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On some Quasigroups of Algebraic Models of Symmetric Spaces III

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In this paper, we observe the fact that symmetric loops treated in the previous papers [1] and [2] are in a special class of homogeneous loops of [3]. It is shown that the homogeneous structures on symmetric loops are in one-to-one correspondence to quasigroups of reflection. Following N. Nobusawa [5], we consider abelian quasigroups of reflection and show that they correspond to homogeneous structures of a certain class of abelian groups. We give also an example of finite symmetric loop of 27 elements due to [5]. In conclusion of this series of notes we give some geometric observations on symmetric loops as affine symmetric spaces, when the natural differentiable structures are assumed on them. For this purpose we consider symmetric Lie loops of [3]. Then, by applying the results of [3] and [4], it will be seen that Lie triple systems can be regarded as the tangent algebras of symmetric Lie loops.

§1. Symmetric loops as homogeneous loops

In [3] we have introduced the concept of homogeneous loops as follows: A loop G is said to be *homogeneous* if it satisfies the following conditions (1) and (2).

(1) G has the left inverse property, i.e., each element x of G has an inverse element x^{-1} for which the equality $x^{-1}(xy)=y$ holds for any y in G.

(2) Any left inner mapping $L_{x,y}$, for x and y in G, is an automorphism of G.

Moreover, if a homogeneous loop G satisfies the following condition (3), G is said to have the *symmetric property*.

(3) The mapping $J(x)=x^{-1}$ of G onto itself is an automorphism of G.

In this section we review the results in [1] and [2] from the viewpoint of homogeneous loops. In the beginning we show the following theorem which provides another definition of symmetric loops. We notice that the condition for G to be power associative is redundant in the definition of the symmetric loop G in [1] and [2].

THEOREM 1. *A loop G is a symmetric loop if and only if it is a homogeneous loop satisfying the following conditions* (1.1–3):

(1.1) *G has the symmetric property.*

(1.2) *G is left alternative, i.e., $L_{x,x}=\mathrm{id}$ holds for any $x\in G$.*

(1.3) *Each element x of G has a unique square root u, $u^2=x$, in G.*

PROOF. The conditions (1.1–3) for a homogeneous loop G imply that G is left power alternative (i.e., left di-associative in the strong sense). In fact, denoting $x^o=e$ (the identity), $x^n=xx^{n-1}$ and $x^{-n}=(x^n)^{-1}$ for $x\in G$ and for any positive integer n, we can show the equality $(f_x)^i=f_{x^i}$ for any integer i, by a method similar to that in the proof of Proposition 1 [2], where f_x denotes the left translation by x. Then the theorem is reduced to Theorem 3 [2]. *q. e. d.*

If a homogeneous loop G has the symmetric property, then the equality $L_{x,x}L_{x,x}=\mathrm{id}$ holds. We do not know whether the symmetric property (1.1) implies (1.2) or not. If G satisfies (1.1) and (1.3), then by Theorem 1 the subset $G'=\{y\in G;\ L_{x,x}y=y$ for all $x\in G\}$ of G is a symmetric subloop of G.

Let $G=G^{(e)}$ be a symmetric loop with the identity e. Then, by Theorem 2 [1], a quasigroup of reflection $(G, *)$ is associated with $G^{(e)}$ under the binary operation

$$x*y=x(xy^{-1}). \tag{1.4}$$

In [3] we considered the homogeneous structure (Definition 1.5 [3]) of a homogeneous loop G which assigns to each element a of G a homogeneous loop $G^{(a)}$, called the transposed loop centered at a, which is isomorphic to $G^{(e)}$. Therefore, if $G^{(e)}$ is a symmetric loop, then $G^{(a)}$ is again a symmetric loop and the same quasigroup of reflection is associated with $G^{(a)}$ as one with $G^{(e)}$. In fact, by Lemma 1.8 [3], the inverse y' of any element y in $G^{(a)}$ is equal to $a(a^{-1}y)^{-1}$, and so, describing (1.4) for $G^{(a)}$, we get $x\overset{(a)}{\cdot}(x\overset{(a)}{\cdot}y')=a((a^{-1}x)((a^{-1}x)(a^{-1}y)^{-1}))=x(xy^{-1})$, where $\overset{(a)}{\cdot}$ denotes the multiplication in $G^{(a)}$. Thus, given a symmetric loop $G^{(e)}$, a unique quasigroup of reflection is associated with the homogeneous structure of $G^{(e)}$. Let ϕ: $\{G^{(a)};\ a\in G\}\to\{H^{(b)};\ b\in H\}$ be a homomorphism of homogeneous structures (cf. Definition 1.5 [3]) of symmetric loops. ϕ induces a homomorphism of any symmetric loop $G^{(a)}$ into $H^{(a')}$, $a'=\phi(a)$. Then, by (1.4), it is clear that the mapping ϕ induces a homomorphism of the quasigroups of reflection corresponding to $G^{(a)}$ and $H^{(a')}$.

