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Normal Systems in an Orthogroup

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In this paper we firstly introduce the concept of a normal system in an orthogroup and the concept of the kernel system of a homomorphism of an orthogroup. The normal systems in an orthogroup are completely determined. In the latter half of the paper, some generalizations of results appear in the group theory are established for orthogroups by using the above-mentioned concepts instead of the concepts of a normal subgroup and the kernel of a homomorphism in the group theory. The complete proofs are almost omitted, and only an outline of the results is given.

§1. Normal systems

Let S be an orthogroup (that is, a semigroup which is a union of groups and whose idempotents form a band), and E_S the band of idempotents of S. Let N be a suborthogroup of S (that is, a subsemigroup which is an orthogroup with respect to the multiplication in S) such that $N \supset E_S$, and σ a band congruence on N (that is, a congruence σ on N such that the factor semigroup N/σ of N mod σ is a band). If there exists a congruence ρ on S such that $x\sigma = x\rho$ for all $x \in N$, where $x\sigma$, $x\rho$ are the σ-class and the ρ-class containing x respectively, then the pair (N, σ) of N, σ is called a *normal system* in S. In this case, the congruence ρ above is uniquely determined (when such ρ exists) and is denoted by σ_S (see [1]). This σ_S will be called the *natural extension* of σ to S. It is easy to see that σ_S is the congruence on S generated by σ. Now, it is also easily seen that any homomorphic image of an orthogroup is also an orthogroup. Let $\varphi: S \to T$ be a homomorphism of an orthogroup S onto an orthogroup T, and φ^i the congruence on S induced by φ (that is, for $a, b \in S$, $a\varphi^i b$ if and only if $a\varphi = b\varphi$). The collection $\{\alpha\varphi^{-1}: \alpha \in E_T$ (the set of idempotents of T)$\}$ is called the *kernel* of φ, and denoted by $\mathrm{Ker}\,\varphi$. The subset $N = \cup\, \mathrm{Ker}\,\varphi = \cup\, \{\alpha\varphi^{-1}: \alpha \in E_T\}$ of S is clearly a suborthogroup of S. The restriction ${}_N\varphi^i$ of φ^i to N (that is, for any $a, b \in N$, $a\,{}_N\varphi^i b$ if and only if $a\varphi^i b$) is a band congruence on N, and $(N, {}_N\varphi^i)$ is a normal system in S since ${}_N\varphi^i$ has φ^i as its natural extension to S. This $(N, {}_N\varphi^i)$ is called the *kernel system* of φ. The identity congruence and the universal congruence on a semigroup S will be denoted by 1_S, 0_S respectively. Thus, $x1_Sy$ if and only if $x = y$; and $x0_Sy$ if and only if $x, y \in S$. If N is a suborthogroup of an orthogroup S and if $(N, 0_N)$ is a normal system in S, then $(N, 0_N)$ is especially called a *normal suborthogroup* of S. A normal suborthogroup $(N, 0_N)$ is simplv denoted by N, if there is no confusion. When $(N,$

σ) is a normal system in an orthogroup S and σ_S is the natural extension of σ to S, the factor semigroup S/σ_S is sometimes denoted by $S/(N, \sigma)$. In particular, if $\sigma=0_N$ then $S/(N, \sigma)$ is simply denoted by S/N. Hereafter, the band of idempotents of an orthogroup S will be denoted by E_S. If H is a suborthogroup of an orthogroup S and if σ is a congruence on S, then the restriction of σ to H will be denoted by ${}_H\sigma$. Therefore, for any $a, b \in H$, $a_H\sigma b$ if and only if $a\sigma b$.

REMARKS. The following results are obvious from the definitions above:

1. *If N is a normal suborthogroup of an orthogroup S, then S/N is a group.*

It is well-known that for any element c of an orthogroup S there exists a unique inverse c^* of c such that $cc^*=c^*c$. Hereafter, this inverse c^* of c will be denoted by c^{-1}.

2. *If S is a group and A is a suborthogroup of S, then A is a group and A is a normal suborthogroup of $N(A)=\{x \in S: x^{-1}Ax=A\}$.*

When N is a suborthogroup of an orthogroup S and σ is a band congruence on N, we can obtain the following results as conditions for (N, σ) to be a normal system in S.

