where $\equiv$ denotes the congruence modulo the group $H^1(S, Z)+\delta\Gamma(S, O^{1,0})$. Then we have

7) $x_1 \equiv 4y_2$, 8) $x_2 \equiv -y_2$, 9) $x_3 \equiv -3y_2$,

$$10)\ 7y_2 \equiv 0, \qquad 11)\ y_3 \equiv 2y_2, \qquad 12)\ y_1 \equiv 3y_2.$$

By the formula (2), [7], the class is V-invariant if and only if

$$\begin{array}{llr}
1') & 2x_1 + x_2 + x_3 - y_1 & \equiv 0, \\
2') & x_2 - x_3 + y_2 - y_3 & \equiv 0, \\
3') & -x_2 + x_3 - y_2 + y_3 & \equiv 0, \\
4') & x_1 + y_1 & \equiv 0, \\
5') & x_1 + y_2 - y_3 & \equiv 0, \\
6') & x_1 + x_2 & \equiv 0,
\end{array}$$

then we have 13) $x_3 \equiv -2x_1$. By 7), 9), 13), $-3y_2 \equiv -8y_2$, then $5y_2 \equiv 0$ and so by 10) $2y_2 \equiv 0$, hence $y_2 \equiv 0$, thus the class is zero. Therefore we have

PROPOSITION 5. $H^1(S, O^*)^G = Z$, where $G = \Gamma_{168}$.

By the analogue to the proof of Proposition 2 in [5], we can prove that the group $H^1(G, H^0(S, O^*))^G$ contains at least $2 \times 3 \times 7 = 42$ elements.

II. Here we want to find a plane algebraic curve which realizes the situation in [5]. It is known that any compact Riemann surface of genus two cannot be expressible as a plane algebraic curve without singularity. The plane curve X': $z_0^6 + z_1^6 + z_1^4 z_2^2 = 0$ has an isolated double point [0, 0, 1]. The involution $[z_0, z_1, z_2] \rightarrow [z_0, z_1, -z_2]$ admits the set of fixed points $[1, \varepsilon_6^j, 0]$, $j = 1, 2, \dots, 6$, where $\varepsilon_6 = \sqrt[6]{-1}$ and [0, 0, 1]. The quotient space of X' is $X = \{\{z_0, z_1, z_2\};\ z_0^6 + z_1^6 + z_1^4 z_2 = 0$ and $\{tz_0, tz_1, t^2 z_2\} \equiv \{z_0, z_1, z_2\}$ for each $t \in C^*\}$. By the resolution of the point [0, 0, 1], we get a hyperelliptic surface of genus two and the projective plane as its quotient space.

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