Conversely, given a quasigroup $(G, *)$ of reflection, a symmetric loop $G^{(a)}$ is defined for any element $a\in G$ by Theorem 1 [1] and the family $\{G^{(a)};\ a\in G\}$ defines a homogeneous structure of symmetric loops on G by Proposition 3 [1]. Since any homomorphism of quasigroups of reflection induces homomorphisms of corresponding symmetric loops on them, taking Theorem 3 [1] into account, we have

THEOREM 2. *By assigning to each symmetric loop a quasigroup of reflection of* Theorem 2 [1], *the bijective functor of the category of homogeneous structures of symmetric loops onto the category of quasigroups of reflection is obtained, and its inverse functor is induced by the assignment of* Theorem 1 [1].

As an immediate consequence of this theorem we have

COROLLARY. *For a symmetric loop G, the automorphism group $AUT(G, *)$ of the quasigroup of reflection $(G, *)$ of G is equal to the automorphism group of the* *homogeneous structure of G, identified under the functor of* Theorem 2.

§ 2. Abelian quasigroups of reflection

N. Nobusawa [5] has treated the *homogeneous symmetric structure* of a finite set G which is a synonym of our *finite* quasigroup of reflection. In this section, we investigate the abelian condition for (not necessary finite) quasigroups of reflection, following [5].

Let $(G, *)$ be a quasigroup of reflection. We denote $a*b=S_a b$ for a, b in G. Notice that the reflection S_a across a is denoted here as an operation on the left, in contrast with [5]. Let $L(G)$ be a subgroup of the group of permutations on G generated by $S_a S_b$ for all a, b in G.

LEMMA 1. *$L(G)$ is the left translation group of the symmetric loop $G^{(e)}$ associated with any element $e \in G$ by* Theorem 1 [1].

PROOF. The loop multiplication of the symmetric loop $G^{(e)}$ is defined by $xy=\bar{x}*(e*y)=S_{\bar{x}}S_e y$, for x, y in G, where $\bar{x}$ is an element determined by $\bar{x}*e=x$ (cf. Theorem 1 [1]). Thus the left translation f_x of the symmetric loop $G^{(e)}$ is expressed by $S_{\bar{x}}S_e$ and so the left translation group of $G^{(e)}$ is contained in $L(G)$. Conversely, for any a, b in G, the equalities $S_b S_a=f_b^{(a)}f_b^{(a)}=f_a f_{a^{-1}b} f_{a^{-1}b} f_a^{-1}$ show that $L(G)$ is a subgroup of the left translation group of $G^{(e)}$, where $f_x^{(a)}$ denotes the left translation in the transposed loop $G^{(a)}$ of $G^{(e)}$. *q. e. d.*

We call a quasigroup of reflection $G=(G, *)$ *abelian* if the group $L(G)$ is an abelian group. As a partial generalization of Theorem 4 [5], we have

THEOREM 3. *For a quasigroup of reflection G the following conditions are mutually equivalent*:

(1) *G is abelian.*
(2) *$L(G)=\{S_a S_e;\ a \in G\}$ for a fixed element e in G.*
(3) *The symmetric loop $G^{(e)}$ associated with an element $e \in G$ is an abelian group.*

PROOF. Since each element a of the symmetric loop $G^{(e)}$ has a unique square root $\bar{a}$ given by $\bar{a}*e=a$, and since the left translation f_a in $G^{(e)}$ is expressed by $S_{\bar{a}}S_e$, it is easily seen that the associative law $f_a f_b=f_{ab}$ holds in $G^{(e)}$ if and only if (2) is satisfied. Then, by the symmetric property of $G^{(e)}$ and by Lemma 1, we see that (2) is equivalent to (3). (1)$\Leftrightarrow$(3) follows from Lemma 1 and the following lemma. *q. e. d.*

LEMMA 2. *Let $G^{(e)}$ be a loop with the identity e. The left translation group of $G^{(e)}$ is abelian if and only if $G^{(e)}$ is an abelian group.*

PROOF. Suppose that the left translation group of $G^{(e)}$ is abelian. Then, we see that $G^{(e)}$ is a commutative loop by applying the composition $f_x f_y$ of any two left translations to the identity e. Moreover, using this property, we get $(xy)z=z(xy)=f_z f_x y =f_x f_z y=x(zy)=x(yz)$ for any x, y and z in $G^{(e)}$. Hence $G^{(e)}$ must be an abelian group. The converse is clear. *q.e.d.*

From Theorems 1–3 it follows that abelian quasigroups of reflection correspond one-to-one to homogeneous structures of abelian groups in which each element has a unique square root. That is, if G is an abelian group in which each element has a unique square root, then, with the binary operation $a*b=a^2b^{-1}$, for a, b in G, $(G, *)$ is an abelian quasigroup of reflection, and any abelian quasigroup of reflection is realized in such a way.