THEOREM 1. *Let S be an orthogroup, and N a suborthogroup such that $N \supset E_S$. Let σ be a band congruence on N. Then, (N, σ) is a normal system in S if and only if it satisfies the following* (1.1):

(1.1) $(a, b) \in \sigma$, $cad \in N$ *for* $c, d \in S^1$ *implies* $cbd \in N$ *and* $(cad, cbd) \in \sigma$.

REMARK. It can be easily proved that the condition (1.1) is equivalent to the following (1.2):

(1.2) (1) For any $c \in S$, $c^{-1}Nc$ (denoted by N^c)$\subset N$,
(2) for any $c \in S$, $\sigma^c=\{(c^{-1}ac, c^{-1}bc): (a, b) \in \sigma\} \subset \sigma$, and
(3) $(a, b) \in \sigma$, $ca \in N$, $c \in S$ imply $cb \in N$ and $(ca, cb) \in \sigma$;
and $(a, b) \in \sigma$, $ac \in N$, $c \in S$ imply $bc \in N$ and $(ac, bc) \in \sigma$.

If the band E_S of idempotents of an orthogroup S satisfies the following condition (1.3), then S is said to be *strictly inversive*:

(1.3) If $e, f \in E_S$, $e \leqq f$ (that is, $ef=fe=e$) and $xx^{-1}=f$, then $ex=xe$.

It is well-known (see [2]) that a strictly inversive orthogroup S can be uniquely decomposed into a band of groups, that is, there exists a unique congruence σ on S such that S/σ is a band and each σ-class is a subgroup of S. This σ is actually the least band congruence on S.

As special cases of Theorem 1, we have the following results.

THEOREM 2. *Let S be an orthogroup, and N a strictly inversive suborthogroup* *of S such that* $N \supset E_S$. *Let* σ *be the least band congruence on N. Then,* (N, σ) *is a normal system in S if and only if* (N, σ) *satisfies* (1), (2) of (1.2).

THEOREM 3. *Let S be a strictly inversive orthogroup, and N a suborthogroup of S such that* $N \supset E_S$. *Let* σ *be the least band congruence on N. Then,* (N, σ) *is a normal system in S if and only if* $N^c \subset N$ *for all* $c \in S$.

COROLLARY. *Let S be a strictly inversive orthogroup, and* (N, ξ) *a normal system in S. Let* σ *be the least band congruence on N. Then,* (N, σ) *is also a normal system in S.*

REMARK. It is also true that if the band E_S of idempotents of a strictly inversive orthogroup S is a normal suborthogroup, then (E_S, ρ) is a normal system in S for any band congruence ρ on E_S.

§2. Isomorphism theorems

In this §, we shall show analogues to the fundamental theorem for homomorphisms, the first isomorphism theorem, the second isomorphism theorem and the Zassenhaus theorem of the group theory.

First, we have

THEOREM 4. *Let S be an orthogroup, and* (N, σ) *a normal system in S. Then,* $f: S \to S/(N, \sigma)$ $(=S/\sigma_S)$ *defined by* $xf = x\sigma_S$ *(the* σ_S*-class containing x),* $x \in S$, *is a surjective homomorphism. In this case, the kernel system of f is* (N, σ), *that is,* $\cup \operatorname{Ker} f = N$ *and* ${}_N f^i = \sigma$.

Conversely, if $g: S \to T$ *is a surjective homomorphism of an orthogroup S to an orthogroup T, then the kernel system* $(N, {}_N g^i)$ *is a normal system in S and* $S/(N, {}_N g^i)$ *is isomorphic to T.*

COROLLARY. *If* $\varphi: S \to G$ *is a surjective homomorphism of an orthogroup S to a group G, then* $A = \operatorname{Ker} \varphi$ *is a normal suborthogroup of S and S/A is isomorphic to A.*

Let (N, σ) and (M, ρ) be normal systems in an orthogroup S. If $N \subset M$, it can be easily proved that (N, σ) is a normal system in M. In general, the following is true.

THEOREM 5. *Let S be an orthogroup, and* (N, σ) *a normal system in S. Let H be a suborthogroup of S such that* $H \supset E_S$. *Then,* $(H \cap N, {}_{H \cap N}\sigma)$ *is a normal system in H.*

Now, if $N \subset M$, $\sigma_M \subset \rho$ and $x\sigma_S = x\sigma_M$ for all $x \in M$ (σ_M, σ_S are the natural extensions of σ to M and S respectively), then we say that (M, ρ) *contains* (N, σ) (with respect to σ). This will be denoted by $(N, \sigma) \underset{\sigma}{\subset} (M, \rho)$. Let $\varphi: S \to T$ be a homomorphism of an orthogroup S onto an orthogroup T, and put $\cup \operatorname{Ker} \varphi = M$. Let (N, σ) be a normal system in T. We have $L=N\varphi^{-1}\subset M$ since $N\supset E_T$. Now, define a band congruence $\sigma\varphi^{-1}$ on L as follows:

(2.1) For $x, y\in L$, $x\sigma\varphi^{-1}y$ if and only if $(x\varphi)\sigma(y\varphi)$.

Then, it is easily proved that $(N\varphi^{-1}, \sigma\varphi^{-1})$ is a normal system in S such that $(M, {}_M\varphi^i)\underset{M\varphi^i}{\subset}(N\varphi^{-1}, \sigma\varphi^{-1})$.

Conversely, let (K, ρ) be a normal system in S such that $(M, {}_M\varphi^i)\underset{M\varphi^i}{\subset}(K, \rho)$. Define a band congruence $\rho\varphi$ on $K\varphi$ as follows:

(2.2) For $u, v\in K\varphi$, $u\rho\varphi v$ if and only if $u=x\varphi$, $v=y\varphi$ and $x\rho y$ for some $x, y\in K$.

Then, it is also easily proved that $(K\varphi, \rho\varphi)$ is a normal system in T.

Now, let $\mathscr{S}=\{(K, \rho): (K, \rho)$ is a normal system in S such that $(M, {}_M\varphi^i)\underset{M\varphi^i}{\subset}(K, \rho)\}$ and $\mathscr{T}=\{(N, \sigma): (N, \sigma)$ is a normal system in $T\}$. Define $\psi: \mathscr{S}\to\mathscr{T}$ and $\phi: \mathscr{T}\to\mathscr{S}$ by $(K, \rho)\psi=(K\varphi, \rho\varphi)$ and $(N, \sigma)\phi=(N\varphi^{-1}, \sigma\varphi^{-1})$ respectively. Then, it can be proved that $\psi\phi=i_{\mathscr{S}}$ (the identity mapping on $\mathscr{S}$) and $\phi\psi=i_{\mathscr{T}}$ (the identity mapping on $\mathscr{T}$). Consequently, both ϕ and ψ are injective and surjective. Therefore, we have

THEOREM 6. *Let $\varphi: S\to T$ be a homomorphism of an orthogroup S onto an orthogroup T, and put* $\cup$ Ker $\varphi=M$. *Let $\mathscr{S}$ be all the normal systems (K, ρ) in S such that $(M, {}_M\varphi^i)\underset{M\varphi^i}{\subset}(K, \rho)$, and $\mathscr{T}$ all the normal systems in T. Then, $\psi: \mathscr{S}\to\mathscr{T}$ defined by $(K, \rho)\psi=(K\varphi, \rho\varphi)$ gives an 1–1 correspondence between $\mathscr{S}$ and $\mathscr{T}$.*

Next, let us consider isomorphism theorems. If σ, ρ are congruences on a semigroup S such that $\sigma\subset\rho$, then ρ/σ denotes the congruence on S/σ defined as follows:

(2.3) For $\bar{a}, \bar{b}\in S/\sigma$ (where $\bar{x}$ is the σ-class containing x), $\bar{a}\ \rho/\sigma\ \bar{b}$ if and only if $a\rho b$.

Then, firstly we have the following result as a generalization of the first isomorphism theorem of the group theory.