§3. Examples of finite symmetric loops

Here, we consider the examples due to N. Nobusawa [5]. A finite quasigroup of reflection (i.e., a homogeneous symmetric set) of 27 elements has been given in §5 [5] as follows:

Let $G=\{1, 2, 3,\dots, 9; 1', 2',\dots, 9'; 1'', 2'',\dots, 9''\}$. The reflections on G are defined as $i*k=2i-k$, $i*k'=(i+k)''$, $i*k''=(k-i)'$; $i'*k=(i+k)''$, $i'*k'=(2i-k)'$, $i'*k''=k-i$; $i''*k=(i-k)'$, $i''*k'=i-k$, $i''*k''=(2i-k)''$, where all integers are considered mod 9. Then $(G, *)$ is a quasigroup of reflection. Associated with $e=1$ of $(G, *)$ we obtain the following symmetric loop which is not an abelian group. The loop multiplication $x\cdot y$ on G is given as $i\cdot k=i+k-1$, $i\cdot k'=(4(i-1)+k)'$, $i\cdot k''=(5(i-1)+k)''$; $i'\cdot k=(i+k-1)'$, $i'\cdot k'=(2i-k+1)''$, $i'\cdot k''=i-k+2$; $i''\cdot k=(i-k+1)''$, $i''\cdot k'=k-i+2$, $i''\cdot k''=(2i-k-1)'$, where all integers are considered mod 9. The left inner mappings $L_{x,y}$'s are not equal to the identity permutation except for $L_{i,j}$ $(i, j=1, 2,\dots, 9)$. The subloop $G_0=\{1, 2,\dots, 9\}$ is an abelian subgroup.

In the table of symmetric structures of a set $\{1, 2, 3, 4, 5\}$ of §6 [5], only homogeneous case is of Type 1. The associated symmetric loop of this type of quasigroup of reflection is the cyclic group of order 5.

§4. Final remarks

In the previous papers [1] and [2], we have investigated the conditions for loops to be global algebraic systems on symmetric spaces (reflection spaces) given by quasigroups of reflection, and arrived at the concept of symmetric loops. Furthermore, in [3], we have considered homogeneous loops, including both of groups and symmetric loops as special classes. Then, by supposing the natural differentiable structure on a homogeneous loop G, we have introduced the concept of homogeneous Lie loops and, giving some geometric interpretations on the multiplication of G, we have developed in [3] and [4] the theory of geodesic homogeneous Lie loops and their tangent algebras, called Lie triple algebras. In these discussions, in order to consider the special case when G is reduced to a symmetric space, it is rather convenient to assume only the symmetric property (1.1) of Theorem 1 for G, than assuming to be a symmetric loop, and so we call G a *symmetric Lie loop* if G is a connected homogeneous Lie loop with the symmetric property (cf. Theorem 6.1 and Definition 6.1 in [3]). In fact, if G is a symmetric Lie loop, then G is left alternative, i.e., (1.2) of Theorem 1 is satisfied, as was shown in Lemma 6.2 [3]. Moreover, G is an affine symmetric homogeneous space by Theorem 6.1 [3], and a curve $x(t)$ through the identity $e=x(0)$ of G is a geodesic curve if and only if $x(t)$ is a 1-parameter subgroup of G (cf. Propositions 5.4–5, Definition 5.1 and Theorem 6.4 in [3]), and so G can be regarded as satisfying the unique square root property (1.3) of Theorem 1 *in local.*

Now, by applying the results in § 7 [3] and in [4], we have the following results on symmetric Lie loops.

THEOREM 4. *Let G be a symmetric Lie loop. The Lie triple algebra $\mathfrak{G}$ of G is a Lie triple system under the ternary operation $[X, Y, Z]=R_e(X, Y)Z$ for tangent vectors X, Y, Z at the identity e, where R denotes the curvature of the canonical connection of G. Two symmetric Lie loops are locally isomorphic if and only if their Lie triple systems are isomorphic.*

PROOF. These results are easily seen by Remarks 6.1, 7.1 and Corollary 7.9 in [3]. *q.e.d.*

By Theorem 2 and Proposition 2 in [4] we have

THEOREM 5. *A connected Lie subloop H of a symmetric Lie loop G is itself a symmetric Lie loop whose Lie triple system $\mathfrak{H}$ is a subsystem of the Lie triple system $\mathfrak{G}$ of G.*

We call H in the above theorem a *symmetric Lie subloop* of G. A symmetric Lie subloop H (resp. subsystem $\mathfrak{H}$) of a symmetric Lie loop G (resp. the Lie triple system $\mathfrak{G}$ of G) is called *left invariant* if H (resp. $\mathfrak{H}$) is invariant under the left inner mapping group of G.

The following theorem is obtained as a corollary to Theorem 1 [4].

THEOREM 6. *Let G be a symmetric Lie loop and $\mathfrak{G}$ its Lie triple system. For any left invariant symmetric Lie subloop H, its Lie triple system $\mathfrak{H}$ is a left invariant subsystem of $\mathfrak{G}$. Conversely, for any left invariant subsystem $\mathfrak{H}$ of $\mathfrak{G}$, there exists a unique left invariant symmetric Lie subloop H of G whose Lie triple system is $\mathfrak{H}$.*

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