THEOREM 7. (*The first isomorphism theorem*).

Let S be an orthogroup, and (N, σ), (M, ρ) normal systems in S such that $(N, \sigma)\underset{\sigma}{\subset}(M, \rho)$. Then,

$$S/(N, \sigma)\big/(M/(N, \sigma), \rho/\sigma_M)\cong S/(M, \rho).$$

Further, by using Theorem 5, we can obtain the following result. If M, N are suborthogroups of an orthogroup S such that $M\supset E_S$ and $N\supset E_S$, then $M\vee N$ denotes the least suborthogroup containing $M\cup N$ (the set union), that is, the suborthogroup generated by $M\cup N$.

THEOREM 8. (*The second isomorphism theorem*).

Let S be an orthogroup, and (N, σ) a normal system in S. Let M be a suborthogroup of S such that $M \supset E_S$. Then,

$$M \vee N/(N, \sigma) \cong M/(M \cap N, {}_{M \cap N}\sigma).$$

REMARK. It is easily proved that if (N, σ) is a normal system in a strictly inversive orthogroup S and if M is a suborthogroup of S such that $M \supset E_S$, then $NM = MN = M \vee N$.

If S is a strictly inversive orthogroup, then every suborthogroup of S is also strictly inversive. Therefore, if N is a suborthogroup of S such that $N \supset E_S$ and if σ is the least band congruence on N then σ gives the decomposition of N into a bandE_S of groups; that is, $N/\sigma \cong E_S$ and each σ-class is a subgroup of S. We shall denote the pair (N, σ) simply by $[N]$; $[N] = (N, \sigma)$.

Now, finally we have the following result as a generalization of the Zassenhaus theorem of the group theory.

THEOREM 9. (*A generalization of the Zassenhaus theorem*)

Let S be a strictly inversive orthogroup, and H, H_1, K, K_1 suborthogroups of S such that $H \supset H_1 \supset E_S$ and $K \supset K_1 \supset E_S$. If $[H_1]$ is a normal system in H and if $[K_1]$ is a normal system in K, then

(1) $H_1 \vee (H \cap K) = H_1(H \cap K)$, $K_1 \vee (K \cap H) = K_1(K \cap H)$, $H_1 \vee (H \cap K_1) = H_1(H \cap K_1)$, $K_1 \vee (K \cap H_1) = K_1(K \cap H_1)$; *and* $[H_1(H \cap K_1)]$ *is a normal system in* $H_1(H \cap K)$ *and* $[K_1(K \cap H_1)]$ *is a normal system in* $K_1(K \cap H)$; *and*

(2) $$H_1(H \cap K)/[H_1(H \cap K_1)] \cong K_1(K \cap H)/[K_1(K \cap H_1)].$$

§ 3. Subdirect decompositions and direct decompositions

Let S be an orthogroup, and (A, σ), (B, ρ) normal systems in S. Suppose that

(3.1) $$\sigma \cap \rho = i_{A \cap B} \text{ (the identity congruence on } A \cap B).$$

Let $<\sigma \cap \rho>_S$ be the congruence on S generated by $\sigma \cap \rho$. Then, $<\sigma \cap \rho>_S = i_S$ is obvious. Since ${}_A(\sigma_S \cap \rho_S) \subset \sigma$ and ${}_B(\sigma_S \cap \rho_S) \subset \rho$, it follows that ${}_{A \cap B}(\sigma_S \cap \rho_S) \subset \sigma \cap \rho = i_{A \cap B}$. Now, let $a(\sigma_S \cap \rho_S)x$ for $a \in A \cap B$ and $x \in S$. Then, $a\sigma_S x$ and $a\rho_S x$. Since (A, σ) is a normal system in S, $a\sigma_S = a\sigma$ holds. Therefore, $x \in A$. Similarly, $x \in B$. Hence, $x \in A \cap B$. Thus, $(A \cap B, {}_{A \cap B}(\sigma_S \cap \rho_S))$ is a normal system in S and the natural extension of ${}_{A \cap B}(\sigma_S \cap \rho_S)$ to S is $\sigma_S \cap \rho_S$. Since ${}_{A \cap B}(\sigma_S \cap \rho_S) = i_{A \cap B}$, we have $\sigma_S \cap \rho_S = i_S$. Hence, the mapping $\psi: S \to S/\sigma_S \times S/\rho_S$ (direct product) defined by $x\psi = (x\sigma_S, x\rho_S)$ is an injective homomorphism. It is obvious that $S\psi p_1 = \{x\psi p_1 : x \in S\} = S/\sigma_S$ and

$S\psi p_2=\{x\psi p_2: x\in S\}=S/\rho_S$, where p_1, p_2 denote the first and the second projections of $S/\sigma_S\times S/\rho_S$ respectively. Thus, we have the following result.

THEOREM 10. *Let S be an orthogroup, and* (A, σ), (B, ρ) *normal systems in S. If* $\sigma\cap\rho=i_{A\cap B}$, *then S is isomorphic to a subdirect product of* S/σ_S *and* S/ρ_S.

REMARK. By using the theorem above, we can obtain the following results:
I. *If S is a strictly inversive orthogroup and if* E_S *is a normal suborthogroup of S, then S is isomorphic to a subdirect product of a group and* E_S.

It should be also noted that E_S is a normal suborthogroup of S if and only if it satisfies the condition

(3.2) $ce\in E_S$, $c\in S$, $e\in E_S$ imply $c\in E_S$, and $ec\in E_S$, $c\in S$, $e\in E_S$ imply $c\in E_S$.

II. *If S is a commutative orthogroup and if S satisfies the condition*

(3.3) $aa^{-1}=bb^{-1}$ *and* $at=bt$, $t\in E_S$ *imply* $a=b$,

then S is isomorphic to a subdirect product of a commutative group and a semilattice.
III. *If S is an orthogroup in which the band of idempotents is isomorphic to the direct product of a rectangular band and a semilattice, then S is isomorphic to a subdirect product of a* (C)-*inversive orthogroup (that is, an orthogroup which is a semilattice of groups) and a rectangular band.*

Finally, again let (A, σ), (B, ρ) be normal systems in an orthogroup S. Consider the following condition for $\sigma\circ\rho$*).

(3.4) For any x, $y\in S$, there exists $w\in S$ such that $wx\,(\sigma\circ\rho)\,wy$ (hence, of course $wx\in A$ and $wy\in B$), $ww^{-1}xx^{-1}\sigma xx^{-1}$ and $ww^{-1}yy^{-1}\rho yy^{-1}$.

Suppose that (A, σ), (B, ρ) satisfy the conditions (3.1) and (3.4). For any x, $y\in S$, there exists $w\in S$ such that $wx\sigma t$ and $t\rho wy$ for some $t\in A\cap B$ and $ww^{-1}xx^{-1}\sigma xx^{-1}$, $ww^{-1}yy^{-1}\rho yy^{-1}$. Now, $ww^{-1}x\sigma_S w^{-1}t$ and $w^{-1}t\rho_S w^{-1}wy$. On the other hand, $w^{-1}wxx^{-1}\sigma xx^{-1}$ implies $w^{-1}wx\sigma_S x$. Similarly, $ww^{-1}y\rho_S y$. Hence, $x\sigma_S w^{-1}wx\sigma_S w^{-1}t\rho_S w^{-1}wy\rho_S y$, that is, $x\,(\sigma_S\circ\rho_S)\,y$. By (3.1), $\sigma_S\cap\rho_S=i_S$. Therefore, $f: S\rightarrow S/\sigma_S\times S/\rho_S$ defined by $xf=(x\sigma_S, x\rho_S)$ is an isomorphism. Thus, we have the following theorem.

THEOREM 11. *Let* (A, σ), (B, ρ) *be normal systems in an orthogroup S. If* (A, σ), (B, ρ) *satisfy the conditions* (3.1), (3.4), *then S is isomorphic to the direct product of* $S/(A, \sigma)$ *and* $S/(B, \rho)$.

References

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*) $\sigma\circ\rho=\{(x, y): x, y\in S, (x, t)\in\sigma \text{ and } (t, y)\in\rho \text{ for some } t\in S\}$.